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# Instanton-type solutions with free $(m+1)$-parameters for the $m$-th member of the first Painlevé hierarchy

By

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**Abstract**

A construction of instanton-type solutions with holomorphic functions as coefficients is discussed for the $m$-th member of the first Painlevé hierarchy with a large parameter. The solutions constructed here contain free $(m+1)$-parameters.

## § 1. Introduction

The first Painlevé hierarchy with a large parameter $\eta$ is a family of systems of non-linear equations whose first member is the traditional first Painlevé equation with $\eta$. For $m = 1, 2, \ldots,$ the $m$-th member $(P_{\mathrm{I}})_m$ of the hierarchy given in [9] consists of $2m$-differential equations with unknown functions $u_j$ and $v_j$ of $t$:

$$(1.1)\qquad \begin{cases} \eta^{-1}\dfrac{du_j}{dt} = 2v_j, & j = 1, 2, \ldots, m, \\[2ex] \eta^{-1}\dfrac{dv_j}{dt} = 2(u_{j+1} + u_1 u_j + w_j), & j = 1, 2, \ldots, m, \end{cases}$$

where $w_j$ is defined recursively by

$$(1.2)\qquad w_j := \frac{1}{2}\sum_{k=1}^{j} u_k u_{j+1-k} + \sum_{k=1}^{j-1} u_k w_{j-k} - \frac{1}{2}\sum_{k=1}^{j-1} v_k v_{j-k} + c_j + \delta_{jm} t.$$

Here $c_j$ is a constant and $\delta_{jm}$ stands for the Kronecker's delta and $u_{m+1}$ is assumed to be zero.

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In the paper [3], T. Aoki, N. Honda and the author have rewritten $(P_{\mathrm{I}})_m$ itself in the form

$$\eta^{-1}\frac{d}{dt}\begin{pmatrix} U\theta \\ V\theta \end{pmatrix} \equiv \begin{pmatrix} 2V\theta \\ -(1+2u_1\theta)(1-U)+\dfrac{1+2C-\theta V^2}{1-U} \end{pmatrix} \tag{1.3}$$

with generating functions defined by

$$U(\theta) := \sum_{k=1}^{\infty} u_k\theta^k,\ \ V(\theta) := \sum_{k=1}^{\infty} v_k\theta^k,\ \ C(\theta) := \sum_{k=1}^{\infty}(c_k+\delta_{km}t)\theta^{k+1}, \tag{1.4}$$

where $\theta$ denotes an independent variable and by $A \equiv B$ we mean that $A-B$ is zero modulo $\theta^{m+2}$. Note that, with the condition that the coefficients of $\theta^{m+1}$ of $U$ and $V$ are zero, instanton-type solutions for (1.3) give ones for (1.1). Hence it suffices to construct instanton-type solutions for (1.3) by multiple-scale analysis.

Now we recall the solution space for (1.3) constructed in [3]. Let $\alpha := -\dfrac{1}{2}$ and $\tau := (\tau_1, \ldots, \tau_m)$ be $m$-independent variables. We denote by $\Omega$ an open subset in $\mathbb{C}_t$ satisfying some conditions (see Section 2) and by $\mathcal{M}(\Omega)[[\theta]]$ (resp. $\mathcal{O}(\Omega)[[\theta]]$) the set of formal power series in $\theta$ with coefficients in multi-valued holomorphic functions with a finite number of branching points and poles (resp. holomorphic functions) on $\Omega$. Then we define the rings

$$\begin{aligned} \mathcal{A}_\alpha(\Omega) &:= (\mathcal{M}(\Omega)[[\theta]])\left[\left[\eta^\alpha e^{\tau_1}, \ldots, \eta^\alpha e^{\tau_m}, \eta^\alpha e^{-\tau_1}, \ldots, \eta^\alpha e^{-\tau_m}\right]\right], \\ \mathcal{A}^{\mathcal{O}}_\alpha(\Omega) &:= (\mathcal{O}(\Omega)[[\theta]])\left[\left[\eta^\alpha e^{\tau_1}, \ldots, \eta^\alpha e^{\tau_m}, \eta^\alpha e^{-\tau_1}, \ldots, \eta^\alpha e^{-\tau_m}\right]\right]. \end{aligned} \tag{1.5}$$

We also define $\hat{\mathcal{A}}_\alpha(\Omega)$ (resp. $\hat{\mathcal{A}}^{\mathcal{O}}_\alpha(\Omega)$) by the subset in $\mathcal{A}_\alpha(\Omega)$ (resp. $\mathcal{A}^{\mathcal{O}}_\alpha(\Omega)$) consisting of a formal power series of order less than or equal to $\alpha$ with respect to $\eta$.

To obtain an instanton-type solution of $(P_{\mathrm{I}})_m$, we computed the system of partial differential equations in $\hat{\mathcal{A}}^2_\alpha(\Omega) := (\hat{\mathcal{A}}_\alpha(\Omega))^2$ associated with (1.3) and constructed its solution $(u, v) \in \hat{\mathcal{A}}^2_\alpha(\Omega)$ with free $2m$-parameters in [3]. In this article, taking parameters suitably, we prove that the solution $(u, v)$ with free $(m+1)$-parameters can be constructed in $(\hat{\mathcal{A}}^{\mathcal{O}}_\alpha(D))^2$ where $D \subset \mathbb{C}_t$ is a specific region described in Section 3.

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## § 2. Preparations

In this section, we briefly review some results in [3] which are needed later. For any $x \in \hat{\mathcal{A}}_\alpha(\Omega)$, we define $\sigma^\theta_i(x)$ (resp. $\sigma^\eta_{j\alpha}(x)$) by the coefficient of $\theta^i$ (resp. $\eta^{j\alpha}$) in $x$

$(i,\, j \geq 1)$. We consider the linearized equation of (1.3) along $(\hat{u}_0,\, \hat{v}_0)$ given by

$$\hat{u}_0 = 1 - \sqrt{\frac{1+2C}{1+2\hat{u}_{1,\,0}\theta}}, \qquad \hat{v}_0 = 0. \tag{2.1}$$

Here $\hat{u}_{1,\,0}$ is taken so that the coefficient of $\theta^{m+1}$ in $\hat{u}_0$ is zero. We define $(u,\, v) \in \hat{\mathcal{A}}^2_\alpha(\Omega)$ by

$$u := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} u_{i,\,j\alpha}(t)\,\theta^i\,\eta^{j\alpha} \quad \text{and} \quad v := \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v_{i,\,j\alpha}(t)\,\theta^i\,\eta^{j\alpha}, \tag{2.2}$$

where $u_{i,\,j\alpha}$ and $v_{i,\,j\alpha}$ $(i,\, j \geq 1)$ denote unknown functions of the variable $t$. Then (1.3) is transformed by a change of $(U,\, V) = \big(\hat{u}_0 + (1-\hat{u}_0)u,\, \hat{v}_0 + (1-\hat{u}_0)v\big)$ into the system of non-linear equations for $(u,\, v)$:

$$\begin{aligned}\left(\eta^{-1}\frac{d}{dt} - Q\right)\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} \equiv & \left(\begin{pmatrix} \eta^{-1}\rho\theta \\ S(u,\, v) \end{pmatrix} - uQ\begin{pmatrix} u\theta \\ v\theta \end{pmatrix}\right) \\ & - \left(u\begin{pmatrix} \eta^{-1}\rho \\ 2\sigma_1^\theta(u)u \end{pmatrix} + \eta^{-1}\rho\begin{pmatrix} u \\ v \end{pmatrix}\right)\theta \\ & + \eta^{-1}u\left(\rho + \frac{d}{dt}\right)\begin{pmatrix} u \\ v \end{pmatrix}\theta \end{aligned} \tag{2.3}$$

with

$$S(u,\, v) := \frac{1}{2}(-v,\, u)Q\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} + 3\sigma_1^\theta(u)u\theta \;\text{ and }\; \rho := \frac{d}{dt}(\log(1-\hat{u}_0)). \tag{2.4}$$

Here the map $Q : (\Theta\theta)^2 \longrightarrow \Theta^2$ is defined by

$$Q\begin{pmatrix} x\,\theta \\ y\,\theta \end{pmatrix} := 2\begin{pmatrix} y\theta \\ (1+2\hat{u}_{1,0}\theta)\,x - \sigma_1^\theta(x)\theta \end{pmatrix} \tag{2.5}$$

for any $x = \sum_{i=1}^{\infty} x_i\theta^i$ and $y = \sum_{i=1}^{\infty} y_i\theta^i$ in $\Theta$, where $\Theta$ is the set of formal power series of $\theta$ without constant terms.

As the principal parts of (2.3) are expressed by the map $Q$, we construct the solution $(u,\, v)$ so that it is a linear combination of eigenvector $A(\lambda)$'s of $Q$. Here $A(\lambda)$ is said to be the eigenvector corresponding to an eigenvalue $\lambda$ of $Q$ if $A(\lambda)$ satisfies $Q(A(\lambda)\theta) = \lambda A(\lambda)\theta$. We can see that the eigenvalue $\lambda$ of $Q$ is a root of the algebraic equation

$$\Lambda(\lambda,\, t) := g(\lambda)^m - \sum_{k=1}^{m} \hat{u}_{k,\,0}\, g(\lambda)^{m-k} = 0, \quad g(\lambda) := \frac{\lambda^2 - 8\hat{u}_{1,\,0}}{4}, \tag{2.6}$$

where $\hat{u}_{k,\,0}$ denote the coefficient of $\theta^k$ in $\hat{u}_0$ given by (2.1). Note that $\Lambda(\lambda,\,t)$ is an even function of $\lambda$. Let $\nu_{\pm 1}(t),\,\ldots,\,\nu_{\pm m}(t)$ be the roots of the algebraic equation (2.6) of $\lambda$ with convention $\nu_k = -\nu_{-k}$ $(1 \le k \le m)$. We denote by $E$ the set of turning points of $(P_{\mathrm{I}})_m$, i.e. the zero set of the discriminant of (2.6). Let $\Omega$ be an open subset in $\mathbb{C} \setminus E$ and we consider our problem on $\Omega$. For any $\psi(\tau_1, \ldots, \tau_m,\, t,\, \theta,\, \eta) \in \hat{\mathcal{A}}_\alpha(\Omega)$, we define the morphism $\iota$ by

$$\iota(\psi) = \psi\left(\eta \int^t \nu_1(s)ds,\, \ldots,\, \eta \int^t \nu_m(s)ds,\, t,\, \theta,\, \eta\right) \tag{2.7}$$

and the operator $P$ is defined by

$$P := \nu_1 \frac{\partial}{\partial \tau_1} + \cdots + \nu_m \frac{\partial}{\partial \tau_m} - Q. \tag{2.8}$$

Then we obtain the partial differential equation associated with (2.3) given by [3]:

$$\begin{aligned} P\begin{pmatrix} u\theta \\ v\theta \end{pmatrix} \equiv & \left(\begin{pmatrix} \eta^{-1}\rho\theta \\ S(u,\,v) \end{pmatrix} + u\, P\begin{pmatrix} u\theta \\ v\theta \end{pmatrix}\right) \\ & - \left(u\begin{pmatrix} \eta^{-1}\rho \\ 2\sigma_1^\theta(u)u \end{pmatrix} + \eta^{-1}\left(\rho + \frac{\partial}{\partial t}\right)\begin{pmatrix} u \\ v \end{pmatrix}\right)\theta \\ & + \eta^{-1}u\left(\rho + \frac{\partial}{\partial t}\right)\begin{pmatrix} u \\ v \end{pmatrix}\theta. \end{aligned} \tag{2.9}$$

Here $S(u,\,v)$ and $\rho$ have been given by (2.4). Let us recall the definition of instanton-type solutions for $(P_{\mathrm{I}})_m$.

**Definition 2.1** ([3]). A formal solution $(U,\,V)$ on $\Omega$ of (1.3) is called of instanton-type if $(U,\,V)$ has the form $(\hat{u}_0,\,\hat{v}_0) + (1 - \hat{u}_0)(\iota(u),\,\iota(v))$ for which $(u,\,v) \in \hat{\mathcal{A}}_\alpha^2(\Omega)$ is a solution of (2.9).

The main theorem in [3] is as follows.

**Theorem 2.1** ([3, Theorem 5.3]). *Let $\Omega$ be an open subset in $\mathbb{C} \setminus E$. Then we have instanton-type solutions for $(P_{\mathrm{I}})_m$ with free $2m$-parameters $(\beta_{-m},\,\ldots,\,\beta_m) \in \mathbb{C}^{2m}[[\eta^{-1}]]$. Especially, we can construct the solution $(u,\,v)$ in $\hat{\mathcal{A}}_\alpha^2(\Omega)$ for (2.9) of the form*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{1 \le |k| \le m} f_k(\tau,\,t;\eta)A(\nu_k) \tag{2.10}$$

*with*

$$A(\nu_k) := \begin{pmatrix} a(\nu_k) \\ \dfrac{\nu_k}{2}\, a(\nu_k) \end{pmatrix}, \quad a(\nu_k) := \frac{\theta}{1 - g(\nu_k)\theta} = \sum_{j=0}^\infty g(\nu_k)^j \theta^{j+1} \tag{2.11}$$

*and*

$$f_k(\tau,\, t;\eta) = \sum_{j=1}^{\infty}\left(\sum_{\ell\geq 0,\, p\in\mathbb{Z}^m,\, 2\ell+|p|=j} f_{k,p,\ell}(t)e^{p\cdot\tau}\right)\eta^{-\frac{j}{2}}. \tag{2.12}$$

*Here $g(\nu_k)$ has been defined by* (2.6).

The following lemma shows the more explicit form of the leading term $f_{k,\,\frac{1}{2}}$ of $f_k$ in (2.10) with respect to $\eta$.

**Lemma 2.1** ([3, Lemma 4.1 and Proposition 4.10]). *We have*

$$f_{k,\,\frac{1}{2}} = \omega_k e^{\tau_k} \quad (1\leq |k|\leq m), \tag{2.13}$$

*where $\omega_k$, $\omega_{-k}$ $(1\leq k\leq m)$ are multi-valued holomorphic functions on $\Omega$ in the form*

$$\begin{aligned}\omega_k &= \beta_k^{(1)}\exp\left(\int^t\left(\frac{1}{\nu_k}\sum_{j=1}^{m}\varphi(k,\,j)\beta_j^{(1)}\beta_{-j}^{(1)}\exp\left(-2\int^t h_j dt\right) - h_k\right)dt\right),\\ \omega_{-k} &= \beta_{-k}^{(1)}\exp\left(\int^t\left(-\frac{1}{\nu_k}\sum_{j=1}^{m}\varphi(k,\,j)\beta_j^{(1)}\beta_{-j}^{(1)}\exp\left(-2\int^t h_j dt\right) - h_k\right)dt\right)\end{aligned} \tag{2.14}$$

*with free $2m$-parameters $(\beta_{-m}^{(1)},\,\ldots,\,\beta_m^{(1)})\in\mathbb{C}^{2m}$. Here $\varphi(k,\,j)$ are rational functions of the variables $\nu_\ell$ $(1\leq \ell\leq m)$ and $h_k$ are multi-valued holomorphic functions of finite determination in $\Omega$ with the conditions*

$$\varphi(k,\,j) = \varphi(-k,\,j) \quad (1\leq j\leq m), \quad h_k = h_{-k}. \tag{2.15}$$

The strict forms of $\varphi(k,\,j)$ and $h_k$ are also given in [3].

## § 3. Existence of instanton-type solutions with holomorphic functions as coefficients

In this section, we prove that we have a solution $(u,\,v)\in(\hat{\mathcal{A}}_\alpha^{\mathcal{O}}(D))^2$ containing free $(m+1)$-parameters for (2.9), where $D(\subset\mathbb{C}_t)$ is a specific region described below. In what follows we use the same notations as those in Section 2. For any $1\leq j\leq m$, we define $D_j$ by

$$D_j := \bigcap_{\substack{i=1,\\ i\neq j}}^{m} D_{j,\,i} \ \text{ with } \ D_{j,\,i} := \{t\in\mathbb{C};\nu_i(t)\neq k\nu_j(t) \text{ for any } k\in\mathbb{R}\setminus\{0\}\}. \tag{3.1}$$

From now on, we consider the case of $j=1$. For any $t\in D_1\backslash E$, the line $L:=\{k\nu_1(t);k\in\mathbb{R}\}$ divides the complex plane $\mathbb{C}$ into two half-planes. Noticing the relation $\nu_{-k}=-\nu_k$

$(1 \leq k \leq m)$, we see that each half-plane contains $m-1$ eigenvalues. We may assume that eigenvalues contained in the same half-plane are $(\nu_2, \ldots, \nu_m)$ and $(\nu_{-2}, \ldots, \nu_{-m})$ respectively. Then, putting $(\beta^{(1)}_{-m}, \ldots, \beta^{(1)}_{-2}) = (0, \ldots, 0)$ in (2.14), we have the leading term $(\sigma^\eta_\alpha(u), \sigma^\eta_\alpha(v))$ of $(u, v)$ with $m+1$ free parameters in the form

$$A(\nu_1)\omega_1 e^{\tau_1} + A(\nu_{-1})\omega_{-1}e^{-\tau_1} + \sum_{k=2}^{m} A(\nu_k)\omega_k e^{\tau_k}. \tag{3.2}$$

Here $\omega_k$ has been defined by (2.14). Generally, putting $(\beta_{-m}, \ldots, \beta_{-2}) = (0, \ldots, 0)$ in $(\beta_{-m}, \ldots, \beta_m)$ constructed in Theorem 2.1, we can construct the solution $(u, v)$ with free $(m+1)$-parameters in $\mathbb{C}^{m+1}[[\eta^{-1}]]$ for (2.9).

Next, let us specify a domain on which the solution $(u, v)$ with holomorphic functions as coefficients is defined. By the definition of the operator $P$, for any $1 \leq i \leq m$ and $(q_1, q_2, \ldots, q_m) \in (\mathbb{Z} \times \mathbb{Z}^{m-1}_{\geq 0})$, we see

$$\begin{aligned} &P(A(\nu_i)e^{q_1\tau_1 + q_2\tau_2 + \cdots + q_m\tau_m}) \\ &\qquad = ((q_1\nu_1 + q_2\nu_2 + \cdots + q_m\nu_m) - \nu_i)\, A(\nu_i)e^{q_1\tau_1 + q_2\tau_2 + \cdots + q_m\tau_m}. \end{aligned} \tag{3.3}$$

Therefore it suffices to take a domain so that $((q_1\nu_1 + q_2\nu_2 + \cdots + q_m\nu_m) - \nu_i)$ is never zero for any $(q_1, q_2, \ldots, q_m) \in (\mathbb{Z} \times \mathbb{Z}^{m-1}_{\geq 0})$ except for $q_i = 1$ and $q_k = 0$ $(k \neq i)$.

Let us denote by $H$ one of half planes divided by the line $L$. Then, as $\nu_i$'s $(2 \leq i \leq m)$ belong to the same half-plane $H$, we have $q_2\nu_2 + \cdots + q_m\nu_m \in H$ for any $(q_2, \cdots, q_m) \in \mathbb{Z}^{m-1}_{\geq 0} \setminus \{0\}$. Hence $q_1\nu_1 + q_2\nu_2 + \cdots + q_m\nu_m$ is never zero for any $(q_1, q_2, \ldots, q_m) \in (\mathbb{Z} \times \mathbb{Z}^{m-1}_{\geq 0}) \setminus \{0\}$. By these observations, we see that the zero set of $((q_1\nu_1 + q_2\nu_2 + \cdots + q_m\nu_m) - \nu_i)$ in (3.3) is contained in the union of subsets defined by the following equations.

$$q_1\nu_1 + q_2\nu_2 + \cdots + q_{i-1}\nu_{i-1} + q_{i+1}\nu_{i+1} + \cdots + q_m\nu_m = \nu_i,\ 2 \leq i \leq m \tag{3.4}$$

with convention $\nu_{m+1} := 0$. Let $K_1$ be a compact subset in $D_1 \setminus E$ and $\widehat{K}_1$ is defined by

$$\widehat{K}_1 := \bigcup_{i=2}^{m} \bigcup_{q} \{t \in K_1; q_1\nu_1(t) + \cdots + q_{i-1}\nu_{i-1}(t) + q_{i+1}\nu_{i+1}(t) + \cdots + q_m\nu_m(t) = \nu_i(t)\}. \tag{3.5}$$

Here $q$ runs through $(q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_m) \in \mathbb{Z} \times \mathbb{Z}^{m-2}_{\geq 0}$. Let $\Phi$ be the projective map from the half-plane $H$ to $\widehat{L} := \{k\sqrt{-1}\nu_1(t); k \in \mathbb{R}\} \cap H$ for any $t \in K_1$. We set $M$ and $m$ by

$$M = \max\{\max_{t \in K_1}|\Phi(\nu_i(t))|\}_{i=2}^{m} \quad \text{and} \quad m = \min\{\min_{t \in K_1}|\Phi(\nu_i(t))|\}_{i=2}^{m},$$

respectively. If $t \in \widehat{K}_1$, we have $\displaystyle\sum_{\substack{j=2,\\ j\neq i}}^{m} |\Phi(q_j\nu_j(t))| = |\Phi(\nu_i(t))|$ for some $2 \le i \le m$. Hence $m(q_2 + \cdots + q_m) \le M$. Therefore the second union with respect to $q$ of (3.5) is finite. As, for $t \in K_1 \setminus \widehat{K}_1$, the $((q_1\nu_1 + q_2\nu_2 + \cdots + q_m\nu_m) - \nu_i)$ in (3.3) never become zero, all coefficients in $f_k$ of (2.10) are holomorphic on a connected component of $K_1 \setminus \widehat{K}_1$. Note that, by the same arguments, we have the similar result as above when we put $(\beta_2, \ldots, \beta_m) = (0, \ldots, 0)$ instead of $(\beta_{-m}, \ldots, \beta_{-2}) = (0, \ldots, 0)$. Summing up, we have the theorem below.

For any compact subset $K_j$ in $D_j \setminus E$ $(1 \le j \le m)$, we set

(3.6)
$$\widehat{K}_j := \bigcup_{\substack{i=1,\\ i\neq j}}^{m} \bigcup_{q} \{t \in K_j; q_1\nu_1(t) + \cdots + q_{i-1}\nu_{i-1}(t) + q_{i+1}\nu_{i+1}(t) + \cdots + q_m\nu_m(t) = \nu_i(t)\},$$

where the $q_j$ runs through $\mathbb{Z}$ and the other $q_k$ $(k \neq j)$ runs through $\mathbb{Z}_{\ge 0}$ and $\nu_{m+1} := 0$.

**Theorem 3.1.** *For any $1 \le j \le m$, we have instanton-type solutions of $(P_{\mathrm{I}})_m$ which are defined on $\Omega_j := K_j \setminus \widehat{K}_j$ with free $(m+1)$-parameters in $\mathbb{C}^{m+1}[[\eta^{-1}]]$. Especially, we can construct the solution $(u, v)$ in $(\hat{\mathcal{A}}^{\mathcal{O}}_{\alpha}(\Omega_j))^2$ for (2.9) of the form (2.10).*

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