

A WEAK EQUIVALENCE AND TOPOLOGICAL ENTROPY .

By Taijiro Ohno .

We will investigate the topological entropies of weakly equivalent topological flows. Let X be a compact metric space. A topological flow $\Phi = \{\phi_t ; -\infty < t < \infty\}$ on X is a one parameter group of homeomorphisms of X which is continuous in (t, x) . Let Φ and Ψ be flows on X and Y respectively. They are weakly equivalent by definition if there is a homeomorphism π of X onto Y such that $\{\pi^{-1} \circ \psi_t \circ \pi(x) ; t \in \mathbb{R}\} = \{\phi_t(x) ; t \in \mathbb{R}\}$. It is a generalization of time changes of flows given by [5]. In [5] Totoki investigated the changes of metrical entropy of metrical flows under time changes. Time changes by integrable additive functions preserve the properties that the entropy is zero, positive finite or infinite respectively. Our first result is analogous to the above one. To simplify we assume that $X = Y$.

Theorem 1. Φ and Ψ are weakly equivalent and they have no fixed point, then we have $c_1 h(\psi_1) \leq h(\phi_1) \leq c_2 h(\psi_1)$ where c_1, c_2 are positive constants of ϕ_1, ψ_1 .

In order to prove the theorem we will apply measure theoretical method. Let us denote by $\mathcal{E}(\Phi)$ ($\mathcal{E}(\phi_1)$) the set of all $\Phi(\phi_1)$ -invariant ergodic probability measures. Let $h_\mu(\phi_1)$ be the metrical entropy of ϕ_1 for $\mu \in \mathcal{E}(\phi_1)$.

Lemma 1. ([6])

$$h(\phi_1) = \sup_{\mu \in \mathcal{E}(\phi_1)} h_\mu(\phi_1) = \sup_{m \in \mathcal{E}(\Phi)} h_m(\phi_1).$$

Under the assumption of the theorem we can and do assume further that ϕ and ψ have the same orbits with the same directions. Then we can define $\theta(t, x)$, $-\infty < t < \infty$, $x \in X$ with the following properties (cf.[2]) ;

$$i) \quad \phi_t(x) = \psi_{\theta(t, x)}(x)$$

$$ii) \quad \theta(t+s, x) = \theta(t, \phi_s(x)) + \theta(s, x)$$

$$iii) \quad \theta(0, x) = 0 \text{ and } \theta(t, x) \text{ is strictly increasing}$$

$$iv) \quad \theta(t, x) \text{ is continuous in } (t, x).$$

Lemma 2. ([5])

For $m \in \mathcal{E}(\Phi)$ if $E_m(\theta(1, x))$ then

$$h_{\hat{m}}(\psi_1) = \frac{1}{E_m(\theta(1, x))} h_m(\phi_1).$$

Lemma 3.

For $m \in \mathcal{E}(\Phi)$,

$$E_{\hat{m}}(f) = \frac{1}{E_m(\theta(1, x))} E_m\left(\int_0^{\theta(1, x)} f(\psi_t(x)) dt\right)$$

define an $\hat{m} \in \mathcal{E}(\Psi)$. The map $\wedge : m \in \mathcal{E}(\Phi) \longrightarrow m \in \mathcal{E}(\Psi)$ is bijective and $G(\Phi, m) = G(\Psi, \hat{m})$

$$\text{where } G(\Phi, m) = \{x \in X ; \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\phi_t(x)) dt = E_m(f)\}$$

for all continuous f .

Corollary 1. Any time changed (weakly equivalent) flow of a strictly ergodic (i.e. minimal and uniquely ergodic) flow is also strictly ergodic.

Now combining Lemma 1, 2 and 3, we can prove the theorem.

We have

$$h(\phi_1) = \sup_{m \in \mathcal{E}(\Phi)} h_m(\phi_1) = \sup_{m \in \mathcal{E}(\Phi)} E_m(\theta(1, x)) h_{\hat{m}}(\psi_1),$$

and so putting $c_1 = \min_{x \in X} \theta(1, x)$ and $c_2 = \max_{x \in X} \theta(1, x)$

$$\begin{aligned} c_1 h(\psi_1) &= c_1 \sup_{\hat{m} \in \mathcal{E}(\Psi)} h_{\hat{m}}(\psi_1) = c_1 \sup_{m \in \mathcal{E}(\Phi)} h_{\hat{m}}(\psi_1) \leq h(\Phi) \\ &\leq c_2 \sup_{m \in \mathcal{E}(\Phi)} h_{\hat{m}}(\psi_1) = c_2 h(\psi_1). \end{aligned}$$

Next we will find that if Φ and Ψ have fixed point $\{x_0\}$ then the result of theorem 1 does not hold. In fact we get a stronger result in the view point of invariant measures.

Theorem 2. There exists a pair of weakly equivalent flows Φ and Ψ with fixed point $\{x_0\}$ such that

- 1) there exists a $\mu \in \mathcal{E}(\Phi)$ such that $0 < h_\mu(\phi_1) < \infty$,
- 2) there is no non-trivial Ψ -invariant ergodic probability measure, that is $\mathcal{E}(\Psi) = \{\delta_{x_0}\}$.

Corollary 2. For Φ and Ψ in Theorem 2, we have $h(\psi_1) = 0$ and $0 < h(\phi_1)$

An example of the pair Φ and Ψ is constructed as the flows under the functions (the special flows or suspensions) . The essential point is that the integrability of $\theta(1,x)$ is broken for any $\mu \in \mathcal{E}(\Phi)$.

In the same way we can construct a pair of weakly equivalent flows one of which has positive finite entropy and the other has infinite entropy. At last we emphasize that these examples can not be extended to differential flows. Because in these examples we have to make the ceiling function sharp. There is a still unsolved and interesting problem, whether the weak equivalence preserves the topological entropy in differentiable flows.

—References—

- [1] R.L.Adler, A.G.Konheim, M.H. McAndrew ; Topological entropy, Trans.Amer.Math.Soc.114 (1965), 309-319.
- [2] M.Baboutoff et W.Stepanoff ; Sur la mesure invariants dans les systemes dynamiques qui ne different que par le temps, Mat. Sb. 7(49), No,1 (1940), 143-164.
- [3] Sh.Ito ; On the topological entropy of a dynamical system, Proc. Japan Acad. ,45 (1969), 838-841.
- [4] V.A.Rohlin . ; Entropy of metric automorphism, Dokl.akad. Nauk USSR, 124, N₀.5 (1959), 980-983. (In Russian)
- [5] H.Totoki : Time changes of flows, Mem.Fac. Sci. Kyushu Univ. Ser.A,20 (1966), 27-55.
- [6] P.Walters ; A variational principle for the pressure of continuous transformations, Amer.J.Math 97, N₀.4 (1975) , 937-971.