数理解析研究所講究録
第 1717 巻 2010 年 55-68

Production of Some Fractional Differintegral Equations in N- Fractional Calculus

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Keywords ; Fractional Calculus, N- Fractional Calculus Operator, Special Differential Equation

Abstract

In this article homogeneous fractional differintegral equations

1) $$\varphi_\gamma - \varphi \cdot a^\gamma \left(1 + \frac{\gamma}{a(z-b)}\right) = 0, \quad (a(z-b) \neq 0),$$

2) $$\varphi_{\gamma+2} - \varphi_{\gamma+1} \cdot a - \varphi_\gamma \cdot \left(\frac{a^2}{a(z-b)+\gamma}\right) = 0, \quad (a(z-b)+\gamma \neq 0),$$

and nonhomogeneous ones

3) $$\varphi_{\gamma+1} - \varphi_\gamma \cdot \frac{\gamma+1}{z-b} = (\cos z)_\gamma \left((z-b) + \frac{\gamma^2+\gamma}{z-b}\right), \quad ((z-b) \neq 0),$$

and

4) $$\varphi_{\gamma+2} - \varphi_{\gamma+1} \cdot \frac{\gamma+2}{z-b} + \varphi_\gamma \cdot \frac{(\gamma+1)(\gamma+2)}{(z-b)^2}$$

$$= -(\sin z)_\gamma (z-b) - (\cos z)_\gamma \cdot \frac{\gamma(\gamma+1)(\gamma+2)}{(z-b)^2}, \quad ((z-b) \neq 0),$$

are discussed in the field of N- fractional calculus; where

$$\varphi \in \boldsymbol{F} = \{ \varphi ; 0 \neq |\varphi_\gamma| < \infty, \gamma \in \boldsymbol{R} \}, \quad (\varphi = \varphi(z)).$$

Particular solutions are given by

$$\varphi = e^{az}(z-b)$$

to the equations 1) and 2), and

$$\varphi = (\sin z)(z-b)$$

to the equations 3) and 4), respectively, without the consideration of the arbitrary constants for integrations.

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+ .

(Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in $D (z \in D)$,

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\, d\zeta \qquad (\nu \notin \mathbf{Z}^-), \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \qquad (m \in \mathbf{Z}^+), \tag{2}$$

where $\quad -\pi \le \arg(\zeta - z) \le \pi$ for C_- , $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+ ,

$\zeta \ne z$, $z \in \mathbf{C}$, $\nu \in \mathbf{R}$, Γ ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^ν be*

$$N^\nu = \left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{d\zeta}{(\zeta - z)^{\nu+1}} \right) \qquad (\nu \notin \mathbf{Z}^-), \quad [\,\textit{Refer to}\,(1)\,] \tag{3}$$

with

$$N^{-m} = \lim_{\nu \to -m} N^\nu \qquad (m \in \mathbf{Z}^+), \tag{4}$$

and define the binary operation $\circ$ *as*

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \qquad (\alpha, \beta \in \mathbf{R}), \tag{5}$$

then the set

$$\{N^\nu\} = \{N^\nu \mid \nu \in \mathbf{R}\} \tag{6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f;\ 0 \ne |f_\nu| < \infty, \nu \in \mathbf{R}\}$, where $f = f(z)$ and $z \in \mathbf{C}$.

(vis. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α .)

Theorem B. " F.O.G. $\{N^\nu\}$ " *is an* " *Action product group which has continuous index* ν " *for the set of* F . (*F.O.G.* ; Fractional calculus operator group) [3]

Theorem C. *Let*

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} \quad (\nu \in \boldsymbol{R}). \tag{7}$$

Then the set S *is a commutative ring for the function* $f \in F$, *when the identity*

$$N^\alpha + N^\beta = N^\gamma \quad (N^\alpha, N^\beta, N^\gamma \in S) \tag{8}$$

holds. [5]

(III) Lemma. We have [1]

(i) $$((z-c)^b)_\alpha = e^{-i\pi\alpha}\frac{\Gamma(\alpha-b)}{\Gamma(-b)}(z-c)^{b-\alpha} \quad \left(\left|\frac{\Gamma(\alpha-b)}{\Gamma(-b)}\right| < \infty\right),$$

(ii) $$(\log(z-c))_\alpha = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c) \quad (|\Gamma(\alpha)| < \infty),$$

where $z-c \neq 0$ for (i) and $z-c \neq 0, 1$ for (ii), (iii) ,

(iv) $$(u \cdot v)_\alpha := \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad \begin{pmatrix} u = u(z), \\ v = v(z) \end{pmatrix}.$$

§ 1. Production of Fractional Differintegral Equations

Theorem 1. *Let*

$$\varphi = \varphi(z) = e^{az}(z-b) \quad (a(z-b) \neq 0). \tag{1}$$

We have then the following homogeneous fractional differintegral equations;

(i) $$\varphi_\gamma - \varphi \cdot a^\gamma \left(1 + \frac{\gamma}{a(z-b)}\right) = 0, \quad (a(z-b) \neq 0), \tag{2}$$

$$\begin{pmatrix} \textit{Fractional differential equation for } \gamma > 0, \\ \textit{Fractional integral equation for } \gamma < 0. \end{pmatrix}$$

and

(ii) $$\varphi_{\gamma+2} - \varphi_{\gamma+1} \cdot a - \varphi_\gamma \cdot \left(\frac{a^2}{a(z-b)+\gamma}\right) = 0, \quad (a(z-b)+\gamma \neq 0), \tag{3}$$

$$\begin{pmatrix} \textit{Fractional differential equation for } \gamma > 0, \\ \textit{Fractional integral equation for } \gamma < -2, \\ \textit{Fractional differintegral equation for } -2 < \gamma < 0. \end{pmatrix}$$

applying N- fractional calculus, for arbitrary γ .

Proof of (i). Operate N- fractional calculus operator N^{γ} to the both sides of (1), we have then

$$N^{\gamma}\varphi = N^{\gamma}(e^{az}(z-b)), \tag{4}$$

that is,

$$\varphi_{\gamma} = (e^{az}(z-b))_{\gamma} = \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)}{k!\,\Gamma(\gamma+1-k)} (e^{az})_{\gamma-k}(z-b)_{k}. \tag{5}$$

$$= (e^{az})_{\gamma}(z-b) + \gamma(e^{az})_{\gamma-1} \tag{6}$$

$$= a^{\gamma}e^{az}(z-b) + \gamma a^{\gamma-1}e^{az}, \tag{7}$$

by Lemma (i v).

Therefore, we have

$$\varphi_{\gamma} - \varphi \cdot \left(a^{\gamma} + \frac{\gamma a^{\gamma-1}}{z-b} \right) = 0 \tag{8}$$

from (7) and (1).

We have then (2) from (8) clearly, for arbitrary γ.

Proof of (i i). We have

$$\varphi_{\gamma} = a^{\gamma}e^{az}\left(z - b + \frac{\gamma}{a} \right) \tag{9}$$

from (7), hence

$$\varphi_{\gamma+1} = a^{\gamma+1}e^{az}\left(z - b + \frac{\gamma+1}{a} \right) \tag{10}$$

and

$$\varphi_{\gamma+2} = a^{\gamma+2}e^{az}\left(z - b + \frac{\gamma+2}{a} \right). \tag{11}$$

Therefore, applying (9), (10) and (11), we obtain

$$\text{LHS of (3)} = a^{\gamma+2}e^{az}\left(z - b + \frac{\gamma+2}{a} \right) - a^{\gamma+2}e^{az}\left(z - b + \frac{\gamma+1}{a} \right)$$

$$- \frac{a^{2}}{a(z-b)+\gamma} \cdot a^{\gamma}e^{az}\left(z - b + \frac{\gamma}{a} \right) \tag{12}$$

$$= a^{\gamma+2}e^{az}\frac{1}{a} - \frac{a^{\gamma+2}}{a(z-b)+\gamma} e^{az}\left(\frac{a(z-b)+\gamma}{a} \right) = 0 , \tag{13}$$

We have then (3) clearly, for arbitrary γ.

Theorem 2. *Let*

$$\varphi = \varphi(z) = (\sin z)(z - b) \quad ((z - b) \neq 0). \tag{14}$$

We have then the following nonhomogeneous fractional differintegral equations;

(i)
$$\varphi_{\gamma+1} - \varphi_\gamma \cdot \frac{\gamma+1}{z-b} = \left(z - b + \frac{\gamma^2+\gamma}{z-b}\right)(\cos z)_\gamma \ , \tag{15}$$

$$\left(\begin{array}{c} \textit{Fractional differential equation for } \gamma > 0, \\ \textit{Fractional integral equation for } \gamma < -1, \\ \textit{Fractional differintegral equation for } -1 < \gamma < 0. \end{array}\right)$$

and

(ii)
$$\varphi_{\gamma+2} - \varphi_{\gamma+1} \cdot \frac{\gamma+2}{z-b} + \varphi_\gamma \cdot \frac{(\gamma+1)(\gamma+2)}{(z-b)^2}$$

$$= -(z-b)(\sin z)_\gamma - \frac{\gamma(\gamma+1)(\gamma+2)}{(z-b)^2}(\cos z)_\gamma \tag{16}$$

$$\left(\begin{array}{c} \textit{Fractional differential equation for } \gamma > 0, \\ \textit{Fractional integral equation for } \gamma < -2, \\ \textit{Fractional differintegral equation for } -2 < \gamma < 0. \end{array}\right)$$

applying N- fractional calculus.

Proof of (i). Operate N- fractional calculus operator N^γ to the both sides of (14), we have then

$$\varphi_\gamma = (\sin z \cdot (z-b))_\gamma \,. \tag{17}$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(\gamma+1)}{k!\,\Gamma(\gamma+1-k)} (\sin z)_{\gamma-k} (z-b)_k \,. \tag{18}$$

$$= (\sin z)_\gamma (z-b) + \gamma (\sin z)_{\gamma-1} (z-b)_1 \tag{19}$$

$$= (\sin z)_\gamma (z-b) + \gamma (\sin z)_{\gamma-1} \ , \tag{20}$$

by Lemma (iv).

Therefore, we have

$$\varphi_{\gamma+1} = (\sin z)_{\gamma+1}(z-b) + (\gamma+1)(\sin z)_\gamma \ , \tag{21}$$

and

$$\varphi_{\gamma+2} = (\sin z)_{\gamma+2}(z-b) + (\gamma+2)(\sin z)_{\gamma+1} \tag{22}$$

from (20) respectively.

Then applying (20) and (21) we obtain

$$\text{LHS of (15)} = (\sin z)_{\gamma+1}(z-b) + (\gamma+1)(\sin z)_{\gamma}$$

$$-(\sin z)_{\gamma}(\gamma+1) - (\sin z)_{\gamma-1} \cdot \frac{\gamma(\gamma+1)}{z-b} \tag{23}$$

$$= \left(z - b + \frac{\gamma^2+\gamma}{z-b}\right)(\cos z)_{\gamma} \,, \tag{24}$$

for arbitrary γ.

Proof of (i i). Applying (20), (21) and (22), we obtain

$$\text{LHS of (16)} = (\sin z)_{\gamma+2}(z-b) + (\gamma+2)(\sin z)_{\gamma+1}$$

$$-\frac{\gamma+2}{(z-b)}\{(\sin z)_{\gamma+1}(z-b) + (\gamma+1)(\sin z)_{\gamma}\}$$

$$+\frac{(\gamma+1)(\gamma+2)}{(z-b)^2}\{(\sin z)_{\gamma}(z-b) + \gamma(\sin z)_{\gamma-1}\} \tag{25}$$

$$= (\sin z)_{\gamma+2}(z-b) + (\sin z)_{\gamma-1}\frac{\gamma(\gamma+1)(\gamma+2)}{(z-b)^2} \tag{26}$$

$$= -(\sin z)_{\gamma}(z-b) - (\cos z)_{\gamma}\frac{\gamma(\gamma+1)(\gamma+2)}{(z-b)^2} \tag{27}$$

for arbitrary γ.

§ 2. N- Fractional Calculus Method to The Equations obtained in Previous Section

Theorem 3. *Let* $\varphi \in \boldsymbol{F} = \{\varphi\,;\, 0 \neq |\varphi_{\gamma}| < \infty, \gamma \in \boldsymbol{R}\}$, $(\varphi = \varphi(z))$, *then the homogeneous fractional differintegral equations*

$$\varphi_{\gamma} - \varphi \cdot a^{\gamma}\left(1 + \frac{\gamma}{a(z-b)}\right) = 0\,, \quad (a(z-b) \neq 0)\,, \tag{1}$$

have a particular solution

$$\varphi = e^{az}(z-b). \tag{2}$$

Proof. Since $\gamma \in \boldsymbol{R}$, setting $\gamma = 1$ in (1), we have

$$\varphi_1 - \varphi \cdot \left(a + \frac{1}{z-b} \right) = 0 . \tag{3}$$

A particular solution to this variable separable form equation is given by (2) omitting the arbitrary constant for integration, clearly. And the function given by (2) satisfies equation (1), as we see in § 1.

Theorem 4. *Let* $\varphi \in \boldsymbol{F} = \{ \varphi ; 0 \neq | \varphi_\gamma | < \infty, \gamma \in \boldsymbol{R} \}$, $(\varphi = \varphi(z))$, *then the homogeneous fractional differintegral equations*

$$\varphi_{\gamma+2} - \varphi_{\gamma+1} a - \varphi_\gamma \cdot \frac{a^2}{a(z-b)+\gamma} = 0, \quad (a(z-b) + \gamma \neq 0) , \tag{4}$$

have a particular solution

$$\varphi = e^{az}(z-b). \tag{2}$$

Proof. Since $\gamma \in \boldsymbol{R}$, setting $\gamma = 0$ in (4), we have

$$\varphi_2 \cdot (z-b) - \varphi_1 \cdot a(z-b) - \varphi \cdot a = 0 . \tag{5}$$

Operate N^ν to the both sides of (5), we have then

$$(\varphi_2 \cdot (z-b))_\nu - (\varphi_1 \cdot a(z-b))_\nu - (\varphi \cdot a)_\nu = 0 . \tag{6}$$

Now we have

$$(\varphi_2 \cdot (z-b))_\nu = \varphi_{2+\nu} \cdot (z-b) + \nu \varphi_{1+\nu} , \tag{7}$$

$$(\varphi_1 \cdot a(z-b))_\nu = a(\varphi_1 \cdot (z-b))_\nu \tag{8}$$

$$= a \varphi_{1+\nu} \cdot (z-b) + a \nu \varphi_\nu . \tag{9}$$

and

$$(\varphi \cdot a)_\nu = \varphi_\nu \cdot a . \tag{10}$$

Therefore, we obtain

$$\varphi_{2+\nu} \cdot (z-b) + \varphi_{1+\nu} \cdot (\nu + ab - az) - \varphi \cdot a(\nu+1) = 0 \tag{11}$$

from (6), applying (7), (9) and (10).

We have then

$$\varphi_1 \cdot (z-b) + \varphi \cdot (ab - 1 - az) = 0 \tag{12}$$

from (11), choosing $\nu = -1$.

A particular solution to this variable separable form equation is given by (2) omitting the arbitrary constant for integration, clearly. And the function (2) satisfies equation (4), as we see in **§ 1.**

Theorem 5. *Let* $\varphi \in \boldsymbol{F} = \{ \varphi \,;\, 0 \neq |\varphi_\gamma| < \infty, \gamma \in \boldsymbol{R} \}$, $(\varphi = \varphi(z))$, *then the nonhomogeneous fractional differintegral equations*

$$\varphi_{\gamma+1} - \varphi_\gamma \cdot \frac{\gamma+1}{z-b} = \left(z - b + \frac{\gamma^2+\gamma}{z-b} \right) (\cos z)_\gamma \qquad ((z-b) \neq 0)\,, \tag{13}$$

have a particular solution

$$\varphi = (\sin z)(z-b)\,. \tag{14}$$

Proof. Since $\gamma \in \boldsymbol{R}$, setting $\gamma = 0$ in (13), we have

$$\varphi_1 - \varphi \cdot \frac{1}{z-b} = (\cos z)(z-b)\,. \tag{15}$$

A particular solution to this linear first order equation is given by (14) without the consideration of arbitrary constant for integration.

Inversely, the function shown by (14) satisfies equation (13) clearly, as we see in **§ 1.** (Refer to Theorem 2. (i).)

Theorem 6. *Let* $\varphi \in \boldsymbol{F} = \{ \varphi \,;\, 0 \neq |\varphi_\gamma| < \infty, \gamma \in \boldsymbol{R} \}$, $(\varphi = \varphi(z))$, *then the nonhomogeneous fractional differintegral equations*

$$\varphi_{\gamma+2} - \varphi_{\gamma+1} \cdot \frac{\gamma+2}{z-b} + \varphi_\gamma \cdot \frac{(\gamma+1)(\gamma+2)}{(z-b)^2}$$

$$= -(z-b)(\sin z)_\gamma - \frac{\gamma(\gamma+1)(\gamma+2)}{(z-b)^2} (\cos z)_\gamma\,, \qquad ((z-b) \neq 0) \tag{16}$$

have a particular solution

$$\varphi = (\sin z)(z-b)\,. \tag{14}$$

Proof. Since $\gamma \in \boldsymbol{R}$, setting $\gamma = 0$ in (16), we have

$$\varphi_2 - \varphi_1 \cdot \frac{2}{z-b} + \varphi \cdot \frac{2}{(z-b)^2} = -(\sin z)(z-b) \tag{17}$$

hence

$$\varphi_2 \cdot (z-b)^2 - \varphi_1 \cdot 2(z-b) + \varphi \cdot 2 = -(\sin z)(z-b)^3 \tag{18}$$

Operate N^{ν} to the both sides of (18), we have then

$$(\varphi_2 \cdot (z-b)^2)_\nu - (\varphi_1 \cdot 2(z-b))_\nu + (\varphi \cdot 2)_\nu = -\left((\sin z)\cdot(z-b)^3\right)_\nu . \qquad (19)$$

Now we have

$$(\varphi_2 \cdot (z-b)^2)_\nu = \sum_{k=0}^{2} \frac{\Gamma(\nu+1)}{k!\,\Gamma(\nu+1-k)} (\varphi_2)_{\nu-k}((z-b)^2)_k , \qquad (20)$$

$$= \varphi_{\gamma+2} \cdot (z-b)^2 + \varphi_{\gamma+1} \cdot 2\nu(z-b) + \varphi_\gamma \cdot \nu(\nu-1), \qquad (21)$$

$$(\varphi_1 \cdot 2(z-b))_\nu = 2(\varphi_1 \cdot (z-b))_\nu \qquad (22)$$

$$= 2\{\varphi_{1+\nu} \cdot (z-b) + \varphi_\nu \cdot \nu\} . \qquad (23)$$

and

$$(\varphi \cdot 2)_\nu = \varphi_\nu \cdot 2 . \qquad (24)$$

Therefore, we obtain

$$\varphi_{2+\nu} \cdot (z-b)^2 + \varphi_{1+\nu} \cdot (z-b)(2\nu-2) + \varphi_\nu \cdot (\nu^2 - 3\nu + 2) = -\left((\sin z)(z-b)^3\right)_\nu \qquad (25)$$

from (19), applying (21), (23) and (24).

Choose ν such that

$$\nu^2 - 3\nu + 2 = (\nu-2)(\nu-1) = 0, \qquad (26)$$

we have then

$$\nu = 1, 2. \qquad (27)$$

(I) When $\nu = 1$, we obtain

$$\varphi_3 \cdot (z-b)^2 = -((\sin z)(z-b)^3)_1 \qquad (28)$$

from (25), hence

$$\varphi_3 = -(\cos z)(z-b) - 3\sin z . \qquad (29)$$

Therefore, we obtain

$$\varphi = -((\cos z)(z-b))_{-3} - 3(\sin z)_{-3} \qquad (30)$$

from (29)

Now we have

$$(\sin z)_{-3} = \cos z \qquad (31)$$

and

$$((\cos z)(z-b))_{-3} = -(z-b)\sin z - 3\cos z . \qquad (32)$$

Then we obtain

$$\varphi = (\sin z)(z-b) . \qquad (14)$$

from (30), (31) and (32), without the consideration of arbitrary constant for integrations.

Inversely, the function shown by (14) satisfies equation (16) clearly, as we see in **§ 1.** (Refer to Theorem 2. (i i).)

(I I) When $\nu = 2$, we obtain

$$\varphi_4 \cdot (z-b)^2 + \varphi_3 \cdot 2(z-b) = -((\sin z)(z-b)^3)_2 \qquad (33)$$

from (25), hence

$$\phi_1 \cdot (z-b)^2 + \phi \cdot 2(z-b) = -((\sin z)(z-b)^3)_2 \qquad (34)$$

(linear first order equations)

from (33), setting

$$\varphi_3 = \phi = \phi(z) . \qquad (35)$$

Therefore, we obtain

$$(\phi \cdot (z-b)^2)_1 = -((\sin z)(z-b)^3)_2 \qquad (36)$$

from (34), hence

$$\phi = -\frac{((\sin z)(z-b)^3)_1}{(z-b)^2} \qquad (37)$$

$$= -(\cos z \cdot (z-b) + 3\sin z) . \qquad (38)$$

Then we obtain

$$\varphi = \phi_{-3} = -(\cos z \cdot (z-b))_{-3} - 3(\sin z)_{-3} \qquad (39)$$

$$= \sin z \cdot (z-b) , \qquad (14)$$

as a particular solution to equation (16), from (35) and (38), without the consideration of arbitrary constants for integrations.

Inversely, the function shown by (14) satisfies equation (16) clearly, as we see in **§ 1.** (Refer to Theorem 2. (i i).)

§ 3. Propositions

After the consideration on the theorems in **§ 1.** and **§ 2.** we obtain the propositions stated below clearly.

Proposition 1. *Let* $\varphi \in F = \{ \varphi ; 0 \neq | \varphi_\gamma | < \infty, \gamma \in \boldsymbol{R} \}$, $(\varphi = \varphi(z))$, *and the fractional differintegral equations be*

$$\varphi_\gamma + \varphi \cdot g(z) = f(z) \tag{1}$$

$$\begin{pmatrix} \textit{Fractional differential equation for } 0<\gamma\ , \\ \textit{Fractional integral equation for } \gamma<0\ . \end{pmatrix}.$$

Then setting $\gamma = 1$, *we obtain*

(i)
$$\varphi_1 + \varphi \cdot g(z) = f(z) \tag{2}$$

(*linear first order equation for* $f(z) \neq 0$)

and

(ii)
$$\varphi_1 + \varphi \cdot g(z) = 0 \tag{3}$$

(*variable separable form equation for* $f(z) = 0$)

from (1).

The particular solutions to equations (2) *and* (3) *are the particular ones to equation* (1) *respectively.*

Note 1. In this case we can't set $\gamma = 0$ in (1), though γ is arbitrary, because (1) is reduced to not a differintegral equation for $\gamma = 0$.

Proposition 2. *Let* $\varphi \in \boldsymbol{F} = \{\varphi\,;\, 0 \neq |\varphi_\gamma| < \infty, \gamma \in \boldsymbol{R}\}$, $(\varphi = \varphi(z))$, *and the nonhomogeneous fractional differintegral equations be*

$$\varphi_{\gamma+2} + \varphi_{\gamma+1} \cdot g(z) + \varphi_\gamma \cdot h(z) = f(z) \tag{4}$$

$$\begin{pmatrix} \textit{Fractional differential equation for } 0<\gamma\ , \\ \textit{Fractional integral equation for } \gamma<-2\ , \\ \textit{Fractional differitegral equation for } -2<\gamma<0. \end{pmatrix}.$$

Set $\gamma = 0$, *then we obtain*

(i)
$$\varphi_2 + \varphi_1 \cdot g(z) + \varphi \cdot h(z) = f(z), \qquad (\textit{for } f(z) \neq 0) \tag{5}$$

and

(ii)
$$\varphi_2 + \varphi_1 \cdot g(z) + \varphi \cdot h(z) = 0, \qquad (\textit{for } f(z) = 0) \tag{6}$$

from (4).

The particular solutions to equations (5) *and* (6) *are the particular ones to equations* (4) *respectively.*

§ 4. Commentary

(I) The linear second order ordinary differential equations, whose solutions are so called " special functions ", are called as " special differential equations (SDE) ".

The SDE shown by (5) and (6) (non-homogeneous and homogeneous) in §3 , can be solved by our " N-fractional calculus method (NFCM) " which are described in § 2 . (Usually, so called SDE is given by (5) or (6) in its form.)

That is,

(i) nonhomogeneous equation § 3. (5) is reduced to linear first order one, and

(ii) homogeneous equation § 3. (6) is reduced to variable separable form one,

respectively., by our NFCM.

Then we can obtain the particular solutions to the original fractional differ-integral equations § 3. (4), when the reduced ones are integrable. ([6] ~ [31])

Hitherto, only the homogeneous SDE are solved by means of Frobenius. However we can solve the nonhomogeneous SDE by our NFCM, as we see in § 2.

Note. **N=Fractional calculus of exponential and trigonometric functions**

We have

(i) $(e^{az})_\gamma = a^\gamma e^{az}$, (ii) $(e^{-az})_\gamma = e^{-i\pi\gamma} a^\gamma e^{-az}$,

(iii) $(\cos az)_\gamma = a^\gamma \cos\left(az + \frac{\pi}{2}\gamma\right)$, (iv) $(\sin az)_\gamma = a^\gamma \sin\left(az + \frac{\pi}{2}\gamma\right)$,

where $a \neq 0$, respectively. ([1] Vol. 1)

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