

A note on Strichartz estimates for the wave equation with orthonormal initial data

By

Neal BEZ*, Shinya KINOSHITA**, and Shobu SHIRAKI***

Abstract

This note is concerned with Strichartz estimates for the wave equation and orthonormal families of initial data. We provide a survey of the known results and present what seems to be a reasonable conjecture regarding the cases which have been left open. We also provide some new results in the maximal-in-space boundary cases.

§ 1. Introduction

The (one-sided) wave propagator $e^{it\sqrt{-\Delta}}$ is given by

$$e^{it\sqrt{-\Delta}}f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x \cdot \xi + t|\xi|)} \widehat{f}(\xi) d\xi$$

for sufficiently nice initial data f and we often write $Uf(t, x) = e^{it\sqrt{-\Delta}}f(x)$. Our interest here are Strichartz estimates of the form

$$(1.1) \quad \left\| \sum_j \lambda_j |Uf_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|\lambda\|_{\ell^\beta}$$

Received November 30, 2022. Revised March 11, 2024.

2020 Mathematics Subject Classification(s): Primary 35L05; Secondary 42B37.

Key Words: Strichartz estimates, Orthonormal systems, Wave equation

Supported by JSPS Kakenhi grant numbers 18KK0073, 19H00644, and 19H01796 (Bez), JP21J00514 (Kinoshita), and 19H01796 and 22H00098 (Shiraki).

*Department of Mathematics, Graduate School of Science and Engineering, Saitama University, Saitama 338-8570, Japan.

Current address: Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan.

e-mail: bez@math.nagoya-u.ac.jp

**Department of Mathematics, Graduate School of Science and Engineering, Saitama University, Saitama 338-8570, Japan.

Current address: Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan.

e-mail: kinoshita@math.nagoya-u.ac.jp

***Department of Mathematics, Graduate School of Science and Engineering, Saitama University, Saitama 338-8570, Japan.

Current address: Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia.

e-mail: shobu.shiraki@math.hr

where $(f_j)_j$ is a (possibly infinite) family of orthonormal functions in the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$, $s = \frac{d}{2} - \frac{d}{r} - \frac{1}{q}$ (we refer the reader forward to the end of this section for the meaning of $\lesssim$). Equivalently, (1.1) can be seen as

$$\left\| \left(\sum_j |Uf_j|^2 \right)^{\frac{1}{2}} \right\|_{L_t^q L_x^r} \lesssim \left(\sum_j \|f_j\|_{\dot{H}^s}^{2\beta} \right)^{\frac{1}{2\beta}},$$

where $(f_j)_j$ is a family of orthogonal functions in $\dot{H}^s(\mathbb{R}^d)$ and s is unchanged. Such an estimate obviously implies the classical Strichartz estimate

$$(1.2) \quad \|Uf\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^s}$$

by considering a family of data for which all but one member is zero. Therefore, in what follows, we restrict attention to exponents q and r which admit a classical Strichartz estimate. Conversely, it is also clear that (1.2) implies (1.1) with $\beta = 1$ by simply using the triangle inequality (and, in particular, not making use of the orthogonality) and so the challenge is to establish (1.1) for a range of β (depending on q and r) which is as large as possible.

Before going forward to describe the current situation regarding (1.1), we pause to briefly clarify which exponents q and r admit a classical Strichartz estimate (1.2). For $d \geq 2$, let us say that $(q, r) \in [2, \infty] \times [2, \infty]$ is *admissible* if

$$\frac{1}{q} \leq \frac{d-1}{2} \left(\frac{1}{2} - \frac{1}{r} \right).$$

Then it is widely known that if (q, r) is admissible and $r < \infty$ then (1.2) holds (see, for example, [16] and the references therein). The boundary case

$$(1.3) \quad \|Uf\|_{L_t^q L_x^\infty} \lesssim \|f\|_{\dot{H}^{\frac{d}{2}-\frac{1}{q}}}$$

presents difficulties. Indeed, it is known that (1.3) fails when $(q, d) = (4, 2)$ and when $q = 2$ for all $d \geq 3$ (see [9, Theorem 3] for the former claim and [14, Theorem 1.1] for the latter). Also (1.3) fails when $q = \infty$ thanks to the failure of the associated Sobolev embedding. For the remaining q (that is, $q \in (4, \infty)$ when $d = 2$, or $q \in (2, \infty)$ when $d \geq 3$) estimate (1.3) holds; see, for example [9, Theorem 3].

Returning to the wider framework in (1.1), Frank and Sabin [11] provided the first progress by establishing this estimate in the sharp admissible case

$$(1.4) \quad \frac{1}{q} = \frac{d-1}{2} \left(\frac{1}{2} - \frac{1}{r} \right)$$

with $q = r = \frac{2(d+1)}{d-1}$ and $\beta \leq \frac{2r}{r+2}$. Formally interpolating with the trivial estimate with $\beta = 1$ at $(q, r) = (\infty, 2)$, this yields (1.1) in the sharp admissible case (1.4) for

$r \in [2, \frac{2(d+1)}{d-1}]$ and $\beta \leq \frac{2r}{r+2}$. As far as we are aware, this interpolation argument seems to require the use of an analytic family of operators (since the regularity exponent s varies); we refer the reader forward to Section 2 for arguments of a similar nature.

To discuss results which have been established since [11], it is very convenient to introduce the notation $\beta_\sigma(q, r)$, where $\sigma > 0$, for the exponent satisfying

$$\frac{\sigma}{\beta_\sigma(q, r)} = \frac{1}{q} + \frac{2\sigma}{r}.$$

Observe that $\beta_\sigma(q, r) = \frac{2r}{r+2}$ in the case $\frac{1}{q} = \sigma(\frac{1}{2} - \frac{1}{r})$, and so the result in [11] may be viewed as the statement that (1.1) holds for $\beta \leq \beta_{\frac{d-1}{2}}(q, r)$ whenever (1.4) and $r \in [2, \frac{2(d+1)}{d-1}]$. It is also trivial to see that $\beta_\sigma(q, r)$ is increasing as a function of σ for each fixed (q, r) .

In terms of upper bounds on the optimal value of the exponent β , it was established in [6] that

$$(1.5) \quad \beta \leq \min \left\{ \beta_{\frac{d}{2}}(q, r), \frac{q}{2} \right\}$$

is necessary in order for (1.1) to be true. For completeness, in the Appendix of this note, we give a sketch of the proof. One interesting consequence is that there is no hope of raising β above 1 in the case where $q = 2$ and so we omit this case from the subsequent discussion. We also note that $\beta_{\frac{d}{2}}(q, r) \leq \frac{q}{2}$ if and only if $\frac{d}{r} \leq \frac{d-1}{q}$.

The result in [11] was extended in [6] and as a result the following is now known to be true.

Theorem 1.1 (see [6, 11]). *Let $d \geq 2$, $q \in (2, \infty)$, $r \in [2, \infty)$, and $\frac{1}{q} \leq \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$. Then the estimate (1.1) holds in each of the following cases.*

- (i) $\beta \leq \beta_{\frac{d-1}{2}}(q, r)$ when $\frac{d-1}{r} > \frac{d-2}{q}$.
- (ii) $\beta < \frac{q}{2}$ when $\frac{d-1}{r} \leq \frac{d-2}{q}$.

It is perhaps beneficial to view this result with Figures 1 and 2 in mind. The points A, B, C, D, E are given as follows.

$$\begin{aligned} A &= \left(\frac{d-2}{2d}, \frac{d-1}{2d} \right), & B &= \left(\frac{d-1}{2(d+1)}, \frac{d-1}{2(d+1)} \right), \\ C &= \left(\frac{1}{2}, 0 \right), & D &= \left(0, \frac{1}{2} \right), & E &= \left(\frac{d-3}{2(d-1)}, \frac{1}{2} \right). \end{aligned}$$

When $(\frac{1}{r}, \frac{1}{q})$ belongs to the interior of the region OAC or the line segment $(A, C]$ (i.e. not including the endpoint at A), then Theorem 1.1 guarantees that (1.1) holds for $\beta \leq \beta_{\frac{d-1}{2}}(q, r)$. For $d \geq 3$ and in the interior of region $ODEA$, or the line segments

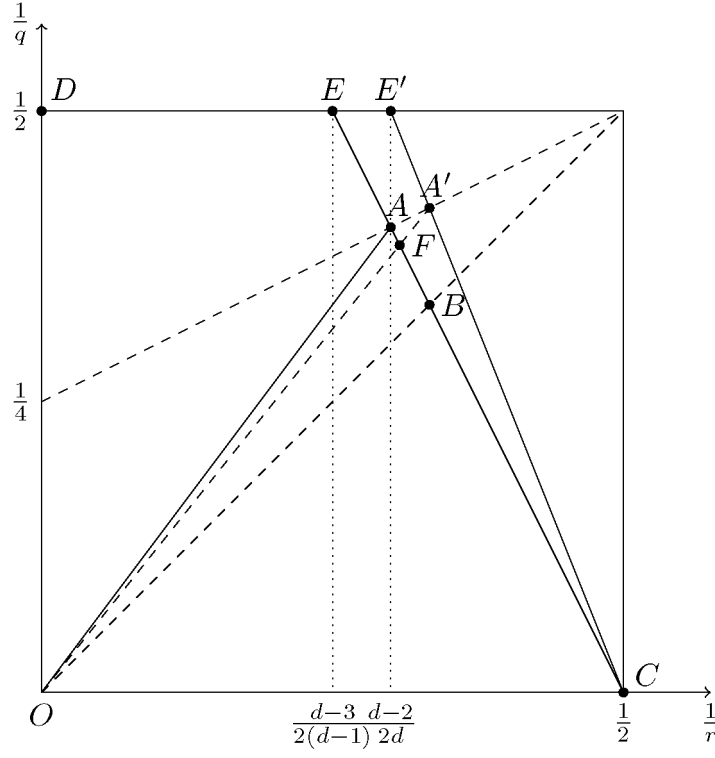


Figure 1. The case $d \geq 3$ (note that $D = E$ when $d = 3$). On the lines $[C, E]$ and $[O, A]$ we have $\frac{1}{q} = \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$ and $\frac{d-2}{q} = \frac{d-1}{r}$, respectively.

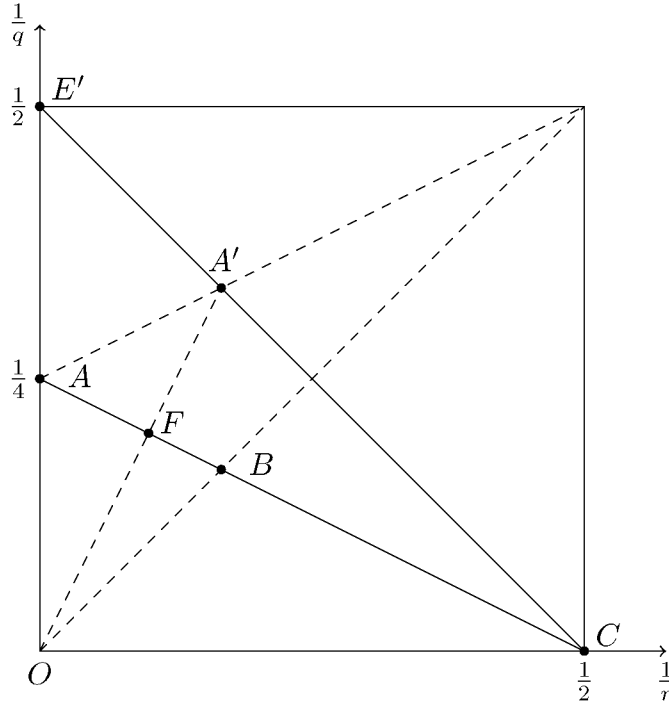


Figure 2. The case $d = 2$. On the line $[C, A]$ we have $\frac{1}{q} = \frac{1}{2}(\frac{1}{2} - \frac{1}{r})$.

$(O, A]$, $[A, E)$, we have that (1.1) holds for $\beta < \frac{q}{2}$. Thanks to (1.5), this means that Theorem 1.1(ii) is sharp up to the critical case $\beta = \frac{q}{2}$.

§ 1.1. A conjecture

The most pressing issue appears to be closing the gap between the necessary condition (1.5) and the sufficient condition $\beta \leq \beta_{\frac{d-1}{2}}(q, r)$ given in Theorem 1.1(i), illustrated in Figures 3 and 4. For this it seems reasonable to us to believe that Theorem 1.1(i) can be improved up to the necessary condition in (1.5). This amounts to the following conjecture.

Conjecture 1.2. *Let $d \geq 2$, $q \in (2, \infty)$, $r \in [2, \infty)$, and $\frac{1}{q} \leq \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$. Then the estimate (1.1) holds in each of the following cases.*

- (i) $\beta \leq \beta_{\frac{d}{2}}(q, r)$ when $\frac{d}{r} > \frac{d-1}{q}$.
- (ii) $\beta < \frac{q}{2}$ when $\frac{d}{r} \leq \frac{d-1}{q}$.

In Figures 1 and 2, the following points also arise:

$$A' = \left(\frac{d-1}{2(d+1)}, \frac{d}{2(d+1)} \right), \quad E' = \left(\frac{d-2}{2d}, \frac{1}{2} \right), \quad F = \left(\frac{(d-1)^2}{2(d^2+1)}, \frac{d(d-1)}{2(d^2+1)} \right).$$

Conjecture 1.2(i) says that (1.1) holds for $\beta \leq \beta_{\frac{d}{2}}(q, r)$ in the interior of OFC and the line segment $[C, F)$, and Conjecture 1.2(ii) says that (1.1) holds for $\beta < \frac{q}{2}$ in the interior of $ODEF$, or (O, F) , or $[F, E)$.

Observe that the line segment $[C, E']$ corresponds to the sharp admissible line

$$\frac{1}{q} = \frac{d}{2} \left(\frac{1}{2} - \frac{1}{r} \right)$$

for the Strichartz estimates associated with the Schrödinger propagator $e^{it\Delta}$. The analogue of (1.1) for Schrödinger propagator takes the form

$$(1.6) \quad \left\| \sum_j \lambda_j |e^{it\Delta} f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \lesssim \|\lambda\|_{\ell^\beta}$$

where $(f_j)_j$ is a family of orthonormal functions in the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$, $s = \frac{d}{2} - \frac{d}{r} - \frac{2}{q}$, and interestingly this family of estimates is much better understood than (1.1). In particular, it is known (see [10, 11, 12, 3]) that for $q \in (2, \infty)$, $r \in [2, \infty)$, and $\frac{1}{q} \leq \frac{d}{2}(\frac{1}{2} - \frac{1}{r})$, the estimate (1.6) holds in each of the following cases.

- (i) $\beta \leq \beta_{\frac{d}{2}}(q, r)$ when $\frac{d}{r} > \frac{d-1}{q}$ (i.e. the interior of $OA'C$ and $[C, A')$).
- (ii) $\beta < \frac{q}{2}$ when $\frac{d}{r} \leq \frac{d-1}{q}$ (i.e. the interior of $ODE'A'$ and the line segments $(O, A']$, $[A', E')$).

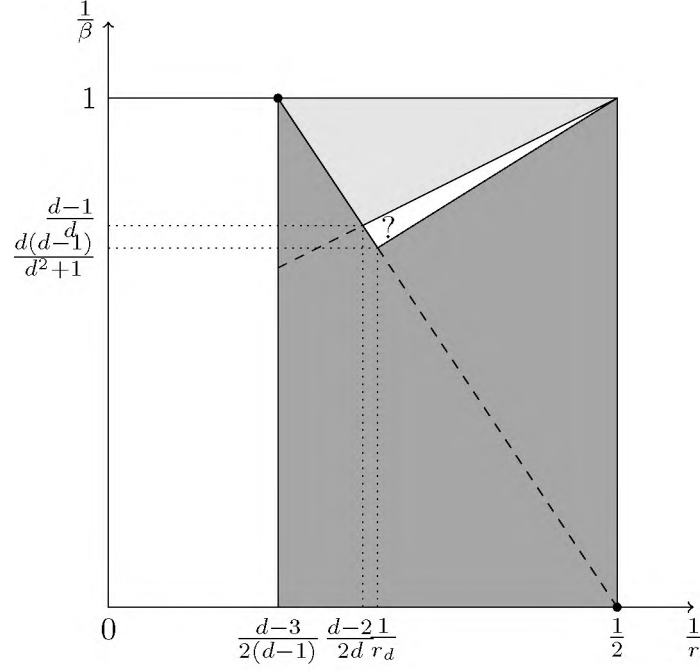


Figure 3. For $d \geq 3$ and in the sharp admissible case $\frac{1}{q} = \frac{d-1}{2}(\frac{1}{2} - \frac{1}{r})$, the gap between the necessary condition (1.5) and Theorem 1.1(i) is represented by the gap between the light gray and dark gray regions. The exponent r_d is given by $r_d = \frac{2(d^2+1)}{(d-1)^2}$.

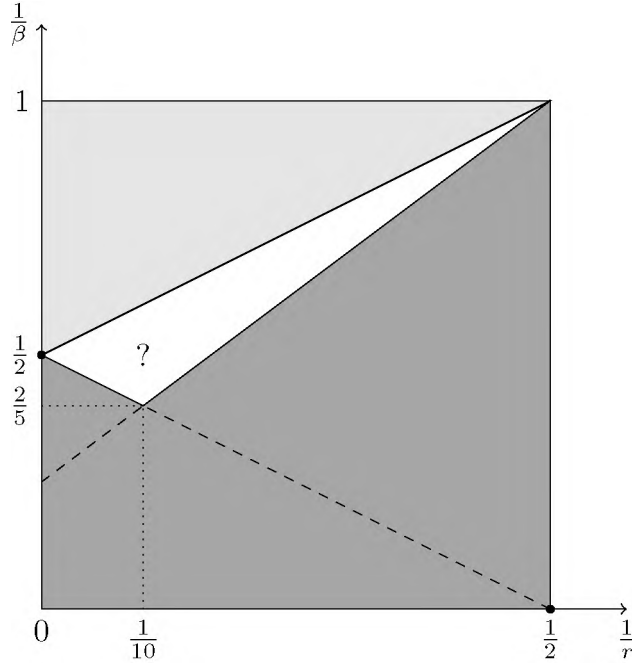


Figure 4. For $d = 2$ and in the sharp admissible case $\frac{1}{q} = \frac{1}{2}(\frac{1}{2} - \frac{1}{r})$, the gap between the necessary condition (1.5) and Theorem 1.1(i) is represented by the gap between the light gray and dark gray regions.

Furthermore, the restriction in (1.5) is also known to be necessary for (1.6) too. Thus, apart from certain critical/boundary cases, we have an almost complete understanding of when the estimates (1.6) are valid. We refer the reader to Frank *et. al* [10] for the origin of this line of study for the Schrödinger equation, and to [11, 12, 3] for further developments on (1.6).

§ 1.2. Boundary cases

The boundary cases $q = \infty$ and $r = \infty$ have been excluded from the statements of Theorem 1.1 and Conjecture 1.2, and for the remainder of this note we mainly focus on the case $r = \infty$. When $q = \infty$, we may simply use the following Sobolev inequality

$$\left\| \sum_j \lambda_j |f_j|^2 \right\|_{L_x^{\frac{r}{2}}} \lesssim \|\lambda\|_{\ell^{\frac{r}{2},1}}$$

proved by Lieb [18]. Here, $(f_j)_j$ is a family of orthonormal functions in $\dot{H}^s(\mathbb{R}^d)$, $s = d(\frac{1}{2} - \frac{1}{r})$ and $r \in (2, \infty)$. Using that U is unitary on $\dot{H}^s(\mathbb{R}^d)$ and the embedding $\ell^\beta \subseteq \ell^{\frac{r}{2},1}$ for $\beta < \frac{r}{2}$, we quickly obtain (1.1) with $q = \infty$, $r \in (2, \infty)$ and $\beta < \frac{r}{2} = \beta_{\frac{d}{2}}(\infty, r)$. In light of the necessary condition (1.5), we see that this result is sharp up to the critical case $\beta = \frac{r}{2}$.

When $r = \infty$, matters appear to be less simple and for the remainder of this note we shall be concerned with the estimate

$$(1.7) \quad \left\| \sum_j \lambda_j |U f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^\infty} \lesssim \|\lambda\|_{\ell^\beta},$$

where $(f_j)_j$ is a family of orthonormal functions in $\dot{H}^{\frac{d}{2} - \frac{1}{q}}(\mathbb{R}^d)$. Also we recall from our discussion of the classical Strichartz estimates (1.2) that we are interested in the range $q \in (4, \infty)$ when $d = 2$, and $q \in (2, \infty)$ when $d \geq 3$. As such, we introduce the notation

$$q_d := \begin{cases} 4 & (d = 2), \\ 2 & (d \geq 3). \end{cases}$$

Our first new result is the following.

Theorem 1.3. *For any $d \geq 2$, the estimate (1.7) holds with $q \in (q_d, \infty)$ and $\beta < \frac{q}{2}$.*

Although it is unclear whether one can extend Theorem 1.3 to the critical case $\beta = \frac{q}{2}$, we are at least able to show the following restricted weak-type estimates.

Theorem 1.4. *For any $d \geq 3$ and $q \in (q_d, \infty)$, the estimate*

$$(1.8) \quad \left\| \sum_j \lambda_j |U f_j|^2 \right\|_{L_t^{\frac{q}{2}, \infty} L_x^\infty} \lesssim \|\lambda\|_{\ell^{\frac{q}{2},1}}$$

holds whenever $(f_j)_j$ is a family of orthonormal functions in $\dot{H}^{\frac{d}{2}-\frac{1}{q}}(\mathbb{R}^d)$.

The rest of the paper is devoted to proving Theorems 1.3 and 1.4, and this is taken up in Sections 2 and 3, respectively. Prior to that, we introduce a small amount of notation.

Notation

For non-negative quantities A, B , the notation $A \lesssim B$ (or $B \gtrsim A$) denotes an estimate $A \leq CB$ for some constant $C > 0$, and we write $A \sim B$ when both $A \lesssim B$ and $B \lesssim A$.

Let $\varphi \in C_0^\infty$ be supported in the annulus $\{\xi \in \mathbb{R}^n : 2^{-1} < |\xi| < 2\}$ and satisfy

$$\sum_{k \in \mathbb{Z}} \varphi(2^{-k}\xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

Define the family of the Littlewood–Paley operators $(P_k)_{k \in \mathbb{Z}}$ by $\widehat{P_k f}(\xi) = \varphi_k(\xi) \widehat{f}(\xi)$, where $\varphi_k(\xi) = \varphi(2^{-k}\xi)$ for each $k \in \mathbb{Z}$. Also, let $\chi \in C_0^\infty$ be supported in the ball $\{\xi \in \mathbb{R}^n : |\xi| < 1\}$ with $\chi(0) = 1$.

For $1 \leq p < \infty$, $1 \leq q \leq \infty$, we write $L^{p,q}(\mathbb{R}^d)$ for the associated Lorentz function space and $\ell^{p,q}$ for the associated Lorentz sequence space. We refer the reader to [13, 22] for the definition and wider discussion on these space. In particular, it is well known that

$$L^{p,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d), \quad L^{p,q_1}(\mathbb{R}^d) \subseteq L^{p,q_2}(\mathbb{R}^d) \quad \text{when } q_1 \leq q_2.$$

§ 2. A sketch proof of Theorem 1.3

In [4], the analogue of (1.7) for the fractional Schrödinger propagators has been obtained.

Theorem 2.1 (see [4]). *Let $a \in (0, \infty) \setminus \{1\}$ and $d \geq 1$. For $q \in (q_{d+1}, \infty)$ and $\beta < \frac{q}{2}$, the estimate*

$$(2.1) \quad \left\| \sum_j \lambda_j |e^{it(-\Delta)^{\frac{a}{2}}} f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^\infty} \lesssim \|\lambda\|_{\ell^\beta}$$

holds whenever $(f_j)_j$ is a family of orthonormal functions in $\dot{H}^{\frac{d}{2}-\frac{a}{q}}(\mathbb{R}^d)$.

One may follow the argument in [4] to prove Theorem 1.3 with very minor modifications, and so we only present a very brief sketch here and refer the reader to [4] for further details.

Key to the argument is the dispersive estimate of the integral kernel associated with the wave propagator:

$$(2.2) \quad \left| \int e^{i(x \cdot \xi + t|\xi|)} |\xi|^{-d+\frac{2}{q}+i\kappa} \chi(\xi) \, d\xi \right| \lesssim C(\kappa) |t|^{-\frac{2}{q}},$$

where $d \geq 2$, $\frac{4}{d-1} < q < \infty$, $\kappa \in \mathbb{R}$, and $C(\kappa) > 0$ satisfies $C(\kappa) \lesssim e^{\varepsilon|\kappa|}$ for an arbitrary small $\varepsilon > 0$. The corresponding estimate for the fractional Schrödinger propagator

$$(2.3) \quad \left| \int e^{i(x \cdot \xi + t|\xi|^a)} |\xi|^{-d+\frac{2a}{q}+i\kappa} \chi(\xi) \, d\xi \right| \lesssim C(\kappa) |t|^{-\frac{2}{q}}$$

holds for $d \geq 1$, $a \in (0, \infty) \setminus \{1\}$, $\frac{4}{a} \leq q < \infty$, $\kappa \in \mathbb{R}$, and $C(\kappa) > 0$ as above (for a proof of (2.2) and (2.3), see for example [15] and [14]). Notice that the condition on q in (2.2) corresponds to the one for (2.3), but d is replaced by $d-1$ and the endpoint $q = \frac{4}{d-1}$ is excluded; the lack of an endpoint estimate here turns out to be harmless (since we are working away from the sharp admissible line).

The argument proceeds using a duality principle observed by Frank–Sabin [11, Lemma 3] in an abstract setting. In our case, we see that (1.7) with U replaced by $\chi(|D|)U$ is equivalent to

$$(2.4) \quad \|WU\chi^2(|D|)|D|^{-(d-1+\frac{2}{q})}U^*\overline{W}\|_{\mathcal{C}^{\beta'}} \lesssim \|W\|_{L_t^{\tilde{q}}L_x^2}^2,$$

where β' denotes the (standard) conjugate of β , and $\tilde{q}$ denotes the (half-)conjugate of q given by

$$\frac{1}{\beta} + \frac{1}{\beta'} = 1, \quad \frac{1}{q} + \frac{1}{\tilde{q}} = \frac{1}{2}.$$

Here, $\mathcal{C}^{\beta'}$ with $\beta' \geq 1$ denotes the Schatten space on $L^2(\mathbb{R}^{d+1})$ defined as

$$\mathcal{C}^{\beta'} = \{T \in \text{Com}(L^2(\mathbb{R}^{d+1})) : \|T\|_{\mathcal{C}^{\beta'}} := \|(\mu_j(T))_{j \in \mathbb{N}}\|_{\ell^{\beta'}} < \infty\},$$

where $\text{Com}(L^2(\mathbb{R}^{d+1}))$ denotes the set of compact operators on $L^2(\mathbb{R}^{d+1})$ and $(\mu_j(T))_j$ denotes the family of the singular values of T (i.e. non-zero eigenvalues of $\sqrt{T^*T}$). Note that establishing (1.7) with $\chi(|D|)U$ suffices since one can rescale to deduce the same estimate with $\chi(\varepsilon|D|)U$ for any $\varepsilon > 0$, and then take a limit $\varepsilon \rightarrow 0$.

The space $\mathcal{C}^1$ consists of trace-class operators and $\mathcal{C}^\infty$ consists of compact operators (endowed with the standard operator norm). Hence $\mathcal{C}^{\beta'}$ with $1 < \beta' < \infty$ is regarded as an intermediate space. Of particular importance to the analysis in this note is that the class $\mathcal{C}^2$ consists of Hilbert–Schmidt operators and that there is a very useful expression of the $\mathcal{C}^2$ norm; if T is given

$$Tf(x) = \int_{\mathbb{R}^{d+1}} K(x, y) f(y) \, dy, \quad f \in L^2(\mathbb{R}^{d+1}),$$

then $\|T\|_{\mathcal{C}^2} = \|K\|_{L^2(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})}$. For more details about Schatten spaces, see Simon's textbook [21].

Our goal is to prove (2.4) for $\tilde{q} \in (2, \tilde{q}_d)$ and $\beta' > \frac{\tilde{q}}{2}$. We write $\mathcal{U} = \chi(|D|)U$ and divide into two cases depending on $\tilde{q}$.

Case 1: $d \geq 3$ and $4 \leq \tilde{q} < \infty$: We aim to get the following $\mathcal{C}^2$ and $\mathcal{C}^\infty$ bilinear estimates: if $2 < \tilde{q}_0 < 4$, $2 < \tilde{q}_1 < \infty$,

$$(2.5) \quad \|W_1 \mathcal{U} |D|^{-(d-1+\frac{2}{\tilde{q}_0})+i\kappa} \mathcal{U}^* \overline{W}_2\|_{\mathcal{C}^2} \lesssim C(\kappa) \|W_1\|_{L_t^{\tilde{q}_0,4} L_x^2} \|W_2\|_{L_t^{\tilde{q}_0,4} L_x^2},$$

$$(2.6) \quad \|W_1 \mathcal{U} |D|^{-(d-1+\frac{2}{\tilde{q}_1})+i\kappa} \mathcal{U}^* \overline{W}_2\|_{\mathcal{C}^\infty} \lesssim C(\kappa) \|W_1\|_{L_t^{\tilde{q}_1,\infty} L_x^2} \|W_2\|_{L_t^{\tilde{q}_1,\infty} L_x^2},$$

where $C(\kappa) \lesssim e^{\varepsilon|\kappa|}$ for an arbitrary small $\varepsilon > 0$. Then, by applying a bilinear version of Stein's analytic interpolation to the above estimates, we may obtain (2.4) for the case $4 \leq \tilde{q} < \infty$. For this purpose, the complex power $i\kappa$ is essential in (2.5) and (2.6) in order to define an appropriate analytic family of operators; we refer the reader to the proof of [4, Proposition 4.4] for further details. One may also notice that, thanks to the Lorentz improvements on the right-hand sides of (2.5) and (2.6), the outcome is also slightly stronger than (2.4), namely, in Case 1 and for $\beta' > \frac{\tilde{q}}{2}$ one obtains

$$\|W \mathcal{U} |D|^{-(d-1+\frac{2}{\tilde{q}})+i\kappa} \mathcal{U}^* \overline{W}\|_{\mathcal{C}^{\beta'}} \lesssim \|W\|_{L_t^{\tilde{q},2\beta'} L_x^2}^2.$$

The proofs of (2.5) and (2.6) are based on the dispersive estimate for the wave propagator (2.2) and the Lorentz refinement of the Hardy–Littlewood–Sobolev inequality due to O'Neil [20]. The interested reader is encouraged to visit [4] where further details regarding the bilinear analytic interpolation of (2.5) and (2.6) can also be found.

Case 2: $d \geq 2$ and $2 < \tilde{q} < 4$: We approach this case by first establishing the somewhat weaker estimates

$$(2.7) \quad \|W_1 \mathcal{U} |D|^{-(d-1+\frac{2}{\tilde{q}})+i\kappa} \mathcal{U}^* \overline{W}_2\|_{\mathcal{C}^{\beta'}} \lesssim C(\kappa) \|W_1\|_{L_t^{\tilde{q},2} L_x^2} \|W_2\|_{L_t^{\tilde{q},2} L_x^2},$$

where $\beta' > \frac{\tilde{q}}{2}$ and $C(\kappa) \lesssim e^{\varepsilon|\kappa|}$ for an arbitrary small $\varepsilon > 0$. In fact, we can offset the loss in (2.7) (in the sense of $L_t^{\tilde{q},2} \subsetneq L_t^{\tilde{q}}$ when $\tilde{q} > 2$) by capitalizing on the gain in (2.5) (in the sense of $L_t^{\tilde{q}_0} \subsetneq L_t^{\tilde{q}_0,4}$ if $\tilde{q}_0 < 4$), and use bilinear analytic interpolation once again to obtain the desired estimate (2.4) in Case 2 (see [4] for further details of this step).

To prove (2.7), we use a frequency-localization argument and use a different bilinear interpolation argument in this spirit of Keel–Tao [16] based on the estimates

$$(2.8) \quad \|W_1 \mathcal{U} P_k |D|^{-2s+i\kappa} P_k \mathcal{U}^* \overline{W}_2\|_{\mathcal{C}^1} \lesssim (2^k)^{d-2s} \|W_1\|_{L_t^2 L_x^2} \|W_2\|_{L_t^2 L_x^2},$$

$$(2.9) \quad \|W_1 \mathcal{U} P_k |D|^{-2s+i\kappa} P_k \mathcal{U}^* \overline{W}_2\|_{\mathcal{C}^2} \lesssim C(\kappa) (2^k)^{d-2s-1+\frac{1}{\tilde{q}_1}+\frac{1}{\tilde{q}_2}} \|W_1\|_{L_t^{\tilde{q}_1} L_x^2} \|W_2\|_{L_t^{\tilde{q}_2} L_x^2},$$

where $\tilde{q}_1, \tilde{q}_2 \in [2, \infty)$ are such that $\frac{1}{\tilde{q}_1} + \frac{1}{\tilde{q}_2} > \frac{1}{2}$, and $C(\kappa) \lesssim e^{\varepsilon|\kappa|}$ for an arbitrary small $\varepsilon > 0$. As in Case 1, (2.9) readily follows from a stronger form of the dispersive estimate

(2.2) with decay $(1 + |t|)^{-(d-1)/2}$. The estimate (2.8) is a manifestation of the frequency localization¹ and Bessel's inequality for orthonormal sequences, and for its proof, it is convenient to make use of the representation

$$\|T\|_{\mathcal{C}^{\beta'}} = \sup_{(\phi, \psi) \in \mathcal{B}} \|\langle T\phi_j, \psi_j \rangle_{L^2(\mathbb{R}^{d+1})}\|_{\ell^{\beta'}}$$

for Schatten norms (see [21, Proposition 2.6]). Here, $\mathcal{B}$ denotes the set of pairs of orthonormal sequences in $L^2(\mathbb{R}^{d+1})$.

§ 3. Proof of Theorem 1.4

Key to our proof of (1.8) is a clever summation idea going back to Bourgain [7] (which has been used in several papers; see also [8, Section 6.2] and [17, Lemma 2.3] for example) which allows us to pass from frequency-local to frequency-global estimates.

Lemma 3.1 (see [3]). *Let $q_0, q_1, r \in [2, \infty]$, $\beta_0, \beta_1 \in [2, \infty]$ and $(g_j)_j$ be a uniformly bounded sequence in $L_t^{q_0} L_x^r \cap L_t^{q_1} L_x^r$. Suppose there exist $\varepsilon_0, \varepsilon_1 > 0$ such that*

$$\left\| \sum_j \lambda_j |P_k g_j|^2 \right\|_{L_t^{\frac{q_0}{2}, \infty} L_x^{\frac{r}{2}}} \lesssim 2^{-\varepsilon_0 k} \|\lambda\|_{\ell^{\beta_0}}$$

and

$$\left\| \sum_j \lambda_j |P_k g_j|^2 \right\|_{L_t^{\frac{q_1}{2}, \infty} L_x^{\frac{r}{2}}} \lesssim 2^{\varepsilon_1 k} \|\lambda\|_{\ell^{\beta_1}}$$

for all $k \in \mathbb{Z}$, then

$$\left\| \sum_j \lambda_j |g_j|^2 \right\|_{L_t^{\frac{q}{2}, \infty} L_x^{\frac{r}{2}}} \lesssim \|\lambda\|_{\ell^{\beta, 1}},$$

where $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\frac{1}{\beta} = \frac{1-\theta}{\beta_0} + \frac{\theta}{\beta_1}$ and $\theta = \frac{\varepsilon_0}{\varepsilon_0 + \varepsilon_1}$.

The above lemma was proved in [3, Proposition 2.2]. Let us give a sketch of the argument when $q_1 = \infty$ (since this case does not immediately follow from the argument in [3] as it stands).

Proof of Lemma 3.1 when $q_1 = \infty$. It is enough to show the desired inequality for $\lambda = \mathbf{1}_E$ (here, $\mathbf{1}_E(j) = 0$ if $j \in E$, and zero otherwise), so that we aim for

$$(3.1) \quad |J_\nu| \lesssim \left(\frac{(\#E)^{\frac{1}{\beta}}}{\nu} \right)^{\frac{q}{2}},$$

¹Note that (2.8) without the frequency localization is not permitted due to the failure of the associated Sobolev embedding.

where $\nu > 0$ and

$$J_\nu := \left\{ t \in \mathbb{R} : \left\| \sum_{j \in E} |g_j(t, \cdot)|^2 \right\|_{L_x^{\frac{r}{2}}} > \nu \right\}.$$

For a fixed $M \in \mathbb{Z}$ (chosen later), set

$$J_{\nu, M}^0 := \left\{ t \in \mathbb{R} : \left\| \sum_{j \in E} \left| \sum_{k \geq M+1} P_k g_j(t, \cdot) \right|^2 \right\|_{L_x^{\frac{r}{2}}} > \frac{\nu}{2} \right\}$$

and

$$J_{\nu, M}^1 := \left\{ t \in \mathbb{R} : \left\| \sum_{j \in E} \left| \sum_{k \leq M} P_k g_j(t, \cdot) \right|^2 \right\|_{L_x^{\frac{r}{2}}} > \frac{\nu}{2} \right\}.$$

Regarding $J_{\nu, M}^0$, let us observe that

$$\left\| \sum_{j \in E} \left| \sum_{k \geq M+1} P_k g_j(t, \cdot) \right|^2 \right\|_{L_t^{\frac{q_0}{2}}, \infty L_x^{\frac{r}{2}}} \lesssim 2^{-\varepsilon_0 M} (\#E)^{\frac{1}{\beta_0}}.$$

In fact, applying Minkowski's inequality twice gives

$$\left\| \sum_{j \in E} \left| \sum_{k \geq M+1} P_k g_j(t, \cdot) \right|^2 \right\|_{L_t^{\frac{q_0}{2}}, \infty L_x^{\frac{r}{2}}} \lesssim \left(\sum_{k \geq M+1} \left\| \sum_j |P_k g_j|^2 \right\|_{L_t^{\frac{q_0}{2}}, \infty L_x^{\frac{r}{2}}}^{\frac{1}{2}} \right)^2,$$

which is further bounded by (up to some constant) $2^{-\varepsilon_0 M} (\#E)^{\frac{1}{\beta_0}}$. Hence, one readily sees that

$$|J_{\nu, M}^0| \lesssim \nu^{-\frac{q_0}{2}} \left\| \sum_{j \in E} \left| \sum_{k \geq M+1} P_k g_j(t, \cdot) \right|^2 \right\|_{L_t^{\frac{q_0}{2}}, \infty L_x^{\frac{r}{2}}}^{\frac{q_0}{2}} \lesssim 2^{-\frac{\varepsilon_0 M q_0}{2}} \nu^{-\frac{q_0}{2}} (\#E)^{\frac{q_0}{2\beta_0}}.$$

A similar calculation reveals that there exists $C > 0$ such that

$$\left\| \sum_{j \in E} \left| \sum_{k \leq M} P_k g_j(t, \cdot) \right|^2 \right\|_{L_t^\infty L_x^{\frac{r}{2}}} \leq C 2^{\varepsilon_1 M} (\#E)^{\frac{1}{\beta_1}},$$

and now we select M to be the largest integer such that

$$C 2^{\varepsilon_1 M} (\#E)^{\frac{1}{\beta_1}} < \frac{\nu}{2}.$$

With this choice, we have $J_{\nu, M}^1 = \emptyset$ and $2^M \sim \nu^{\frac{1}{\varepsilon_1}} (\#E)^{-\frac{1}{\beta_1 \varepsilon_1}}$. Since $|J_\nu| \leq |J_{\nu, M}^0|$, we obtain (3.1) after a straightforward computation. $\square$

Proof of Theorem 1.4. Let $(f_j)_{j \in \mathbb{N}}$ be a family of orthonormal functions in $L^2(\mathbb{R}^d)$. First note that

$$(3.2) \quad \left\| \sum_j \lambda_j |U|D|^{-s} P_k f_j|^2 \right\|_{L_t^\infty L_x^\infty} \lesssim (2^k)^{d-2s} \|\lambda\|_{\ell^\infty}.$$

This is the estimate dual to (2.8) (which can easily be verified directly by an application of Bessel's inequality as in [3]).

First we consider the case $d \geq 4$ in which case it is known (see for example [9, Theorem 1]) that the classical Strichartz estimate (1.3) holds with f replaced by $P_0 f$. Thus, by rescaling,

$$(3.3) \quad \|U|D|^{-s} P_k f_j\|_{L_t^2 L_x^\infty} \lesssim (2^k)^{\frac{d-1}{2}-s}$$

holds for each f_j , and therefore the triangle inequality implies

$$(3.4) \quad \left\| \sum_j \lambda_j |U|D|^{-s} P_k f_j|^2 \right\|_{L_t^1 L_x^\infty} \lesssim (2^k)^{d-1-2s} \|\lambda\|_{\ell^1}.$$

Although (3.2) and (3.4) are valid for all $s \in \mathbb{R}$, if we fix $q \in (2, \infty)$ and apply these estimates with $s_q = \frac{d}{2} - \frac{1}{q}$, then Lemma 3.1 immediately gives the desired estimate

$$\left\| \sum_j \lambda_j |U|D|^{-s_q} f_j|^2 \right\|_{L_t^{\frac{q}{2}, \infty} L_x^\infty} \lesssim \|\lambda\|_{\ell^{\frac{q}{2}, 1}}.$$

The case when $d = 3$ requires a bit more work to show (1.8) for $2 < q < \infty$ since (3.3) with $d = 3$ is known to be false (see [19]). Thus, instead of (3.4), we aim for

$$(3.5) \quad \left\| \sum_j \lambda_j |U|D|^{-s} P_k f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^\infty} \lesssim (2^k)^{3-\frac{2}{q}-2s} \|\lambda\|_{\ell^{\frac{q}{2}}}$$

with q larger than but arbitrarily close to 2 (here the implicit constant blows up as $q \rightarrow 2$). It is clear that Lemma 3.1 combined with (3.2) and (3.5) yields (1.8).

The remainder of the argument is devoted to proving (3.5). First, by rescaling, we may assume $k = 0$. Using duality [11, Lemma 3], it suffices to show

$$(3.6) \quad \|WS\overline{W}\|_{\mathcal{C}^{2m}} \lesssim \|W\|_{L_t^{2m+1} L_x^2}^2,$$

where $S := UP_0 P_0^* U^*$ and m a sufficiently large natural number. The inspiration to consider dyadic exponents arose from arguments in [5] which rested on certain $\mathcal{C}^2$ and $\mathcal{C}^4$ estimates; see [5, Lemma 3.2] or the remarks at the end of [5, Section 3].

To prove (3.6), first recall that the $\mathcal{C}^2$ -norm can be expressed by the L^2 -norm of its integral kernel, and in addition, from the definition of singular values, it holds that $\|T\|_{\mathcal{C}^{2m+1}}^2 = \|T^* T\|_{\mathcal{C}^{2m}}$. Therefore, since S is self-adjoint, we have

$$\|WS\overline{W}\|_{\mathcal{C}^{2m}}^{2^{m-1}} = \|(WS\overline{W})^{2^{m-1}}\|_{\mathcal{C}^2}.$$

Hence our task is to get an L^2 bound for the integral kernel $\mathcal{K}_{2^{m-1}}$ of $(WS\overline{W})^{2^{m-1}}$. The notation for the variables in the forthcoming argument requires a little care and our choices will become clear as we proceed.

Let us write $\zeta_j = (t_j, x_j) \in \mathbb{R}^{1+3}$. Then

$$(W\overline{S\overline{W}})F(\zeta_1) = \int_{\mathbb{R}^4} F(\zeta_2) \mathcal{K}_1(\zeta_1, \zeta_2) d\zeta_2$$

with $\mathcal{K}_1(\zeta_1, \zeta_2) = W(\zeta_1)\overline{W}(\zeta_2)K(\zeta_1, \zeta_2)$ and

$$K(\zeta_1, \zeta_2) = \int_{\mathbb{R}^3} |\varphi(\xi)|^2 e^{i((x_1-x_2)\cdot\xi + (t_1-t_2)|\xi|)} d\xi.$$

To handle this, we shall invoke the following frequency-localized dispersive estimate (see e.g. [15]):

$$(3.7) \quad \sup_{\zeta_1, \zeta_2} |K(\zeta_1, \zeta_2)| \lesssim \langle t_1 - t_2 \rangle^{-1},$$

where $\langle t \rangle := 1 + |t|$.

For $m = 2$ we have

$$(W\overline{S\overline{W}})^2 G(\zeta_1) = \int_{\mathbb{R}^4} G(\zeta_3) \left(\int_{\mathbb{R}^4} W(\zeta_1) |W(\zeta_2)|^2 \overline{W}(\zeta_3) K(\zeta_1, \zeta_2) K(\zeta_2, \zeta_3) d\zeta_2 \right) d\zeta_3$$

and from this we have an expression for $\mathcal{K}_2(\zeta_1, \zeta_3)$. Inductively,

$$\begin{aligned} \mathcal{K}_{2^m}(\zeta_1, \zeta_{2^m+1}) &= \int_{\mathbb{R}^4} \mathcal{K}_{2^{m-1}}(\zeta_1, \zeta_{2^{m-1}+1}) \mathcal{K}_{2^{m-1}}(\zeta_{2^{m-1}+1}, \zeta_{2^m+1}) d\zeta_{2^{m-1}+1} \\ &= \int_{(\mathbb{R}^4)^{2^{m-1}}} W(\zeta_1) \prod_{j=2}^{2^m} |W(\zeta_j)|^2 \overline{W}(\zeta_{2^m+1}) \prod_{k=1}^{2^m} K(\zeta_k, \zeta_{k+1}) \prod_{\ell=2}^{2^m} d\zeta_\ell \end{aligned}$$

for all $m \geq 1$, and consequently

$$|\mathcal{K}_{2^m}(\zeta_1, \zeta_{2^m+1})| \lesssim |W(\zeta_1)| \left(\int_{\mathbb{R}^{2^m-1}} \prod_{j=2}^{2^m} h(t_j) \prod_{k=1}^{2^m} \langle t_k - t_{k+1} \rangle^{-1} \prod_{\ell=2}^{2^m} dt_\ell \right) |W(\zeta_{2^m+1})|,$$

where $h(t) := \|W(t, \cdot)\|_{L_x^2}^2$.

Now we are ready to estimate the relevant Schatten norm of $W\overline{S\overline{W}}$:

$$\begin{aligned} &\|W\overline{S\overline{W}}\|_{\mathcal{S}^{2^m}}^{2^m} \\ &= \int_{(\mathbb{R}^4)^2} |\mathcal{K}_{2^{m-1}}(\zeta_1, \zeta_{2^{m-1}+1})|^2 d\zeta_1 d\zeta_{2^{m-1}+1} \\ &\lesssim \int_{\mathbb{R}^2} h(t_1) \left(\int_{\mathbb{R}^{2^{m-1}-1}} \prod_{j=2}^{2^{m-1}} h(t_j) \prod_{k=1}^{2^{m-1}} \langle t_k - t_{k+1} \rangle^{-1} \prod_{\ell=2}^{2^{m-1}} dt_\ell \right)^2 h(t_{2^{m-1}+1}) dt_1 dt_{2^{m-1}+1} \end{aligned}$$

By expanding the square and an appropriate choice of labelling², we see that

$$(3.8) \quad \|W\overline{SW}\|_{\mathcal{C}^{2^m}}^2 \lesssim \int_{\mathbb{R}^{2^m}} \prod_{j=1}^{2^{m+1}} f_j(L_j \mathbf{t}) \, d\mathbf{t},$$

where $L_j : \mathbb{R}^{2^m} \ni \mathbf{t} \mapsto v_j \cdot \mathbf{t} \in \mathbb{R}$ with

$$v_j = \begin{cases} e_j & \text{if } j = 1, \dots, 2^m, \\ e_{j-2^m} - e_{j+1-2^m} & \text{if } j = 2^m + 1, \dots, 2^{m+1}, \end{cases}$$

under the relation $e_{2^m+1} = e_1$, and

$$f_j(t) = \begin{cases} h(t) & \text{if } j = 1, \dots, 2^m, \\ \langle t \rangle^{-1} & \text{if } j = 2^m + 1, \dots, 2^{m+1}. \end{cases}$$

One may notice that the term on the right-hand side of (3.8) has the structure of the left-hand side of the Brascamp–Lieb inequality and, in fact, we shall invoke the following characterization of finiteness of the Brascamp–Lieb constant (in the rank-one case) due to Barthe [1]. Let $n, M \in \mathbb{N}$ and for $I = \{1, \dots, M\}$ we denote by $\mathbf{1}_I$ the vector of $\mathbb{R}^M$ such that j -th element of $\mathbf{1}_I$ is 1 if $j \in I$ and 0 elsewhere.

Lemma 3.2 (see Proposition 3 in [1]). *There exists $C < \infty$ such that*

$$\int_{\mathbb{R}^n} \prod_{j=1}^M f_j(x \cdot v_j) \, dx \leq C \prod_{j=1}^M \|f_j\|_{L^{p_j}(\mathbb{R})}$$

holds for all non-negative f_j in $L^{p_j}(\mathbb{R})$ if and only if the following holds. The vector $\frac{1}{p} = (\frac{1}{p_1}, \dots, \frac{1}{p_M})$ belongs to the convex hull of $\mathbf{1}_I$ for subsets such that $(v_i)_{i \in I}$ forms a basis of $\mathbb{R}^n$.

We will use this lemma with $n = 2^m$ and $M = 2^{m+1}$. Let $I_0 = \{2^m + 1, 2^m + 2, \dots, 2^{m+1}\}$ so that $\mathbf{1}_{I_0} = (0, \dots, 0; 1, \dots, 1)$; i.e. the first half of elements are all 0 and the rests are all 1 (the separator “;” is positioned so that the components have been divided into exactly two equal-sized parts). For integers i, j with $1 \leq i \leq 2^m < j \leq 2^{m+1}$ the transposition operator swapping i -th element and j -th element, denoted by $[i, j]$, provides

$$[i, j]\mathbf{1}_{I_0} = (0, \dots, 0, 1, 0, \dots, 0; 1, \dots, 1, 0, 1, \dots, 1).$$

From the definition of $(v_j)_j$, if $1 \leq i \leq 2^m$, the vectors

$$(v_j)_{j \in (\{i\} \cup (I_0 \setminus \{2^m+i\}))}$$

²If we use $t_2, \dots, t_{2^m-1}, t_2^*, \dots, t_{2^m-1}^*$ for the variables arising from expanding the square, then the claimed formula follows by relabelling $t_j = t_{2^m+2-j}^*$ for $j = 2^{m-1} + 2, \dots, 2^m$.

form a basis of $\mathbb{R}^{2^m}$. In $\mathbb{R}^{2^{m+1}}$, considering 2^m points $[1, 2^m + 1]\mathbf{1}_I, \dots, [2^m, 2^{m+1}]\mathbf{1}_I$, one may discover that the middle point of the plane spanned by those vectors, namely,

$$(2^{-m}, \dots, 2^{-m}; 1 - 2^{-m}, \dots, 1 - 2^{-m})$$

lies on the boundary of the convex hull of characteristic vectors. Hence, if we set

$$(p_1^{-1}, \dots, p_{2^m}^{-1}; p_{2^m+1}^{-1}, \dots, p_{2^{m+1}}^{-1}) = (2^{-m}, \dots, 2^{-m}; 1 - 2^{-m}, \dots, 1 - 2^{-m}),$$

then Lemma 3.2 implies that

$$\int_{\mathbb{R}^{2^m}} \prod_{j=1}^{2^{m+1}} f_j(L_j \mathbf{t}) \, d\mathbf{t} \lesssim \prod_{j=1}^{2^{m+1}} \|f_j\|_{L_t^{p_j}} \lesssim \|W\|_{L_t^{2^{m+1}} L_x^2}^{2^{m+1}},$$

which ends the proof. $\square$

Appendix: Necessary conditions

We provide a justification of the necessary condition (1.5) for (1.1). The proof here is not new and may be found in [6, Proposition 9].

The necessity of $\beta \leq \frac{q}{2}$

The argument we give here is the same as in the proof of [6, Proposition 9]; since $r < \infty$ was a blanket assumption throughout [6], here we consider the only case $r = \infty$.

Consider the family $(f_j)_{j \geq 1}$ given by

$$f_j(x) = U g(-j, x)$$

where the function g satisfies

$$\widehat{g}(\xi) = c \frac{1_{[\pi, 3\pi]}(|\xi|)}{|\xi|^{s + \frac{d-1}{2}}}$$

and the constant $c > 0$ will be chosen shortly. Parseval's identity and changing to polar coordinates reveals

$$\langle f_j, f_k \rangle_{\dot{H}^s} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(j-k)|\xi|} |\widehat{g}(\xi)|^2 |\xi|^{2s} \, d\xi = \frac{c^2 |\mathbb{S}^{d-1}|}{(2\pi)^d} \int_{\pi}^{3\pi} e^{i(k-j)\rho} \, d\rho$$

and so $(f_j)_{j \geq 1}$ is an orthonormal family in $\dot{H}^s(\mathbb{R}^d)$ upon an appropriate choice of c .

Also, assuming $\lambda_j > 0$ for each j we have

$$\begin{aligned} \left\| \sum_j \lambda_j |U f_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^\infty}^{\frac{q}{2}} &\geq \int_{\mathbb{R}} \left(\sum_j \lambda_j |U g(t - j, 0)|^2 \right)^{\frac{q}{2}} \, dt \\ &\geq \sum_{n \geq 1} \lambda_n^{\frac{q}{2}} \int_n^{n+\varepsilon_0} |U g(t - n, 0)|^q \, dt \\ &= C \sum_{n \geq 1} \lambda_n^{\frac{q}{2}} \end{aligned}$$

where $\varepsilon_0 > 0$ is a sufficiently small constant and

$$C = \int_0^{\varepsilon_0} |Ug(t, 0)|^q dt = c^q \int_0^{\varepsilon_0} \left| \int_{|\xi| \in [\pi, 3\pi]} e^{it|\xi|} \frac{d\xi}{|\xi|^{s+\frac{d-1}{2}}} \right|^q dt.$$

The constant C is clearly finite, and a sufficiently small choice of ε_0 guarantees $|t|\xi| \leq \frac{1}{10}$ for $\xi \in [\pi, 3\pi]$ and thus

$$\begin{aligned} C &\geq c^q \int_0^{\varepsilon_0} \left| \int_{|\xi| \in [\pi, 3\pi]} \cos(t|\xi|) \frac{d\xi}{|\xi|^{s+\frac{d-1}{2}}} \right|^q dt \\ &\geq \left(\frac{c}{2}\right)^q \int_0^{\varepsilon_0} \left(\int_{|\xi| \in [\pi, 3\pi]} \frac{d\xi}{|\xi|^{s+\frac{d-1}{2}}} \right)^q dt > 0. \end{aligned}$$

Hence, if (1.1) holds we deduce $\|\lambda\|_{\frac{q}{2}} \lesssim \|\lambda\|_\beta$ and this implies $\beta \leq \frac{q}{2}$.

The necessity of $\beta \leq \beta_{\frac{q}{2}}(q, r)$

Let $B(x, r)$ be the ball in $\mathbb{R}^d$ centred at x with radius r . Consider the family $(f_j)_{j=1}^N$ given by

$$\widehat{f_j}(\xi) = 1_{B(0,1)}(R(\xi - v_j)).$$

Here, $v_1, \dots, v_N \in \mathbb{R}^d$ are fixed vectors satisfying $|v_j| \in [1, 2]$ and chosen so that the balls $\{B(v_j, \frac{1}{R})\}_{j=1}^N$ are disjoint. In particular, this means that the functions $(\widehat{f_j})_{j=1}^N$ have disjoint support and thus $(f_j)_{j=1}^N$ is an orthogonal family. Moreover, we choose a maximal collection of vectors $v_1, \dots, v_N$ under the above constraints, which means $N \sim R^d$.

Next we claim that $|Uf_j| \gtrsim R^{-d} 1_{B(0, c'R)}$ for an appropriately small choice of the constant c' . To see this, first assume that $v_j = \rho e_d$ for some $\rho \in [1, 2]$, and write

$$\begin{aligned} x \cdot \xi + t|\xi| &= \bar{x} \cdot \bar{\xi} + (x_d + t)(\xi_d - \rho) + t(|\xi| - \xi_d) + \rho(x_d + t) \\ &= \bar{x} \cdot \bar{\xi} + (x_d + t)(\xi_d - \rho) + t \frac{|\bar{\xi}|^2}{|\xi| + \xi_d} + \rho(x_d + t) =: \Phi_{t,x}(\xi) + \rho(x_d + t). \end{aligned}$$

Here, we use the notation $x = (\bar{x}, x_d)$. If c' is chosen sufficiently small, then for $(t, x) \in B(0, c'R)$ and ξ in the support of $\widehat{f_j}$ (in particular, $|\bar{\xi}| \leq \frac{1}{R}$, $|\xi_d - \rho| \leq \frac{1}{R}$) we get

$$|\Phi_{t,x}(\xi)| \leq \frac{1}{10}$$

and thus

$$|Uf_j(t, x)| = \left| \int_{B(\rho e_d, \frac{1}{R})} e^{i\Phi_{t,x}(\xi)} d\xi \right| \gtrsim \frac{1}{R^d}$$

as claimed. Taking $\lambda_j = \|f_j\|_{H^s}^2$ and $g_j = \|f_j\|_{H^s}^{-1} f_j$ for each $j = 1, \dots, N$, and recalling that $N \sim R^d$, we easily deduce that

$$\left\| \sum_{j=1}^N \lambda_j |Ug_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} = \left\| \sum_{j=1}^N |Uf_j|^2 \right\|_{L_t^{\frac{q}{2}} L_x^{\frac{r}{2}}} \gtrsim R^{\frac{2}{q} + \frac{2d}{r} - d}.$$

If we assume (1.1) then

$$R^{\frac{2}{q} + \frac{2d}{r} - d} \lesssim \left(\sum_{j=1}^N \|f_j\|_{\dot{H}^s}^{2\beta} \right)^{\frac{1}{\beta}} \lesssim R^{\frac{d}{\beta} - d}$$

since $\|f_j\|_{\dot{H}^s}^2 \sim R^{-d}$. Therefore $\frac{2}{q} + \frac{2d}{r} \leq \frac{d}{\beta}$ and this yields the necessary condition $\beta \leq \beta_{\frac{d}{2}}(q, r)$.

Acknowledgement. The first author would like to express his appreciation to Sanghyuk Lee and Shohei Nakamura for a number of extremely helpful discussions related to this work. The second author shows his deep gratitude to Yutaka Terasawa and Satoshi Masaki for the opportunity at the RIMS conference “Harmonic Analysis and Partial Differential Equations” in 2022. The third author also thanks Mitsuru Sugimoto for his continuous encouragement. The authors would like to thank the anonymous referee for valuable comments and suggestions.

References

- [1] Barthe, F., On a reverse form of the Brascamp-Lieb inequality, *Invent. Math.*, **134** (1998) 355–361.
- [2] Bergh, J., Löfström, J., Interpolation Spaces, Springer, Berlin, 1976.
- [3] Bez, N., Hong, Y., Lee, S., Nakamura, S., Sawano, Y., On the Strichartz estimates for orthonormal systems of initial data with regularity, *Adv. Math.*, **354** (2019) 106736, 37 pp.
- [4] Bez, N., Kinoshita, S., Shiraki, S., Boundary Strichartz estimates and pointwise convergence for orthonormal systems, *Trans. London Math. Soc.*, **11** (2024) e70002.
- [5] Bez, N., Lee, S., Nakamura, S., Maximal estimates for the Schrödinger equation with orthonormal initial data, *Selecta Math.*, **26** (2020) Article 52.
- [6] Bez, N., Lee, S., Nakamura, S., Strichartz estimates for orthonormal families of initial data and weighted oscillatory integral estimates, *Forum of Math. Sigma*, **9** (2021) e1.
- [7] Bourgain, J., Estimations de certaines fonctions maximales, *C. R. Acad. Sci. Paris Sér. I Math.*, **310** (1985) 499–502.
- [8] Carbery, A., Seeger, A., Wainger, S., Wright, J., Classes of singular integral operators along variable lines, *J. Geom. Anal.*, **9** (1999) 583–605.
- [9] Fang, D., Wang, C., Some remarks on homogeneous estimates for wave equation, *Nonlinear Anal.* **65** (2006) 697–706.
- [10] Frank, R., Lewin, M., Lieb, E. H., Seiringer, R., Strichartz inequality for orthonormal functions, *J. Eur. Math. Soc.*, **16** (2014) 1507–1526.
- [11] Frank, R., Sabin, J., Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform Sobolev estimates, *Amer. J. Math.*, **139** (2017) 1649–1691.
- [12] Frank, R., Sabin, J., The Stein–Tomas inequality in trace ideals, Séminaire Laurent Schwartz – EPD et applications (2015–2016), Exp. No. XV, 12 pp., 2016.

- [13] Grafakos, L., Classical Fourier analysis, third edition, Graduate Texts in Mathematics, vol. 249, Springer, New York, (2014).
- [14] Guo, Z., Li, J., Nakanishi, K., Yan, L., On the boundary Strichartz estimates for wave and Schrödinger equations, *J. Differential Equations*, **265** (2018) 5656–5675.
- [15] Guo, Z., Peng, L., Wang, B., Decay estimates for a class of wave equations, *J. Funct. Anal.*, **254** (2008) 1642–1660.
- [16] Keel, M., Tao, T., Endpoint Strichartz estimates, *Amer. J. Math.*, **120** (1998) 955–980.
- [17] Lee, S., Some sharp bounds for the cone multiplier of negative order in $\mathbb{R}^3$, *Bull. London Math. Soc.*, **35** (2003) 373–390.
- [18] Lieb, E. H., An L^p bound for the Riesz and Bessel potentials of orthonormal functions, *J. Funct. Anal.*, **51** (1983) 159–165.
- [19] Montgomery-Smith, S. J., Time decay for the bounded mean oscillation of solutions of the Schrödinger and wave equations, *Duke Math. J.*, **91** (1998) 393–408.
- [20] O’Neil, R., Convolution operators and $L(p, q)$ spaces, *Duke Math. J.*, **30** (1963) 129–142.
- [21] Simon, B., Trace ideals and their applications, Vol. 35 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1979.
- [22] Stein, E. M., Weiss, G., Introduction to Fourier Analysis on Euclidean Spaces, Princeton Mathematical Series, No. 32, Princeton University Press, (1971).