

An explicit upper bound of the argument of Dirichlet L -functions on the generalized Riemann hypothesis

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Abstract

We prove an explicit upper bound of the function $S(t, \chi)$, defined by the argument of Dirichlet L -functions attached to a primitive Dirichlet character $\chi \pmod{q > 1}$. An explicit upper bound of the function $S(t)$, defined by the argument of the Riemann zeta-function, have been obtained by A. Fujii [1]. Our result is obtained by applying the idea of Fujii's result on $S(t)$. The constant part of the explicit upper bound of $S(t, \chi)$ in this paper does not depend on χ . Our proof does not cover the case $q = 1$ and indeed gives a better bound than the one of Fujii that covers the case $q = 1$.

1 Introduction

The argument of the Riemann zeta-function on the critical line defined by

$$S(t) = \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + it \right)$$

when t is not the ordinate of a zero of $\zeta(s)$. This is obtained by continuous variation along the straight lines connecting 2 , $2 + it$, and $\frac{1}{2} + it$, starting with the value zero. Also, when t is the ordinate of a zero of $\zeta(s)$, we define

$$S(t) = \frac{1}{2} \{S(t+0) + S(t-0)\}.$$

Now, we consider the argument of Dirichlet L -functions. Let $L(s, \chi)$ be the Dirichlet L -function, where $s = \sigma + it$ is a complex variable, associated with a primitive Dirichlet character $\chi \pmod{q > 1}$. Here, we denote the non-trivial zeros of $L(s, \chi)$ by $\rho(\chi) = \beta(\chi) + i\gamma(\chi)$, where $\beta(\chi)$ and $\gamma(\chi)$ are real numbers. Then, when t is not the ordinate of a zero of $L(s, \chi)$, we define

$$S(t, \chi) = \frac{1}{\pi} \arg L \left(\frac{1}{2} + it, \chi \right).$$

This is given by continuous variation along the straight line $s = \sigma + it$, as σ varies from $+\infty$ to $\frac{1}{2}$, starting with the value zero. Also, when t is the ordinate of a zero of $L(s, \chi)$, we define

$$S(t, \chi) = \frac{1}{2} \{S(t+0, \chi) + S(t-0, \chi)\}.$$

Selberg proved

$$S(t, \chi) = O(\log q(t+1))$$

and under the generalized Riemann hypothesis (GRH)

$$S(t, \chi) = O \left(\frac{\log q(t+1)}{\log \log q(t+3)} \right).$$

in Selberg [2]. The purpose of the present article is to prove the following result.

Theorem 1. *Assuming GRH. Then, for $q > 1$*

$$|S(t, \chi)| < 0.804 \cdot \frac{\log q(t+1)}{\log \log q(t+3)} + O\left(\frac{\log q(t+3)}{(\log \log q(t+3))^2}\right).$$

The constant 0.804 obviously does not depend on χ . And we don't know anything concerning the optimality. Also, the implied constant of the error term does not depend on q . Our result does not include the case of the function $S(t)$ since we assume $q > 1$. An explicit upper bound of the function $S(t)$ is obtained by A. Fujii [1], where the value is 0.83.

The basic policy of the proof of this theorem is based on A. Fujii [1]. In the proof, $S(t, \chi)$ is separated by three parts M_1 , M_2 and M_3 . Fujii's idea of [1] is applied to all parts. But we need Lemma 1, which is an explicit formula for $\frac{L'}{L}(s, \chi)$. This lemma is an analogue of Selberg's result.

2 Some notations and a lemma

Here we introduce the following notations.

Let $s = \sigma + it$. We suppose that $\sigma \geq \frac{1}{2}$ and $t \geq 2$. Let x be a positive number satisfying $4 \leq x \leq t^2$. Also, we put

$$\sigma_1 = \frac{1}{2} + \frac{1}{\log x}$$

and

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{for } 1 \leq n \leq x, \\ \Lambda(n) \frac{\log \frac{x^2}{n}}{\log x} & \text{for } x \leq n \leq x^2, \end{cases}$$

with

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with a prime } p \text{ and an integer } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using these notations, we prove the following lemma.

Lemma 1. *Assume the GRH. Let $t \geq 2$ and $x > 0$ such that $4 \leq x \leq t^2$. Then for $\sigma \geq \sigma_1 = \frac{1}{2} + \frac{1}{\log x}$, there exist ω and ω' such that $|\omega| \leq 1$ and $-1 \leq \omega' \leq 1$, we have*

$$\begin{aligned} \frac{L'}{L}(\sigma + it, \chi) = & - \sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma+it}} \chi(n) - \frac{x^{\frac{1}{2}-\sigma} \left(1 + x^{\frac{1}{2}-\sigma}\right) \omega}{1 - \frac{1}{e} \left(1 + \frac{1}{e}\right) \omega'} \Re \left(\sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma_1+it}} \chi(n) \right) \\ & + \frac{x^{\frac{1}{2}-\sigma} \left(1 + x^{\frac{1}{2}-\sigma}\right) \omega}{1 - \frac{1}{e} \left(1 + \frac{1}{e}\right) \omega'} \cdot \frac{1}{2} \log q(t+1) + O(x^{\frac{1}{2}-\sigma}). \end{aligned}$$

This is an analogue of Lemma 2 of A. Fujii [1].

Lemma 2. *Let $a = 0$ if $\chi(-1) = 1$, and $a = 1$ if $\chi(-1) = -1$. Then, for $x > 1$, $s \neq -2q - a$ ($q = 0, 1, 2, \dots$) and $s \neq \rho(\chi)$, we have*

$$\begin{aligned} \frac{L'}{L}(s, \chi) = & - \sum_{n < x^2} \frac{\Lambda_x(n)}{n^s} \chi(n) + \frac{1}{\log x} \sum_{q=0}^{\infty} \frac{x^{-2q-a-s} - x^{-2(2q+a+s)}}{(2q+a+s)^2} \\ & + \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2}. \end{aligned}$$

Lemma 2 is similar to Lemma 15 of Selberg [2]. We write here only a sketch of the proof of Lemma 2. If $a = \max(1, \sigma)$, we have

$$\sum_{n < x^2} \frac{\Lambda_x(n)}{n^s} \chi(n) = \frac{1}{2\pi i \log x} \int_{a-\infty i}^{a+\infty i} \frac{x^{z-s} - x^{2(z-s)}}{(z-s)^2} \cdot \frac{L'}{L}(z, \chi) dz.$$

We consider residues which we encounter when we move the path of integration to the left. At the point $z = s$, the residue is $-(\log x) \frac{L'}{L}(s, \chi)$. At the zeros $-2q - a$ ($q = 0, 1, 2, \dots$), the residues are $\frac{x^{-2q-a-s} - x^{-2(2q+a+s)}}{(2q+a+s)^2}$. At the zeros $s = \rho$ of $L(s, \chi)$, the residues are $\frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2}$. Thus, we obtain Lemma 2. *Proof of Lemma 1.* Assume the GRH. In Lemma 2, since for $\sigma \geq \sigma_1 = \frac{1}{2} + \frac{1}{\log x}$

$$\left| \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2} \right| \leq x^{\frac{1}{2}-\sigma} \left(1 + x^{\frac{1}{2}-\sigma} \right) \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t-\gamma)^2},$$

we have

$$\frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(s-\rho)^2} = x^{\frac{1}{2}-\sigma} \left(1 + x^{\frac{1}{2}-\sigma} \right) \omega \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t-\gamma)^2},$$

where $|\omega| \leq 1$. Hence by Lemma 2, we have for $\sigma \geq \sigma_1$

$$\begin{aligned} \frac{L'}{L}(\sigma + it, \chi) &= - \sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma+it}} \chi(n) + O\left(\frac{x^{\frac{1}{2}-\sigma}}{\log x}\right) \\ &\quad + x^{\frac{1}{2}-\sigma} \left(1 + x^{\frac{1}{2}-\sigma} \right) \omega \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t-\gamma)^2}. \end{aligned} \quad (1)$$

In particular, since $x^{\frac{1}{2}-\sigma} \leq x^{-\frac{1}{\log x}} = \frac{1}{e}$ for $\sigma \geq \sigma_1$, we get

$$\begin{aligned} \Re \frac{L'}{L}(\sigma_1 + it, \chi) &= -\Re \left(\sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma_1+it}} \chi(n) \right) + O\left(\frac{1}{\log x}\right) \\ &\quad + \frac{1}{e} \left(1 + \frac{1}{e} \right) \omega' \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t-\gamma)^2}, \end{aligned} \quad (2)$$

where $-1 \leq \omega' \leq 1$.

Here, since by p. 46 of Selberg [2]

$$\Re \frac{L'}{L}(s, \chi) = \Re \left(-\frac{1}{2} \log \frac{q}{\pi} - \frac{1}{2} \log \left(\frac{s+a}{2} \right) \right) + \sum_{\gamma} \frac{\sigma - \frac{1}{2}}{(\sigma - \frac{1}{2})^2 + (t-\gamma)^2} + O(1),$$

we get for $t \geq 2$

$$\Re \frac{L'}{L}(\sigma_1 + it, \chi) = -\frac{1}{2} \log q(t+1) + \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t-\gamma)^2} + O(1). \quad (3)$$

By (2) and (3) we have

$$\begin{aligned} &\left(1 - \frac{1}{e} \left(1 + \frac{1}{e} \right) \omega' \right) \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t-\gamma)^2} \\ &= -\Re \left(\sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma_1+it}} \chi(n) \right) + \frac{1}{2} \log q(t+1) + O\left(\frac{1}{\log x}\right) + O(1). \end{aligned}$$

Inserting the above inequality to (1), we obtain Lemma 1. □

3 Proof of Theorem 1

The quantity $S(t, \chi)$ is separated into the following three parts.

$$\begin{aligned} S(t, \chi) &= -\frac{1}{\pi} \left\{ \Im \int_{\sigma_1}^{\infty} \frac{L'}{L}(\sigma + it, \chi) d\sigma + \Im \left\{ \left(\sigma_1 - \frac{1}{2} \right) \frac{L'}{L}(\sigma_1 + it, \chi) \right\} \right. \\ &\quad \left. - \Im \int_{\frac{1}{2}}^{\sigma_1} \left\{ \frac{L'}{L}(\sigma_1 + it, \chi) - \frac{L'}{L}(\sigma + it, \chi) \right\} d\sigma \right\} \\ &= -\frac{1}{\pi} \Im (M_1 + M_2 + M_3), \end{aligned}$$

say.

First, we estimate M_1 . By Lemma 1 we have

$$\begin{aligned} M_1 &= \int_{\sigma_1}^{\infty} \left\{ - \sum_{n < x^2} \frac{\Lambda_n(x)}{n^{\sigma+it}} \chi(n) - \frac{x^{\frac{1}{2}-\sigma} \left(1 + x^{\frac{1}{2}-\sigma} \right) \omega}{1 - \frac{1}{e} \left(1 + \frac{1}{e} \right) \omega'} \Re \left(\sum_{n < x^2} \frac{\Lambda_n(x)}{n^{\sigma_1+it}} \chi(n) \right) \right. \\ &\quad \left. + \frac{x^{\frac{1}{2}-\sigma} \left(1 + x^{\frac{1}{2}-\sigma} \right) \omega}{1 - \frac{1}{e} \left(1 + \frac{1}{e} \right) \omega'} \cdot \frac{1}{2} \log q(t+1) + O \left(x^{\frac{1}{2}-\sigma} \right) \right\} d\sigma \\ &= - \sum_{n < x^2} \frac{\Lambda_n(x)}{n^{\sigma_1+it} \log n} \chi(n) + \eta_1(t), \end{aligned} \tag{4}$$

say. Here,

$$\begin{aligned} |\eta_1(t)| &\leq \frac{1}{1 - \frac{1}{e} \left(1 + \frac{1}{e} \right)} \left| \Re \left(\sum_{n < x^2} \frac{\Lambda_n(x)}{n^{\sigma_1+it}} \chi(n) \right) - \frac{1}{2} \log q(t+1) \right| \\ &\quad \times \int_{\sigma_1}^{\infty} x^{\frac{1}{2}-\sigma} \left(1 + x^{\frac{1}{2}-\sigma} \right) d\sigma + O \left(\int_{\sigma_1}^{\infty} x^{\frac{1}{2}-\sigma} d\sigma \right) \\ &\leq \frac{\left(\frac{1}{e} + \frac{1}{2e^2} \right)}{1 - \frac{1}{e} \left(1 + \frac{1}{e} \right)} \cdot \frac{1}{2} \cdot \frac{\log q(t+1)}{\log x} + O \left(\frac{1}{\log x} \left| \sum_{n < x^2} \frac{\Lambda_n(x)}{n^{\sigma_1+it}} \chi(n) \right| \right), \end{aligned} \tag{5}$$

say.

Next, applying Lemma 1 to M_2 , we get

$$|M_2| \leq \frac{\left(\frac{1}{e} + \frac{1}{e^2} \right)}{1 - \frac{1}{e} \left(1 + \frac{1}{e} \right)} \cdot \frac{1}{2} \cdot \frac{\log q(t+1)}{\log x} + O \left(\frac{1}{\log x} \left| \sum_{n < x^2} \frac{\Lambda_n(x)}{n^{\sigma_1+it}} \chi(n) \right| \right), \tag{6}$$

say.

Next we estimate M_3 . By Lemma 16 of Selberg [2] we get

$$\begin{aligned} |\Im(M_3)| &\leq \left| \int_{\frac{1}{2}}^{\sigma_1} \sum_{\gamma} \frac{(t - \gamma) \left\{ \left(\sigma - \frac{1}{2} \right)^2 - \left(\sigma_1 - \frac{1}{2} \right)^2 \right\}}{\left\{ \left(\sigma_1 - \frac{1}{2} \right)^2 + (t - \gamma)^2 \right\} \left\{ \left(\sigma - \frac{1}{2} \right)^2 + (t - \gamma)^2 \right\}} d\sigma \right| + O \left(\frac{1}{\log x} \right) \\ &< \int_{\frac{1}{2}}^{\sigma_1} \sum_{\gamma} N(\gamma, \sigma) d\sigma + O \left(\frac{1}{\log x} \right), \end{aligned}$$

say.

Here, we put $\mathfrak{N} = \int_{\frac{1}{2}}^{\sigma_1} \sum_{\gamma} N(\gamma, \sigma) d\sigma$. Then, we have

$$\begin{aligned} \mathfrak{N} &< \sum_{\gamma} \frac{\left(\sigma_1 - \frac{1}{2} \right)^2}{\left(\sigma_1 - \frac{1}{2} \right)^2 + (t - \gamma)^2} \int_{\frac{1}{2}}^{\infty} \frac{|t - \gamma|}{\left(\sigma - \frac{1}{2} \right)^2 + (t - \gamma)^2} d\sigma \\ &\leq \frac{\pi}{2 \log x} \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{\left(\sigma_1 - \frac{1}{2} \right)^2 + (t - \gamma)^2} \end{aligned}$$

since $\sigma < \sigma_1$ for M_3 .

Here, by (2) and (3) we get

$$\begin{aligned} \sum_{\gamma} \frac{\sigma_1 - \frac{1}{2}}{(\sigma_1 - \frac{1}{2})^2 + (t - \gamma)^2} &= \frac{1}{1 - \frac{1}{e} \left(1 + \frac{1}{e}\right) \omega'} \cdot \frac{1}{2} \log q(t+1) \\ &+ O\left(\left|\sum_{n < x^2} \frac{\Lambda_n(x)}{n^{\sigma_1 + it}} \chi(n)\right|\right) + O\left(\frac{1}{(\log x)^2}\right). \end{aligned}$$

So,

$$\begin{aligned} \Re &= \frac{\pi}{4} \cdot \frac{1}{1 - \frac{1}{e} \left(1 + \frac{1}{e}\right) \omega'} \cdot \frac{1}{\log x} \cdot \log q(t+1) \\ &+ O\left(\frac{1}{\log x} \left|\sum_{n < x^2} \frac{\Lambda_n(x)}{n^{\sigma_1 + it}} \chi(n)\right|\right) + O\left(\frac{1}{(\log x)^3}\right). \end{aligned}$$

Hence we have

$$\begin{aligned} |\Im(M_3)| &\leq \frac{\pi}{4} \cdot \frac{1}{1 - \frac{1}{e} \left(1 + \frac{1}{e}\right) \omega'} \cdot \frac{1}{\log x} \cdot \log q(t+1) \\ &+ O\left(\frac{1}{\log x} \left|\sum_{n < x^2} \frac{\Lambda_n(x)}{n^{\sigma_1 + it}} \chi(n)\right|\right) + O\left(\frac{1}{\log x}\right) \\ &= \eta_4(t) + O\left(\frac{1}{\log x} \left|\sum_{n < x^2} \frac{\Lambda_n(x)}{n^{\sigma_1 + it}} \chi(n)\right|\right) + O\left(\frac{1}{\log x}\right), \end{aligned} \quad (7)$$

say.

Finally, we estimate the sums on right-hand sides of (4), (5), (6) and (7). By definition of $\Lambda_x(n)$ we have

$$\left|\sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma_1 + it}} \chi(n)\right| \leq \sum_{n < x} \frac{\Lambda(n)}{n^{\frac{1}{2}}} + \sum_{x \leq n \leq x^2} \frac{\Lambda(n) \log \frac{x^2}{n}}{n^{\frac{1}{2}}} \cdot \frac{1}{\log x} \ll \frac{x}{\log x}.$$

Similarly,

$$\left|\sum_{n < x^2} \frac{\Lambda_x(n)}{n^{\sigma_1 + it} \log n} \chi(n)\right| \ll \frac{x}{(\log x)^2}.$$

So, we see

$$\begin{aligned} |M_1| &\leq \frac{\left(\frac{1}{e} + \frac{1}{2e^2}\right)}{1 - \frac{1}{e} \left(1 + \frac{1}{e}\right)} \cdot \frac{1}{2} \cdot \frac{\log q(t+1)}{\log x} + O\left(\frac{x}{(\log x)^2}\right), \\ |M_2| &\leq \frac{\left(\frac{1}{e} + \frac{1}{e^2}\right)}{1 - \frac{1}{e} \left(1 + \frac{1}{e}\right)} \cdot \frac{1}{2} \cdot \frac{\log q(t+1)}{\log x} + O\left(\frac{x}{(\log x)^2}\right), \end{aligned}$$

and

$$|M_3| \leq \eta_4(t) + O\left(\frac{x}{(\log x)^2}\right).$$

For $\eta_1(t)$, $\eta_2(t)$, $\eta_3(t)$ and $\eta_4(t)$, taking $x = \log q(t+3) \sqrt{\log q(t+3)}$ we have

$$\begin{aligned} |S(t, \chi)| &< \frac{1}{\pi} \cdot \frac{1}{1 - \frac{1}{e} \left(1 + \frac{1}{e}\right)} \left\{ \frac{\left(\frac{1}{e} + \frac{1}{2e^2}\right)}{2} + \frac{\left(\frac{1}{e} + \frac{1}{e^2}\right)}{2} + \frac{\pi}{4} \right\} \frac{\log q(t+1)}{\log x} \\ &+ O\left(\frac{x}{(\log x)^2}\right) \\ &= 0.803986 \cdots \frac{\log q(t+1)}{\log \log q(t+3)} + O\left(\frac{\log q(t+3)}{(\log \log q(t+3))^2}\right). \end{aligned}$$

Therefore we obtain the theorem. □

References

- [1] A. Fujii, An explicit estimate in the theory of the distribution of the zeros of the Riemann zeta function, *Comment. Math. Univ. Sancti Pauli*, **53**, (2004), 85-114.
- [2] A. Selberg, Contributions to the theory of Dirichlet's L-function, *Avh. Norske Vir. Akad. Oslo I*:1, (1946), No. 3, 1-62.
- [3] A. Selberg, *Collected Works*, vol I, 1989, Springer.
- [4] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, Second Edition; Revised by D. R. Heath-Brown. Clarendon Press Oxford, 1986.