

Microlocalization of Topological Boundary Value Morphism and Regular-Specializable Systems

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Introduction

In microlocal analysis, it is one of the main subjects to give an appropriate formulation of the boundary value problems for hyperfunction or microfunction solutions to a system of linear partial differential equations with analytic coefficients (that is, a coherent (left) $\mathcal{D}$ -Module, here in this article, we shall write *Module* with a capital letter, instead of *sheaf of modules*). If the system is *regular-specializable*, the *nearby-cycle* of the system can be defined in the theory of $\mathcal{D}$ -Modules. After the results by Kashiwara and Oshima [K-O], Oshima [Os] and Schapira [Sc 2], [Sc 3], for any hyperfunction solutions to regular-specializable system Monteiro Fernandes [MF 1] defined a boundary value morphism which takes values in hyperfunction solutions to the nearby-cycle of the system instead of the *induced* system. This morphism is injective (cf. [MF 2]) and a generalization of the non-characteristic boundary value morphism (for the non-characteristic case, see Komatsu and Kawai [Ko-K], Schapira [Sc 1] and further Kataoka [Kat]). Moreover recently Laurent and Monteiro Fernandes [L-MF 2] reformulated this boundary value morphism and discussed the solvability under a kind of hyperbolicity condition (the *near-hyperbolicity*). However, since this morphism is defined only for hyperfunction solutions, a microlocal boundary value problem is not considered. Therefore in this article, we shall state a microlocalization of their result in the framework of Oaku [Oa 2] and Oaku-Yamazaki [O-Y].

The details of this article will be given in our forthcoming paper [Y].

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1 Notation

We denote the set of integers, of real numbers and of complex numbers by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ respectively as usual. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z}; n \geq 1\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

All the manifolds are assumed to be paracompact. Let $\tau: E \rightarrow Z$ a vector bundle over a manifold Z . Then, set $\dot{E} := E \setminus Z$ and $\dot{\tau}$ the restriction of τ to $\dot{E}$. Let M be an $(n+1)$ -dimensional real analytic manifold and N a one-codimensional closed real analytic submanifold of M . Let X and Y be complexifications of M and N respectively such that Y is a closed submanifold of X and that $Y \cap M = N$. Moreover, we assume the existence of a partial complexification of M in X ; that is, there exists a $(2n+1)$ -dimensional real analytic submanifold L of X containing both M and Y such that the triplet (N, M, L) is locally isomorphic to $(\mathbb{R}^n \times \{0\}, \mathbb{R}^{n+1}, \mathbb{C}^n \times \mathbb{R})$ by a local coordinate system $(z, \tau) = (x + \sqrt{-1}y, t + \sqrt{-1}s)$ of X around each point of N . We say such a coordinate system *admissible*. We shall mainly follow the notation in Kashiwara-Schapira [K-S]; we denote the normal deformations of N and Y in M and L by $\widetilde{M}_N$ and $\widetilde{L}_Y$ respectively and regard $\widetilde{M}_N$ as a closed submanifold of $\widetilde{L}_Y$. We have the following commutative diagram:

$$\begin{array}{ccccccc}
 T_N M & \xhookrightarrow{s_M} & \widetilde{M}_N & \xleftarrow{j_M} & \Omega_M & & \\
 \downarrow \tau_N & & \downarrow p_M & \swarrow \tilde{p}_M & \downarrow & & \\
 N & \xhookrightarrow{\quad} & M & \xleftarrow{i_M} & X & & \\
 \downarrow \tau_Y & & \downarrow & & \downarrow & & \\
 T_Y L & \xhookrightarrow{s_L} & \widetilde{L}_Y & \xleftarrow{j_L} & \Omega_L & & \\
 \downarrow \tau_Y & & \downarrow p_L & \swarrow \tilde{p}_L & \downarrow & & \\
 Y & \xhookrightarrow{i_Y} & L & \xleftarrow{i_L} & X & &
 \end{array}$$

and by admissible coordinates we have locally the following relation:

$$\begin{array}{ccccc}
 N = \mathbb{R}_x^n \times \{0\} & \xhookrightarrow{\quad} & M = \mathbb{R}_x^n \times \mathbb{R}_t & & \\
 \downarrow & & \downarrow i & \searrow i_M & \\
 Y = \mathbb{C}_z^n \times \{0\} & \xhookrightarrow{i_Y} & L = \mathbb{C}_z^n \times \mathbb{R}_t & \xhookrightarrow{i_L} & X = \mathbb{C}_z^n \times \mathbb{C}_\tau.
 \end{array}$$

With these coordinates, we often identify $T_Y X$ and $T_Y L$ with X and L respectively.

The projection $\tau_Y: T_Y L \rightarrow Y$ and $s_L: T_Y L \rightarrow \widetilde{L}_Y$ induce natural mappings:

$$T_N^* Y \xleftarrow{\tau_{Y\pi}} T_N M \times_N T_N^* Y \xrightarrow{\iota_{Y'}} T_{T_N M}^* T_Y L \xleftarrow{\iota_{s_L}'} T_N M \times_{\widetilde{M}_N} T_{\widetilde{M}_N}^* \widetilde{L}_Y \xrightarrow{s_{L\pi}} T_{\widetilde{M}_N}^* \widetilde{L}_Y,$$

and by these mappings, we identify $T_{T_N M}^* T_Y L$ with $T_N M \times_N T_N^* Y$ and $T_N M \times_{\widetilde{M}_N} T_{\widetilde{M}_N}^* \widetilde{L}_Y$.

$T_Y L \setminus T_Y Y$ has two components with respect to its fiber. We denote one of them by $T_Y L^+$ and represent (at least locally) by fixing an admissible coordinate system

$$T_Y L^+ = \{(z, t) \in T_Y L; t > 0\}.$$

Moreover set $T_N M^+ := T_Y L^+ \cap T_N M$. Set an open embedding $f: T_Y L^+ \hookrightarrow T_Y L$ and $f_N := f|_{T_N M^+}: T_N M^+ \hookrightarrow T_N M$. We regard $T_N M^+ \times_N T_N^* Y$ as an open set of $T_{T_N M}^* T_Y L$. Moreover f induces mappings:

$$\begin{array}{ccc} T_{T_N M^+}^* T_Y L^+ & \xleftarrow{\quad} & T_N M^+ \times_{T_N M} T_{T_N M}^* T_Y L \xrightarrow{f_\pi} T_{T_N M}^* T_Y L \\ & & \downarrow \quad \quad \quad \circlearrowleft \quad \quad \quad \downarrow \\ & & T_N M^+ \times_N T_N^* Y \xrightarrow{f_N \times \text{id}} T_N M \times_N T_N^* Y. \end{array}$$

Hence we identify $T_{T_N M^+}^* T_Y L^+$ with $T_N M^+ \times_N T_N^* Y$, and f_π with $f_N \times \text{id}$.

Let $\pi_{N,M}: T_{\widetilde{M}_N}^* \widetilde{L}_Y \rightarrow \widetilde{M}_N$ and $\pi_{N|M}: T_{T_N M}^* T_Y L \rightarrow T_N M$, be the natural projections. We denote as usual by ν and μ the Sato specialization and microlocalization functors respectively.

2 General Boundary Values

By using an admissible coordinate system we define a continuous section $\sigma: Y \rightarrow \dot{T}_Y X$ by $z \mapsto (z, 1)$. Similarly we define $\iota\sigma: Y \rightarrow \dot{T}_Y^* X$ by $z \mapsto (z, 1)$. In general, let Z be a complex manifold, $\tau: E \rightarrow Z$ a complex vector bundle. Then, denote by $\mathbf{D}_{\mathbb{C}^\times}^b(E)$ the subcategory of $\mathbf{D}^b(E)$ consisting of $\mathbb{C}^\times$ -conic objects.

2.1 Theorem. *For any object $\mathcal{F}$ of $\mathbf{D}^b(X)$ such that $\nu_Y(\mathcal{F}) \in \text{Ob}(\mathbf{D}_{\mathbb{C}^\times}^b(T_Y X))$, there exists the following natural isomorphism:*

$$f_\pi^{-1} \mu_{T_N M}(\nu_Y(i_L^! \mathcal{F})) \simeq f_\pi^{-1} \tau_{Y\pi}^{-1} \mu_N(\sigma^{-1} \nu_Y(\mathcal{F})) \otimes \omega_{L/X}.$$

2.2 Definition. For any object $\mathcal{F}$ of $\mathbf{D}^b(X)$ such that $\nu_Y(\mathcal{F}) \in \text{Ob}(\mathbf{D}_{\mathbb{C}^\times}^b(T_Y X))$, we define by virtue of Kashiwara-Schapira [K-S] and Theorem 2.1:

$$\begin{aligned} \beta: f_\pi^{-1} s_{L\pi}^{-1} \mu_{\widetilde{M}_N}(\mathbf{R}j_{L*} \widetilde{p}_L^{-1} i_L^! \mathcal{F}) &\rightarrow f_\pi^{-1} \mu_{T_N M}(\nu_Y(i_L^! \mathcal{F})) \\ &\simeq f_\pi^{-1} \tau_{Y\pi}^{-1} \mu_N(\sigma^{-1} \nu_Y(\mathcal{F})) \otimes \omega_{L/X}. \end{aligned}$$

2.3 Definition (Laurent-Monteiro Fernandes [L-MF 2]). We say an object $\mathcal{F}$ of $\mathbf{D}^b(X)$ is *near-hyperbolic* at $x_0 \in N$ (in dt -codirection) if there exist positive constants C and ε_1 such that

$$\begin{aligned} \text{SS}(\mathcal{F}) \cap \{(z, \tau; z^*, \tau^*) \in T^* X; |z - x_0|, |\tau| < \varepsilon_1, 0 < \text{Re } \tau\} \\ \subset \{(z, \tau; z^*, \tau^*) \in T^* X; |\text{Re } \tau^*| < C(|\text{Im } z^*|(|\text{Im } z| + |\text{Im } \tau|) + |\text{Re } z^*|)\} \end{aligned}$$

holds by an admissible coordinate system. Here $\text{SS}(\mathcal{F})$ denotes the *microsupport* of $\mathcal{F}$.

2.4 Theorem. Let $\mathcal{F}$ be a object of $\mathbf{D}^b(X)$. Assume that $\nu_Y(\mathcal{F}) \in \text{Ob}(\mathbf{D}_{\mathbb{C}^\times}^b(T_Y X))$ and $\mathcal{F}$ is near-hyperbolic at $x_0 \in N$. Then, for any $p^* \in T_{T_N M}^* T_Y L^+$

$$\beta: s_{L^\pi}^{-1} \mu_{\widetilde{M}_N}(\mathbf{R}j_{L*} \widetilde{p}_L^{-1} i_L^! \mathcal{F})_{p^*} \rightarrow \mu_N(\sigma^{-1} \nu_Y(\mathcal{F}))_{\tau_{Y\pi}(p^*)} \otimes \omega_{L/X}$$

is an isomorphism.

3 Regular-Specializable Systems

In this section, we shall recall the basic results concerning the regular-specializable $\mathcal{D}$ -Module and its nearby-cycle.

As usual, we denote by $\mathcal{D}_X$ the sheaf on X of holomorphic differential operators, and by $\{\mathcal{D}_X^{(m)}\}_{m \in \mathbb{N}_0}$ the usual order filtration on $\mathcal{D}_X$.

3.1 Definition. Denote by $\mathcal{I}_Y$ the defining Ideal of Y in $\mathcal{O}_X$ with a convention that $\mathcal{I}_Y^j = \mathcal{O}_X$ for $j \leq 0$. The V -filtration $\{V_Y^k(\mathcal{D}_X)\}_{k \in \mathbb{Z}}$ (along Y) is a filtration on $\mathcal{D}_X|_Y$ defined by

$$V_Y^k(\mathcal{D}_X) := \bigcap_{j \in \mathbb{Z}} \{P \in \mathcal{D}_X|_Y; P \mathcal{I}_Y^j \subset \mathcal{I}_Y^{j-k}\}.$$

Let us denote by ϑ the Euler operator. Note that $\vartheta \in V_Y^0(\mathcal{D}_X) \setminus V_Y^{-1}(\mathcal{D}_X)$ and that ϑ can be represented by $\tau \partial_\tau$ by admissible coordinates.

3.2 Definition. A coherent $\mathcal{D}_X|_Y$ -Module $\mathcal{M}$ is said to be *regular-specializable (along Y)* if there exist locally a coherent $\mathcal{O}_X$ -sub-Module $\mathcal{M}_0$ of $\mathcal{M}$ and a non-zero polynomial $b(\alpha) \in \mathbb{C}[\alpha]$ such that the following conditions are satisfied:

- (1) $\mathcal{M}_0$ generates $\mathcal{M}$ over $\mathcal{D}_X$; that is, $\mathcal{M} = \mathcal{D}_X \mathcal{M}_0$;
- (2) $b(\vartheta) \mathcal{M}_0 \subset (\mathcal{D}_X^{(m)} \cap V_Y^{-1}(\mathcal{D}_X)) \mathcal{M}_0$, where m is the degree of $b(\alpha)$.

In what follows, we shall omit the phrase “along Y ” since Y is fixed.

3.3 Remark. (1) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X|_Y$ -Module for which Y is non-characteristic. Then, it is easy to see that $\mathcal{M}$ is regular-specializable.

(2) Kashiwara-Kawai [K-K] proved that every regular-holonomic $\mathcal{D}_X|_Y$ -Module is regular-specializable.

3.4 Proposition. If $\mathcal{M}$ is a regular-specializable $\mathcal{D}_X|_Y$ -Module, $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_Y(\mathcal{O}_X))$ and $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_Y(\mathcal{O}_X))$ are objects of $\mathbf{D}_{\mathbb{C}^\times}^b(T_Y^* X)$ and $\mathbf{D}_{\mathbb{C}^\times}^b(T_Y X)$ respectively.

Let $\iota: Y \rightarrow X$ be the natural inclusion. Then the *induced system*, or the *inverse image* in the sense of $\mathcal{D}$ -Modules is defined by $\mathbf{D}\iota^* \mathcal{M} := \mathcal{O}_Y \overset{L}{\otimes}_{\iota^{-1}\mathcal{O}_X} \iota^{-1} \mathcal{M}$.

For any regular-specializable $\mathcal{D}_X$ -Module $\mathcal{M}$, the *nearby-cycle* $\Psi_Y(\mathcal{M})$ of $\mathcal{M}$ and the *vanishing-cycle* $\Phi_Y(\mathcal{M})$ of $\mathcal{M}$ in the theory of $\mathcal{D}$ -Modules can be defined. For the definitions of $\Psi_Y(\mathcal{M})$ and $\Phi_Y(\mathcal{M})$, we refer to Laurent [L], Mebkhout [Me]. We shall recall the following two results:

3.5 Proposition (Laurent [L], Mebkhout [Me]). *Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$ -Module. Then, $\Psi_Y(\mathcal{M})$, $\Phi_Y(\mathcal{M})$ and each cohomology of $D\iota^* \mathcal{M}$ are coherent $\mathcal{D}_Y$ -Modules. Moreover, there exists the following distinguished triangle:*

$$\Phi_Y(\mathcal{M}) \xrightarrow{\text{Var}} \Psi_Y(\mathcal{M}) \rightarrow D\iota^* \mathcal{M} \xrightarrow{+1}.$$

Here, $\text{Var} := \varphi(\vartheta)\tau$ with $\varphi(\zeta) := (e^{2\pi\sqrt{-1}\zeta} - 1)/\zeta$.

3.6 Theorem (Laurent [L]). *Let $\mathcal{C}_{Y|X}^{\mathbb{R}}$ be the sheaf of real holomorphic microfunctions on T_Y^*X as usual. Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$ -Module. Then, there exists the following isomorphism of distinguished triangles:*

$$\begin{array}{ccccc} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \sigma^{-1}\nu_Y(\mathcal{O}_X)) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, {}^t\sigma^{-1}\mathcal{C}_{Y|X}^{\mathbb{R}}) \xrightarrow{+1} \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ R\mathcal{H}om_{\mathcal{D}_Y}(D\iota^* \mathcal{M}, \mathcal{O}_Y) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{O}_Y) & \longrightarrow & R\mathcal{H}om_{\mathcal{D}_Y}(\Phi_Y(\mathcal{M}), \mathcal{O}_Y) \xrightarrow{+1}. \end{array}$$

3.7 Remark. (1) The isomorphism (the Cauchy-Kovalevskaja type theorem)

$$R\mathcal{H}om_{\mathcal{D}_Y}(D\iota^* \mathcal{M}, \mathcal{O}_Y) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)|_Y$$

holds for Fuchsian systems in the sense of Laurent-Monteiro Fernandes [L-MF 1].

(2) Recently Mandai [Man] extended the definition of boundary values to a general Fuchsian differential equation in the complex domain.

4 Boundary Values for Regular-Specializable System

We denote by $\mathcal{O}_X$, $\mathcal{B}_M$ and $\mathcal{C}_M$ the sheaf of *holomorphic functions* on X , of *hyperfunctions* on M and of *microfunctions* on T_M^*X respectively.

4.1 Definition (Oaku [Oa 2], Oaku-Yamazaki [O-Y]). We set:

$$\mathcal{C}_{N|M} := s_{L\pi}^{-1} \mu_{\widetilde{M}_N}(\mathbf{R}j_{L*} \widetilde{p}_L^{-1} i_L^! \mathcal{O}_X) \otimes or_{M/X}[n+1].$$

We can regard $\mathcal{C}_{N|M}$ as a microlocalization of $\nu_N(\mathcal{B}_M)$:

4.2 Proposition. (1) $\mathcal{C}_{N|M}$ is concentrated in degree zero; that is, $\mathcal{C}_{N|M}$ is regarded as a sheaf on $T_{T_N^*M}^* T_Y L$. Further $\mathcal{C}_{N|M}|_{T_N^*M} = \nu_N(\mathcal{B}_M)$ holds.

(2) There exists the following exact sequence on T_N^*M :

$$0 \rightarrow \nu_Y(\mathcal{B}\mathcal{O}_L)|_{T_N^*M} \rightarrow \nu_N(\mathcal{B}_M) \rightarrow \dot{\pi}_{N|M*} \mathcal{C}_{N|M} \rightarrow 0.$$

Here $\mathcal{B}\mathcal{O}_L := \mathcal{H}_L^1(\mathcal{O}_X) \otimes or_{L/X}$ is the sheaf of *hyperfunctions with holomorphic parameters* on L . Note that $\nu_Y(\mathcal{B}\mathcal{O}_L)$ is concentrated in degree zero.

4.3 Definition. Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$ -Module. By Proposition 3.4, $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ satisfies the assumption of Theorem 2.1. Thus, by Definition 2.2 and Theorem 3.6, we define:

$$\beta: f_\pi^{-1} \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \rightarrow f_\pi^{-1} \tau_{Y\pi}^{-1} \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).$$

4.4 Theorem. (1) *The morphism β gives a monomorphism*

$$\beta^0: f_\pi^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) \hookrightarrow f_\pi^{-1} \tau_{Y\pi}^{-1} \mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N).$$

(2) *The restriction of β^0 to the zero-section $T_N M^+$ coincides with the boundary value morphism in the sense of Monteiro Fernandes [MF 1].*

4.5 Definition. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X|_Y$ -Module. Then we say $\mathcal{M}$ is *near-hyperbolic* at $x_0 \in N$ (in dt -codirection) if $\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$ is near-hyperbolic in the sense of Definition 2.3. Here, we remark that $\mathrm{SS}(\mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)) = \mathrm{char}(\mathcal{M})$.

The following theorem is a direct consequence of Theorem 2.4:

4.6 Theorem. *Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$ -Module. Assume that $\mathcal{M}$ is near-hyperbolic at $x_0 \in N$. Then, for any $p^* \in T_{T_N M^+}^* T_Y L^+$*

$$\beta: \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})_{p^*} \rightarrow \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N)_{\tau_{Y\pi}(p^*)}$$

is an isomorphism.

4.7 Remark. Let $\mathcal{C}_{N|M}^F$ be the sheaf of F -mild microfunctions on $T_{T_N M}^* T_Y L$, and set $\tilde{\mathcal{C}}_{N|M}^A := \mathcal{H}^n(\mu_N(\mathcal{O}_X|_Y)) \otimes \mathrm{or}_{N/Y}$ (see Oaku [Oa 1], [Oa 2], and Oaku-Yamazaki [O-Y]). Let $\mathcal{M}$ be a regular-specializable $\mathcal{D}_X|_Y$ -Module. Set $\mathcal{M}_Y := \mathcal{H}^0(\mathbf{D}\iota^* \mathcal{M}) = \mathcal{O}_Y \otimes_{\iota^{-1}\mathcal{O}_X} \mathcal{M}$. By the argument in Oaku-Yamazaki [O-Y] we have the following commutative diagram:

$$\begin{array}{ccccc} f_\pi^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}^F) & \twoheadrightarrow & f_\pi^{-1} \tau_{Y\pi}^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M}^A) & \xrightarrow{\sim} & f_\pi^{-1} \tau_{Y\pi}^{-1} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N) \\ \downarrow & \circlearrowleft & \downarrow & \circlearrowleft & \downarrow \\ f_\pi^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}) & \twoheadrightarrow & f_\pi^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M}) & \xrightarrow{\sim} & f_\pi^{-1} \tau_{Y\pi}^{-1} \mathcal{H}om_{\mathcal{D}_Y}(\Psi_Y(\mathcal{M}), \mathcal{C}_N), \end{array}$$

that is, the boundary value morphism

$$\gamma^F: f_\pi^{-1} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M}^F) \hookrightarrow f_\pi^{-1} \tau_{Y\pi}^{-1} \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N)$$

and β^0 are compatible. In particular, if Y is non-characteristic for $\mathcal{M}$, then it is known that $\Psi_Y(\mathcal{M}) \simeq \mathbf{D}\iota^* \mathcal{M} \simeq \mathcal{M}_Y$ and by Oaku [Oa 2] (cf. Oaku-Yamazaki [O-Y]) we have:

$$\tilde{\gamma}_{N|M}: \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \tilde{\mathcal{C}}_{N|M}) \simeq \tau_{Y\pi}^{-1} \mathbf{R}\mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M}_Y, \mathcal{C}_N).$$

In this case we see that β^0 is equivalent to the non-characteristic boundary value morphism (see Kataoka [Kat] and Oaku [Oa 2]). In particular, the restriction of β^0 to the zero-section $T_N M^+$ is equivalent to Komatsu-Kawai [Ko-K] and Schapira [Sc 1]. Further, if Y is non-characteristic for $\mathcal{M}$ and $\pm dt \in T_N^* M$ is hyperbolic for $\mathcal{M}$, then the nearly-hyperbolic condition is satisfied and β is an isomorphism.

4.8 Example. Assume that $X = \mathbb{C}^{n+1}$ and so on by an admissible coordinate system.

(1) Let $b(\alpha)$ be a non-zero polynomial with degree m , and $Q \in \mathcal{D}_X^{(m)} \cap V_Y^{-1}(\mathcal{D}_X)$. Set $\mathcal{M} := \mathcal{D}_X / \mathcal{D}_X (b(\vartheta) + Q)$. Then $\mathcal{M}$ is regular-specializable. Assume that $b(\alpha) = \prod_{j=1}^{\mu} (\alpha - \alpha_j)^{\nu_j}$ ($\alpha_i - \alpha_j \notin \mathbb{Z}$ for $1 \leq i \neq j \leq \mu$, note that $\sum_{j=1}^{\mu} \nu_j = m$). Then a direct calculation shows that $\Psi_Y(\mathcal{M}) \simeq \mathcal{D}_Y^{\oplus m}$, and β^0 is equivalent to γ in Oaku [Oa 2]: Let $p^* = (x_0, t_0; \sqrt{-1}\langle \xi_0, dx \rangle)$ be a point of $T_{T_N M^+}^* T_Y L^+$, and $f(x, t)$ a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})$ at p^* . Then, we can see that $f(x, t)$ has a defining function

$$F(z, \tau) = \sum_{j=1}^{\mu} \sum_{k=1}^{\nu_j} F_{jk}(z, \tau) \tau^{\alpha_j} (\log \tau)^{k-1}.$$

Here each $F_{jk}(z, \tau)$ is holomorphic on a neighborhood of $\{(z, 0) \in X; |x_0 - z| < \varepsilon, \text{Im } z \in \Gamma\}$ with a positive constant ε and an open convex cone Γ such that $\xi_0 \in \text{Int}(\Gamma^\circ)$ (the interior of the dual cone Γ° of Γ). Then, $\beta^0(f)$ is equivalent to $\{\text{sp}_N(F_{jk}(x + \sqrt{-1}\Gamma 0, 0)); 1 \leq k \leq \nu_j, 1 \leq j \leq \mu\}$. Moreover, if the principal symbol of $b(\vartheta) + Q$ is written as $\tau^m P(z, \tau; z^*, \tau^*)$ for a hyperbolic polynomial P at dt -codirection, the nearly-hyperbolic condition is satisfied. Note that this operator is a special case of Fuchsian hyperbolic operators due to Tahara [T].

(2) Take an operator $A(z; \partial_z) \in \mathcal{D}_Y^{(1)}$ at the origin and set $A^0 := \text{id}$ and $A^{(j)} := \frac{1}{j!} A \circ A^{(j-1)} \in \mathcal{D}_Y^{(j)}$ for $j \geq 1$. Let $p^* = (0, 1; \sqrt{-1}\langle \xi, dx \rangle)$ be a point of $T_{T_N M^+}^* T_Y L^+$ and set $p_0 := (0; \sqrt{-1}\langle \xi, dx \rangle) \in T_N^* Y$. Set $P := (\vartheta - \alpha_1)(\vartheta - \alpha_2) - \tau A(z; \partial_z)\vartheta \in \mathcal{D}_X|_Y$, where $(\alpha_1, \alpha_2) \in \mathbb{C}^{\oplus 2}$. Consider $\mathcal{M} := \mathcal{D}_X / \mathcal{D}_X P = \mathcal{D}_X u$, where $u := 1 \bmod P$. Let $f(x, t)$ be a germ of $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}_{N|M})$ at p^* . Then:

(i) If $(\alpha_1, \alpha_2) = (-1, 0)$, then

$$\begin{aligned} \Phi_Y(\mathcal{M}) &= \frac{V_Y^0(\mathcal{D}_X)u + V_Y^1(\mathcal{D}_X)(\vartheta + 1)u}{V_Y^{-1}(\mathcal{D}_X)u + V_Y^0(\mathcal{D}_X)(\vartheta + 1)u} = \mathcal{D}_Y[u] + \mathcal{D}_Y[\partial_\tau(\vartheta + 1)u] \simeq \mathcal{D}_Y^{\oplus 2}, \\ \Psi_Y(\mathcal{M}) &= \frac{V_Y^{-1}(\mathcal{D}_X)u + V_Y^0(\mathcal{D}_X)(\vartheta + 1)u}{V_Y^{-2}(\mathcal{D}_X)u + V_Y^{-1}(\mathcal{D}_X)(\vartheta + 1)u} = \mathcal{D}_Y[\tau u] + \mathcal{D}_Y[(\vartheta + 1)u] \simeq \mathcal{D}_Y^{\oplus 2}, \end{aligned}$$

and $\text{Var}: ([u], [\partial_\tau(\vartheta - 1)u]) \mapsto ([\tau u], 0)$. Hence $\mathcal{M}_Y \simeq \mathcal{D}_Y[(\vartheta + 1)u] \simeq \mathcal{D}_Y$. In this case $f(x, t)$ has the following defining function:

$$F(z, \tau) = U_0(z) + \frac{U_{-1}(z)}{\tau} - \sum_{j=1}^{\infty} \frac{A^{(j)}U_{-1}(z)}{j-1} \tau^{j-1} - AU_{-1}(z) \log \tau,$$

and $\beta^0(f(x, t))$ is given by $\{\text{sp}_N(U_i)(x)\}_{i=-1,0}$ at p_0 . If $f(x, t)$ is F -mild at p_0 , then $U_{-1}(z) = 0$ and $\gamma^F(f(x, t)) = \{f(x, +0)\} = \{\text{sp}_N(U_0)(x)\}$.

(ii) If $(\alpha_1, \alpha_2) = (0, 1)$, then:

$$\begin{aligned}\Phi_Y(\mathcal{M}) &= \frac{V_Y^1(\mathcal{D}_X)u + V_Y^2(\mathcal{D}_X)\vartheta u}{V_Y^0(\mathcal{D}_X)u + V_Y^1(\mathcal{D}_X)\vartheta u} = \mathcal{D}_Y[\partial_\tau u] + \mathcal{D}_Y[\partial_\tau^2 \vartheta u] \simeq \mathcal{D}_Y^{\oplus 2}, \\ \Psi_Y(\mathcal{M}) &= \frac{V_Y^0(\mathcal{D}_X)u + V_Y^1(\mathcal{D}_X)\vartheta u}{V_Y^{-1}(\mathcal{D}_X)u + V_Y^0(\mathcal{D}_X)\vartheta u} = \mathcal{D}_Y[u] + \mathcal{D}_Y[\partial_\tau \vartheta u] \simeq \mathcal{D}_Y^{\oplus 2},\end{aligned}$$

and $\text{Var}[\partial_\tau u] = \text{Var}[\partial_\tau^2 \vartheta u] = 0$. Hence $\mathcal{M}_Y \simeq \mathcal{D}_Y[u] + \mathcal{D}_Y[\partial_\tau \vartheta u] \simeq \mathcal{D}_Y^{\oplus 2}$. In this case $f(x, t)$ has the following defining function:

$$F(z, \tau) = U_0(z) + \sum_{j=0}^{\infty} \frac{A^{(j)}U_1(z)}{j+1} \tau^{j+1},$$

and $f(x, t)$ is always F -mild. Hence $\beta^0(f(x, t))$ at p_0 coincides with $\gamma^F(f(x, t)) = \{\partial_t^i f(x, +0)\}_{i=0,1} = \{\text{sp}_N(U_i)(x)\}_{i=0,1}$ (if $\tau \neq 0$, $\mathcal{M}$ is isomorphic to $\mathcal{D}_X / \mathcal{D}_X(\partial_\tau^2 - A(z; \partial_z) \partial_\tau)$ for which Y is non-characteristic).

(iii) If $(\alpha_1, \alpha_2) = (1, 1)$, then

$$\begin{aligned}\Phi_Y(\mathcal{M}) &= \frac{V_Y^2(\mathcal{D}_X)u}{V_Y^1(\mathcal{D}_X)u} = \mathcal{D}_Y[\partial_\tau^2 u] + \mathcal{D}_Y[\partial_\tau^2(\vartheta - 1)u] \simeq \mathcal{D}_Y^{\oplus 2}, \\ \Psi_Y(\mathcal{M}) &= \frac{V_Y^1(\mathcal{D}_X)u}{V_Y^0(\mathcal{D}_X)u} = \mathcal{D}_Y[\partial_\tau u] + \mathcal{D}_Y[\partial_\tau(\vartheta - 1)u] \simeq \mathcal{D}_Y^{\oplus 2}.\end{aligned}$$

and $\text{Var}: ([\partial_\tau^2 u], [\partial_\tau^2(\vartheta - 1)u]) \mapsto (2\pi\sqrt{-1}[\partial_\tau(\vartheta - 1)u], 0)$. Hence $\mathcal{M}_Y \simeq \mathcal{D}_Y[\partial_\tau u] \simeq \mathcal{D}_Y$. In this case $f(x, t)$ has the following defining function:

$$F(z, \tau) = \sum_{j=0}^{\infty} A^{(j)}U_0(z) \tau^{j+1} - \sum_{j=1}^{\infty} \sum_{k=1}^j \frac{A^{(j)}U_1(z)}{k} \tau^{j+1} + \sum_{j=0}^{\infty} A^{(j)}U_1(z) \tau^{j+1} \log \tau,$$

and $\beta^0(f(x, t))$ is given by $\{\text{sp}_N(U_i)(x)\}_{i=0,1}$ at p_0 . If $f(x, t)$ is F -mild at p_0 , then $U_0(z) = 0$ and $\gamma^F(f(x, t)) = \{\partial_t f(x, +0)\} = \{\text{sp}_N(U_1)(x)\}$.

(iv) If $(\alpha_1, \alpha_2) = (1, 2)$, then:

$$\begin{aligned}\Phi_Y(\mathcal{M}) &= \frac{V_Y^2(\mathcal{D}_X)u + V_Y^3(\mathcal{D}_X)(\vartheta - 1)u}{V_Y^1(\mathcal{D}_X)u + V_Y^2(\mathcal{D}_X)(\vartheta - 1)u} = \mathcal{D}_Y[\partial_\tau^2 u] + \mathcal{D}_Y[\partial_\tau^3(\vartheta - 1)u] \simeq \mathcal{D}_Y^{\oplus 2}, \\ \Psi_Y(\mathcal{M}) &= \frac{V_Y^1(\mathcal{D}_X)u + V_Y^2(\mathcal{D}_X)(\vartheta - 1)u}{V_Y^0(\mathcal{D}_X)u + V_Y^1(\mathcal{D}_X)(\vartheta - 1)u} = \mathcal{D}_Y[\partial_\tau u] + \mathcal{D}_Y[\partial_\tau^2(\vartheta - 1)u] \simeq \mathcal{D}_Y^{\oplus 2},\end{aligned}$$

and $\text{Var}: ([\partial_\tau^2 u], [\partial_\tau^3(\vartheta - 1)u]) \mapsto (0, 2A[\partial_\tau u])$. Hence

$$\mathcal{M}_Y \simeq \frac{\mathcal{D}_Y[\partial_\tau u] + \mathcal{D}_Y[\partial_\tau^2(\vartheta - 1)u]}{\mathcal{D}_Y A[\partial_\tau u]}.$$

In this case $f(x, t)$ has the following defining function:

$$F(z, \tau) = \sum_{j=0}^{\infty} A^{(j)} U_2(z) \tau^{j+2} + U_1(z) \tau - \sum_{j=2}^{\infty} \sum_{k=1}^{j-1} \frac{j A^{(j)} U_1(z)}{k} \tau^{j+1} \\ + \left(\sum_{j=0}^{\infty} (j+1) A^{(j+1)} U_1(z) \tau^j \right) \tau^2 \log \tau,$$

and $\beta^0(f(x, t))$ is given by $\{\mathrm{sp}_N(U_i)(x)\}_{i=1,2}$ at p_0 . $f(x, t)$ is F -mild under the condition that $AU_1(z) = 0$, and in this case $\gamma^F(f(x, t))$ at p_0 is given by $\gamma^F(f_3(x, t)) = \{\partial_t^i f(x, +0)\}_{i=1,2} = \{\mathrm{sp}_N(U_1)(x), 2\mathrm{sp}_N(U_2)(x)\}$ with $A\partial_t f(x, +0) = A\mathrm{sp}_N(U_1)(x) = 0$.

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