

TOWARDS A MODULI THEORETIC CHARACTERIZATION OF A RATIONAL PRIME Q-FANO 3-FOLD OF GENUS SIX WITH ONE $\frac{1}{2}(1, 1, 1)$ -SINGULARITY

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1. DEFINITIONS, MOTIVATIONS, . . .

Definition 1.1. A projective 3-fold X is called a *Q-Fano 3-fold*, or simply, *Fano 3-fold* if X has only terminal singularities and the anti-canonical divisor $-K_X$ is ample.

The *genus* $g(X)$ of a Fano 3-fold X is defined to be $h^0(-K_X) - 2$. Note that, in case X is smooth, $2g(X) - 2 = (-K_X)^3$ by the Riemann-Roch theorem and the Kodaira vanishing theorem.

A Fano 3-fold is called *prime* if the group of numerical equivalence classes of $\mathbb{Q}$ -Cartier Weil divisors is generated by the anti-canonical class. A quartic hypersurface in $\mathbb{P}^4$ is a simple but interesting example of a prime Fano 3-fold.

Aim 1.2. Find as many as possible (prime) Fano 3-folds X which can be recovered from data on curves C ‘characteristic’ for X , ideally, as a moduli space of some objects on C .

S. Mukai found beautiful examples of smooth prime Fano 3-folds for which such characterizations are possible. I will explain his discovery. Before that, let me show how to find candidates of C to recover X . It is done by variants of the Fano-Iskovskih double projection from a line on a prime Fano 3-fold (Fano, Iskovskih, Takeuchi).

Example 1.3. See [IP99] or [Take89] for details. Let X be a smooth prime Fano 3-fold with very ample $-K_X$ in this example. I embed X in $\mathbb{P}^{g(X)+1}$ by the anti-canonical linear system.

- (1) ($g(X) = 7$) Let q be a general conic (with respect to the anti-canonical embedding) on X and consider the rational map defined by the linear system $|-2K_X - 3q|$, the sublinear system of $|-2K_X|$ consisting of the members with multiplicities 3 along q . The image turns out to be Q^3 , a smooth quadric 3-fold. This rational map is

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the composite of the three elementary rational maps as follows:

$$\begin{array}{ccc} & Y \dashrightarrow Y' & \\ f \swarrow & & \searrow f' \\ X & \dashrightarrow & Q^3, \\ & \Phi_{|-2K_X-3q|} & \end{array}$$

where f is the blow-up along q , $Y \dashrightarrow Y'$ a flop, and f' is the blow-up of Q^3 along a smooth curve C_q of genus 7 and degree 10. For this example, I will take C_q as C .

- (2) ($g(X) = 9$) Let l be a general line and consider the rational map defined by the linear system $|-K_X - 2l|$.¹ Similarly to the above case, I obtain the following diagram:

$$\begin{array}{ccc} & Y \dashrightarrow Y' & \\ f \swarrow & & \searrow f' \\ X & \dashrightarrow & \mathbb{P}^3, \\ & \Phi_{|-K_X-2l|} & \end{array}$$

where f is the blow-up along l , $Y \dashrightarrow Y'$ a flop, and f' is the blow-up of $\mathbb{P}^3$ along a smooth curve C_l of genus 3 and degree 7. For this example, I will take C_l as C .

- (3) ($g(X) = 12$) In this example, I consider also the rational map defined by the linear system $|-K_X - 2l|$ for a general line l and I obtain the following diagram:

$$\begin{array}{ccc} & Y \dashrightarrow Y' & \\ f \swarrow & & \searrow f' \\ X & \dashrightarrow & B_5, \\ & \Phi_{|-K_X-2l|} & \end{array}$$

where f is the blow-up along l , $Y \dashrightarrow Y'$ a flop, B_5 is a smooth quintic del Pezzo 3-fold,² and f' is the blow-up of B_5 along a smooth curve C_l of genus 0 and degree 5.³ For this example, I will **not** take C_l as C . Instead I will take the Hilbert scheme of lines on X , which I can compute by the diagram noting general lines are transformed to general lines on B_5 intersecting C . Consequently, C is a plane quartic curve and is smooth if X is general in the moduli.

¹This rational map is the so-called double projection from a line.

²A quintic del Pezzo 3-fold is a Fano 3-fold such that $-K_X = 2H$, where $H \in \text{Pic } X$ and $H^3 = 5$.

³The degree is with respect to H .

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Keep in mind the notation of Example 1.3. Mukai's theorem (with comments) is the following (see [Muk01]):

Theorem 1.4. (1) ($g(X) = 7$)

$$X \simeq \{[\mathcal{E}] \mid \mathcal{E} \text{ is a rank 2 semi-stable vector bundle on } C \\ \text{with } \det \mathcal{E} = K_C \text{ and } h^0(\mathcal{E}) \geq 5\}.$$

This is an example of non-abelian Brill-Noether locus.

- (2) ($g(X) = 9$) In this case, X cannot be recovered from C because the moduli number of X is 12^4 and the moduli number of C is 6. Thus some data on C is needed. Mukai showed the following and used it as a data to recover X :

*The Hilbert scheme of conics on X is the smooth surface $\mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a Nagata stable vector bundle of rank 2.*⁵

$$X \simeq \{[\mathcal{E}] \mid \mathcal{E} \text{ is a rank 2 semi-stable vector bundle on } C \\ \text{with } \det \mathcal{E} = \det \mathcal{F} + K_C \text{ and } \dim \operatorname{Hom}(\mathcal{F}, \mathcal{E}) \geq 3\}.$$

This is an example of another kind of non-abelian Brill-Noether locus.

- (3) ($g(X) = 12$) Though the moduli numbers of X and C are the same, X cannot be recovered from C . As a data to recover X , Mukai obtained the following:

*There exists a unique theta-characteristic θ on C with $h^0(\theta) = 0$ such that inside $C \times C$,*⁶

$$\{([l_1], [l_2]) \mid l_1 \neq l_2, l_1 \cap l_2 \neq \emptyset\} = \{([l_1], [l_2]) \mid h^0(\theta + [l_1] - [l_2]) > 0\}.$$

A classic result of Scorza asserts that there exists a unique quartic curve Γ living in the same $\mathbb{P}^2$ as C associated to the pair (C, θ) (see [DK93]). Let F be a defining equation of Γ .

X is isomorphic to the closure in $\operatorname{Hilb}^6 \tilde{\mathbb{P}}^2$ of the following:

$$\{\langle \tilde{l}_1, \dots, \tilde{l}_6 \rangle \mid l_1^4 + \dots + l_6^4 = F\},$$

where l_i is a linear form on $\mathbb{P}^2$ and $\tilde{l}_i$ is the point of $\tilde{\mathbb{P}}^2$ corresponding to l_i .

Remark. Mukai conjectured that there is a similar characterization of a prime Fano 3-fold X of genus 10 (see [Mukb]). In this case, C is a smooth curve of genus 2, which is the center of the blow-up of Q^3

⁴This can be computed by the diagram in Example 1.3.

⁵A vector bundle $\mathcal{F}$ of rank 2 is called *Nagata stable* if $\sigma^2 \geq 3 = g(C)$ for any section σ of $\mathbb{P}(\mathcal{F})$.

⁶Recall that C is the Hilbert scheme of lines on X .

appearing in the decomposition of the double projection from a general line.

2. SINGULAR FANO 3-FOLDS

I am tempted to find more examples of Fano 3-folds with characterizations as in Theorem 1.4 and more curves which are characteristic for some Fano 3-folds. In my thesis [Taka02a] and [Taka02b], I classified prime Fano 3-folds X with $g(X) \geq 2$ and with only $\frac{1}{2}$ -singularities.⁷ More precisely, I classified the type of the following diagram, which is a variant of the double projection from a line as in the previous section.

$$\begin{array}{ccc} & Y & \dashrightarrow Y' \\ & \swarrow f & \searrow f' \\ X & & X', \end{array}$$

where f is the blow-up at a $\frac{1}{2}$ -singularity, $Y \dashrightarrow Y'$ is a flop or a composite of a flop and a flip, f' is a non-small extremal contraction. I present two examples. I denote by N the number of $\frac{1}{2}$ -singularities.

Example 2.1. (1) ($g(X) = 8, N = 2$). The diagram is as follows:

$$\begin{array}{ccc} & Y & \dashrightarrow Y' \\ & \swarrow f & \searrow f' \\ X & & B_5, \end{array}$$

where $Y \dashrightarrow Y'$ is a composite of a flop and a flip, and f' is the blow-up along $C \simeq \mathbb{P}^1$ with $\deg C = 6$.

This diagram is very similar to the smooth prime Fano 3-fold of genus 12 (Example 1.3 (3)). Actually there are more similarities. I studied this case more in detail with Francesco Zucconi in Udine. I briefly explain our results. Assume that X is general in the moduli.

- The Hilbert scheme of ‘lines’⁸ is isomorphic to a smooth complete intersection of a smooth quadric and a cubic in $\mathbb{P}^3$. I choose as C this curve of genus 4.

⁷A $\frac{1}{2}$ -singularity is, by definition, analytically isomorphic to the origin of $\mathbb{C}^3/(x, y, z) \sim (-x, -y, -z)$, where (x, y, z) is the coordinate of $\mathbb{C}^3$. Usually this is called a $\frac{1}{2}(1, 1, 1)$ -singularity.

⁸Here, by a line, I mean a curve with degree 1 with respect to $-K_X$ and with arithmetic genus 0. There is a degenerate line, which is the union of two $\mathbb{P}^1$'s with degree $\frac{1}{2}$ with respect to $-K_X$.

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- As in the case of the smooth prime Fano 3-fold of genus 12, there exists a unique theta-characteristic θ on C with $h^0(\theta) = 0$ such that inside $C \times C$,

$$\{([l_1], [l_2]) \mid l_1 \neq l_2, l_1 \cap l_2 \neq \emptyset\} = \{([l_1], [l_2]) \mid h^0(\theta + [l_1] - [l_2]) > 0\}.$$

A classic result of Scorza and complementary works by Dolgachev and Kanev assert that there exists a unique quartic surface Γ living in the same $\mathbb{P}^3$ as C associated to the pair (C, θ) . Let F be a defining equation of Γ .

- The Hilbert scheme S of ‘conics’ is the smooth surface obtained by blowing up $\mathbb{P}^2$ at 6 points lying on a smooth conic, and S is a weak del Pezzo surface of degree 3. Denote by $\bar{S}$ the anti-canonical model of S .
- As a characterization of X , we conjecture the following:

Conjecture 2.2. ‘An explicit birational model’ of X can be embedded in $\text{Hilb}^{10}S$ as the closure of the locus

$$\{\langle \tilde{l}_1, \dots, \tilde{l}_{10} \rangle \mid l_1^4 + \dots + l_{10}^4 = F, \tilde{l}_i \in S\},$$

where l_i is a linear form on $\mathbb{P}^3$ and $\tilde{l}_i$ is the point of $\check{\mathbb{P}}^3$ corresponding to l_i .

- (2) ($g(X) = 6, N = 1$) There are two type of Fano 3-folds with these invariants, one of which is birational to a smooth cubic 3-fold, another is rational. I only describe the latter case. The diagram is as follows:

$$\begin{array}{ccc} & Y & \dashrightarrow Y' \\ & \swarrow f & \searrow f' \\ X & & Q^3, \end{array}$$

where $Y \dashrightarrow Y'$ is a flop, and f' is the blow-up along a smooth curve C with $g(X) = 6$ and $\deg C = 9$. I will choose as a characteristic curve for X this C and I will go back to this case in the next section.

By looking at the list of Fano 3-folds with $g(X) \geq 2$ and with only $\frac{1}{2}$ -singularities, I obtain the following range of genus of curves as the genus of characteric curves:

$$g(C) = 1, 2, 3, 4, 5, 6, 7, 8, 9.$$

Thus I hope that Fano 3-folds are useful for the study of curves with small genus.

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3. RATIONAL FANO 3-FOLD WITH GENUS 6 AND WITH ONE $\frac{1}{2}$ -SINGULARITY

From now on, let X be

a rational Fano 3-fold of genus 6 and with one $\frac{1}{2}$ -singularity.

Assume that X is general in the moduli.

First I describe the diagram in Example 2.1 (2) more in detail. It is easy to show the following:

- The composite of the embedding $C \hookrightarrow Q^3 \hookrightarrow \mathbb{P}^4$ is defined by the linear system $|K_C - p|$, where p is a point of C .
- There exists a pencil of quadrics in $\mathbb{P}^4$ containing C . The intersection of the quadrics in the pencil is a smooth del Pezzo surface S of degree 4. S is the strict transform of the f -exceptional divisor.
- There exist 5 tri-secants lines of C , which are contained in S . These are the images of flopping curves for $Y' \dashrightarrow Y$.
- C is isomorphic to a complete intersection in $G(5, 2)$ defined by 4 hyperplanes and 1 quadric hypersurface. By [Muk93], this is equivalent to that C has no g_4^1 , g_5^2 and C is not bi-elliptic.

The following is the main result of this article with comments:

Proposition 3.1. (A) In this case, X cannot be recovered from C because the moduli number of X is 17^9 and the moduli number of C is 15. Thus some data on C is needed as in the case of the smooth prime Fano 3-fold of genus nine.

(A1) *The Hilbert scheme $\mathcal{H}_{5/2}$ of $\frac{5}{2}$ -curves¹⁰ on X is the smooth surface $\mathbb{P}(\mathcal{F})$, where $\mathcal{F}$ is a stable and globally generated vector bundle of rank 2 obtained as follows: let $\mathcal{F}_0$ be the restriction of the universal quotient bundle on $G(5, 2)$ (now I consider that C is embedded in $G(5, 2)$). $\mathcal{F}$ fits into the exact sequence*

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_0 \rightarrow k(p) \rightarrow 0.$$

(A2) *X can be recovered from $\mathcal{F}$.*

Unfortunately in (A2), I did not succeed in recovering X as a moduli.

(B) As for the recovery as a moduli of X , I have the following weaker result than expected.

⁹This can be computed by the diagram in Example 2.1 (2).

¹⁰by a $\frac{5}{2}$ -curve, I mean a curve with degree $\frac{5}{2}$ with respect to $-K_X$ and with arithmetic genus 0.

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Let $g: \tilde{\mathbb{P}}^4 \rightarrow \mathbb{P}^4$ be the blow-up of $\mathbb{P}^4$ along C , and $h: \tilde{\mathbb{P}}^4 \rightarrow Z$ the anti-flipping contraction of the strict transforms of 5 tri-secant lines of C . Let

$$M := \{[\mathcal{E}] \mid \mathcal{E} \text{ is a rank 3 semi-stable vector bundle on } C \\ \text{with } \det \mathcal{E} = K_C - p \text{ and } h^0(\mathcal{E}) \geq 4\}.$$

There exists a finite birational morphism $Z \rightarrow M$.

Once I can prove M is normal, I have $Z \simeq M$. Since the anti-canonical model $\bar{Y}$ of Y is contained in Z , I believe that $\bar{Y}$ can be characterized as a moduli by using $\mathcal{F}$.

4. OUTLINE OF THE PROOF OF PROPOSITION 3.1

For (A), it suffices to prove the following:

Let C be a general smooth curve of genus 6. In particular, C has no g_4^1 and g_5^2 and C is not bi-elliptic. Let p be a general point of C . Finally let $\mathcal{F}$ be a stable and globally generated bundle of rank 2 on C obtained as in the statement of Proposition 3.1 (A). Then there is an embedding $C \hookrightarrow Q^3$ such that by blowing up Q^3 along C , Q^3 can be birationally transformed to a Fano 3-fold of genus 6 as in the diagram in Example 2.1 (2).

I only show the following diagram, from which the assertion is easily verified:

$$\begin{array}{ccc} C & \longrightarrow & G(H^0(\mathcal{F}), 2) \\ \Phi_{|K_C - p|} \downarrow & & \downarrow \text{Plücker} \\ \mathbb{P}^4 & \longrightarrow & \mathbb{P}^5, \end{array}$$

where $C \rightarrow G(H^0(\mathcal{F}), 2) \simeq G(4, 2)$ is defined by

$$x \mapsto C \rightarrow (H^0(\mathcal{F}) \rightarrow \mathcal{F}_x) \in G(H^0(\mathcal{F}), 2).$$

I define $Q^3 := G(H^0(\mathcal{F}), 2) \cap \mathbb{P}^4$.

I will explain why $\mathbb{P}(\mathcal{F}) \simeq \mathcal{H}_{5/2}$. By the diagram in Example 2.1 (2), I can show that a general $\frac{5}{2}$ -curve on X is a birational transform of a general line on Q^3 intersecting C . Thus I explain how to attach to a point $s \in \mathbb{P}(\mathcal{F})$ a line l_s on Q^3 intersecting C . For a point $s \in \mathbb{P}(\mathcal{F})$, set

$$V_s := \{\sigma \in H^0(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)) \mid s \in (\sigma)_0\} \subset H^0(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)) \simeq H^0(\mathcal{F}).$$

Note that $\dim V_s = 3$ since $\mathcal{F}$ is globally generated. Set

$$l_s := G(2, V_s) \cap Q^3 \subset G(2, H^0(\mathcal{F})) = G(H^0(\mathcal{F}), 2),$$

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which is a line since $G(2, V_s) \simeq \mathbb{P}^2$ and Q^3 does not contain a plane. Let $u := \pi(s)$, where $\pi: \mathbb{P}(\mathcal{F}) \rightarrow C$ is the natural projection. Note that $u = \ker(H^0(\mathcal{F}) \rightarrow \mathcal{F}_u)$ in $G(2, H^0(\mathcal{F}))$. Thus $u \in l_s \cap C$ since

$$\ker(H^0(\mathcal{F}) \rightarrow \mathcal{F}_u) = \{\sigma \in H^0(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)) \mid \pi^{-1}(u) \subset (\sigma)_0\} \subset V_s.$$

Now I explain the proof of Proposition 3.1 (B), which depends on the two propositions.

I start from the preparation for the first proposition. Let U_t be the 4-dimensional subspace of $H^0(K - p)$ corresponding to $t \in \mathbb{P}^4 = \mathbb{P}^*H^0(K - p)$. Define $\mathcal{E}_t^\vee$ by

$$0 \rightarrow \mathcal{E}_t^\vee \rightarrow U_t \otimes \mathcal{O}_C \rightarrow K - p.$$

If $t \notin C$, then $U_t \otimes \mathcal{O}_C \rightarrow K - p$ is surjective, thus $\det \mathcal{E}_t = K_C - p$. If $t \in C$, then $\text{Im}(U_t \otimes \mathcal{O}_C \rightarrow K - p) = K - p - t$, thus $\det \mathcal{E}_t = K_C - p - t$. Actually, Mukai constructs in [Muka] the vector bundle $\tilde{\mathcal{E}}$ on $\tilde{\mathbb{P}}^4 \times C$ such that for $t' \in \tilde{\mathbb{P}}^4$, if $t := g(t') \notin C$, then $\tilde{\mathcal{E}}_{t'} \simeq \mathcal{E}_t$, or if $t \in C$, then $\tilde{\mathcal{E}}_{t'}$ fits into the exact sequence

$$0 \rightarrow \mathcal{E}_t \rightarrow \tilde{\mathcal{E}}_{t'} \rightarrow k(t) \rightarrow 0.$$

Thus $\det \tilde{\mathcal{E}}_{t'} = K_C - p$ for any $t' \in \tilde{\mathbb{P}}^4$.

Proposition 4.1. *$\tilde{\mathcal{E}}_{t'}$ is semi-stable for any $t' \in \tilde{\mathbb{P}}^4$, and $\tilde{\mathcal{E}}_{t'}$ is strictly semi-stable if and only if one of the following equivalent condition hold:*

(1) *there exists an exact sequence as follows:*

$$0 \rightarrow \delta - p \rightarrow \tilde{\mathcal{E}}_{t'} \rightarrow \mathcal{G} \rightarrow 0,$$

where δ is a g_4^1 and $\mathcal{G}$ is a stable vector bundle of rank 2 uniquely determined by

$$0 \rightarrow \mathcal{G}^\vee \rightarrow H^0(K - \delta) \otimes \mathcal{O}_C \rightarrow K - \delta \rightarrow 0.$$

(2) *t' is on the strict transform of a tri-secant line of C .*

The correspondence between a g_4^1 in (1) and a tri-secant line in (2) is given as follows: for δ in (1), the unique member $|\delta - p|$ lies on a tri-secant line, and vice versa.

In particular, the S -equivalent classes of $\tilde{\mathcal{E}}_{t'}$ is constant on the strict transform of a tri-secant line.

Proposition 4.2. *Let $\mathcal{E} \in M_C(3, K - p, 1)$,*

$$ev_{\mathcal{E}} := H^0(\mathcal{E}) \otimes \mathcal{O}_C \rightarrow \mathcal{E} \text{ and } \mathcal{E}_1 := \text{Im } ev_{\mathcal{E}}.$$

Then $\dim H^0(C, \mathcal{E}) = 4$ and $\text{rk } \mathcal{E}_1 = 3$. Moreover one of the following holds:

(1) *$ev_{\mathcal{E}}$ is surjective. In this case, $\mathcal{E}$ defines a point of $\mathbb{P}^4 \setminus C$.*

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(2) $\mathrm{rk} \mathcal{E}_1 = 3$, $h^0(\mathcal{E}_1^\vee) = 0$ and there exists an exact sequence as follows:

$$0 \rightarrow \mathcal{E}_1^\vee \rightarrow \mathcal{O}_C^{\oplus 4} \rightarrow K_C - p - x \rightarrow 0$$

for a point $x \in C$.

(3) $\mathrm{rk} \mathcal{E}_1 = 3$ and $h^0(\mathcal{E}_1^\vee) > 0$, and there exists an exact sequence as follows:

$$0 \rightarrow \mathcal{G} \rightarrow \mathcal{E} \rightarrow \delta - p \rightarrow 0,$$

where δ is a g_4^1 and $\mathcal{G}$ is a stable vector bundle of rank 2 uniquely determined by

$$0 \rightarrow \mathcal{G}^\vee \rightarrow H^0(K - \delta) \otimes \mathcal{O}_C \rightarrow K - \delta \rightarrow 0.$$

I omit the proof of these propositions. I just mention that the proof are based on the so-called Castelnuovo's trick of the following type.

Lemma 4.3. *Let $\mathcal{E}$ be a rank 2 vector bundle on a smooth curve. Set $r := h^0(\mathcal{E})$ and $s := \dim \mathrm{Im} (\wedge^2 H^0(\mathcal{E}) \rightarrow H^0(\wedge^2 \mathcal{E}))$. If $\dim G(2, r) = 2(r - 2) \geq s$, then there exists a 2-dimensional subspace V of $H^0(\mathcal{E})$ such that $\mathrm{Im} (V \otimes \mathcal{O}_C \rightarrow \mathcal{E})$ is invertible.*

reference??

I continue the outline of the proof of Proposition 3.1 (B). The vector bundles in the cases (1) and (2) of Proposition 4.2 appear as $\tilde{\mathcal{E}}_{t'}$ for some t' . The vector bundles in the case (3) are new but S -equivalent to strictly semi-stable $\tilde{\mathcal{E}}_{t'}$ in Proposition 4.1. Hence we have the surjective morphism $\iota: \tilde{\mathbb{P}}^4 \rightarrow M$. The fact $h^0(\mathcal{E}) = 4$ for $[\mathcal{E}] \in M$ (Proposition 4.2) implies that $\mathcal{E}_{t_1} \not\cong \mathcal{E}_{t_2}$ for two points t_1, t_2 on $\tilde{\mathbb{P}}^4 \setminus C$ since U_{t_i} can be recovered by $\mathcal{E}_{t_i}$ as $U_{t_i} = H^0(\mathcal{E}_{t_i})^\vee$. Thus ι is birational. Moreover strictly semi-stable bundle in M are parameterized by the points on the strict transforms of tri-secants and their S -equivalence classes are constant on each strict transform, ι descends on Z . Since $\rho(Z) = 1$, the morphism ι is finite.

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