

On the indices and integral bases of abelian biquadratic fields

By

Toru Nakahara

1. Introduction. Let K be an algebraic number field over the rationals Q with finite degree. Z and O_K denote the ring of rational integers and the integer ring of K respectively. If $O_K = Z[\alpha]$ for some number α in K , it is called that O_K has a power basis. For a number ξ in O_K we denote by $\text{Ind } \xi$ the group index $(O_K : Z[\xi])$ if ξ is a primitive element of K and 0 otherwise. Then the index $m(K)$ of any field K is defined by $\text{g.c.d. } \{ \text{Ind } \xi ; \xi \in O_K \}$. The minimum index $\tilde{m}(K)$ of any K is defined by $\min \{ \text{Ind } \eta ; \eta \in O_K, Q(\eta) = K \}$. In §2 we shall give an estimate of the index $m(K)$ without using the decomposition theory of primes when K is any abelian biquadratic field. In §3 we shall investigate some relations between $m(K)$ and an integral basis related to a problem of Hasse and construct such a field K that the minimum index $\tilde{m}(K)$ is greater than any given integer N applying a method of M. Hall[2].

2. An estimate of the indices. By [8] it is well known that if a prime p divides the index $m(K)$, then p is smaller than the degree $[K : Q]$.

In our situation we obtain more precisely the next lemma.

Lemma 1. For any abelian biquadratic field K over Q it holds that if the number $2^e 3^{e'}$ exactly divides the index $m(K)$,

then $e \leq 2$ and $e' \leq 1$. Especially if the discriminant $d(K)$ of a field K is even, then $e = 0$.

Proof. i) The cyclic cases. Let χ be a biquadratic character with odd conductor n determined by the biquadratic residue symbol. k_n denotes the n -th cyclotomic field $Q(\zeta_n)$, herein $\zeta_n = \exp(2\pi i/n)$. Let G be the Galois group of k_n/Q . The group $\langle \chi \rangle$ is a cyclic subgroup with order 4 of the character group of G . Let K denote the subfield of k_n corresponding to the kernel H of χ . Then we have $K = Q(\eta)$ with the Gauss period $\eta = \sum_{x \in H} \zeta_n^x$. We fix an element σ in G such that

$\chi(\sigma) = i$, and denote $\sigma(\xi)$, $\sigma^2(\xi)$, $\sigma^3(\xi)$ by ξ' , ξ'' , ξ''' respectively for ξ in K .

First we consider the case of odd conductor n . Since the set $\{1, \eta, \eta', \eta''\}$ makes an integral basis of K , it is enough for computation of the $\text{Ind } \xi$ to choose $\xi = x\eta + y\eta' + z\eta''$ for ξ in K . Let $n = \ell m$ be square-free for odd integers $\ell = a^2 + 4b^2$, m where any prime factor of ℓ is congruent to 1 modulo 4 and $\lambda = a + 2bi \equiv 1 \pmod{2(1-i)}$. Then by using the Gauss sum $\tau(\chi) = \sum_{x \in G} \chi(x) \zeta_n^x$ attached to χ and the Jacobi sum $\tau(\chi)^2 / \tau(\chi^2)$

we obtain $\text{Ind } \xi = \sqrt{|d(\xi)/d(K)|} = |cN\alpha_n|$, where $\alpha_n = (cm + d\sqrt{\ell})/2$, $c = ((x-z)^2 - y^2)b - (x-z)ya$, $d = ((x-y+z)^2 - \chi(-1) \times ((x-z)^2 + y^2)m)/2$. Herein $d(\xi)$, N mean the discriminant of a number ξ , the norm with respect to $Q(\sqrt{\ell})/Q$ respectively.

i)₁ If $\ell \equiv 1 \pmod{8}$ and $b \equiv 0, 4 \pmod{8}$ (resp. $b \equiv \pm 2 \pmod{8}$), then for $\xi = 2\eta + \eta' - \eta''$ (resp. $2\eta \pm \eta'$ from $\chi(-1) = \begin{cases} 1, \\ -1, \end{cases}$

$m \equiv 1 \pmod{4}$ }
 $m \equiv -1 \pmod{4}$) we have $\text{Ind } \xi \equiv 4 \pmod{8}$. i)₂ If $\ell \equiv 5 \pmod{8}$,

then for $\xi = \eta + \eta' - \eta''$ we get $\text{Ind } \xi \equiv 1 \pmod{2}$. i)₃ If $m \equiv 0 \pmod{3}$ and $a \equiv 0 \pmod{3}$ (resp. $a \not\equiv 0 \pmod{3}$), then we choose $\xi = \eta$ (resp. $\eta + \eta'$). Thus we have $\text{Ind } \xi \not\equiv 0 \pmod{3}$. i)₄ If $m \not\equiv 0 \pmod{3}$ and $a \equiv 0 \pmod{3}$, then $b \not\equiv 0 \pmod{3}$ and for $\xi = \eta$ $4N\alpha_n \equiv b^2 |1 - 2\chi(-1)m| \pmod{9}$ holds. When $1 - 2\chi(-1)m \equiv 0 \pmod{9}$, we reset $\xi = 2\eta + 3\eta'$. Then $4N\alpha_n \equiv b^2 |1 - 8\chi(-1)m| \not\equiv 0 \pmod{9}$. i)₅ The case of $m \not\equiv 0$ and $a \not\equiv 0 \pmod{3}$. For $\xi = \eta$ if $4^2 N\alpha_n \equiv |4b^2 m^2 - (1 - \chi(-1)m)^2 \ell| \equiv 0 \pmod{9}$ holds, then we reset $\xi = 2\eta + \eta''$. If $4^2 N\alpha_n \equiv |4b^2 m^2 - (-\chi(-1)m)^2 \ell| \equiv 0 \pmod{9}$, then we have $1 - 2\chi(-1)m \equiv 0 \pmod{9}$ from $(\ell, 3) = 1$. Then we take again $\xi = 3\eta + 2\eta''$. If $4^2 N\alpha_n \equiv |4b^2 m^2 - (7 - \chi(-1)m)^2 \ell| \equiv 0 \pmod{9}$, then $1 + \chi(-1)m \equiv 0 \pmod{9}$ must hold. This is a contradiction. By i)₁₋₅ we have $e \leq 2$ and $e' \leq 1$.

Next we estimate the case of even conductor. At first we consider the case of $\chi = \chi_o^{(\nu)} \chi_\ell \psi_m$, $n = 16\ell m$, $\ell m \equiv 1 \pmod{2}$, where $\chi_o^{(\nu)}(x) = (-1)^{\nu(x-1)/2} i^{(x^2-1)/8}$ are the even and the odd biquadratic characters with conductor 16 for $\nu = 0$ and 1 respectively, and χ_ℓ , ψ_m are the biquadratic, the quadratic characters with conductors ℓ , m respectively. From $\chi((n/2) + 1) = -1$, it follows that $\eta'' = \sigma^2(\eta) = \sum_{x \in H} \zeta_n^{((n/2)+1)x} = - \sum_{x \in H} \zeta_n^x = -\eta$. However it is known that $\{1, \eta, \eta', \sqrt{f}/2\}$ is an integral basis of K , where $d(K) = fn^2$ and $f = 8\ell$ is the conductor of $Q(\sqrt{8\ell})$ [3]. Then for $\xi = x\eta + y\eta' + z(\sqrt{f}/2)$ we obtain $\text{Ind } \xi = |cN\alpha_f|$, where $\alpha_f = cm + d(\sqrt{f}/2)$, $c = -2xy(a-b) + (x^2 - y^2)(a+2b)$, $d = 2z^2 - \chi(-1)(x^2 + y^2)m$. For $\xi_1 = \eta$, we have

$$\text{Ind } \xi_1 = |a + 2b|m^2 |(a + 2b)^2 - 2\ell| \equiv 1 \pmod{2}. \quad \text{If } a + 2b \equiv \pm 3 \pmod{9}$$

and $m \not\equiv 0 \pmod{3}$, then $\text{Ind } \xi_1 \equiv \pm 3 \pmod{9}$. If $a + 2b \equiv \pm 3 \pmod{9}$ (resp. $\not\equiv 0 \pmod{3}$) and $m \equiv 0 \pmod{3}$, then for $\xi_2 = \eta + (\sqrt{f}/2)$ we get $\text{Ind } \xi_2 = |a + 2b| |(a + 2b)^2 m^2 - 8\ell| \equiv \pm 3 \pmod{9}$ (resp. $\not\equiv 0 \pmod{3}$). If $a + 2b \equiv 0 \pmod{9}$ and $m \equiv 0 \pmod{3}$, then $a - 2b \not\equiv 0 \pmod{3}$ holds. Thus for $\xi_3 = \eta + \eta' + (\sqrt{f}/2)$ we have $\text{Ind } \xi_3 \equiv 2|a - 2b|\ell \not\equiv 0 \pmod{3}$. In the case of $(a + 2b)m \not\equiv 0 \pmod{3}$, we have $\text{Ind } \xi_1 \not\equiv 0 \pmod{3}$ for $a \not\equiv -b \pmod{3}$, and put $\xi_4 = \eta + \eta'$, then $\text{Ind } \xi_4 \equiv \pm 3 \pmod{9}$ for $a \equiv -b \pmod{3}$. If $a + 2b \equiv 0 \pmod{9}$ and $m \not\equiv 0 \pmod{3}$, then we have $\text{Ind } \xi_4 \not\equiv 0 \pmod{9}$. Therefore we obtain $e = 0$ and $e' \leq 1$. Secondly we treat the case of $\chi = \chi_\ell \psi_m$, $n = \ell m$, $m \equiv 0 \pmod{2}$. In this case the set $\{1, \eta, \eta', \eta''\}$ is also not an integral basis of K . But $\{1, \eta, \eta', (1 + \sqrt{\ell})/2\}$ is an integral basis, where $d(K) = \ell n^2$, $f = \ell[3]$. Then for $\xi = x\eta + y\eta' + z(1 + \sqrt{\ell})/2$ we have $\text{Ind } \xi = |cN\alpha_f|$, where $\alpha_f = cm + d\sqrt{f}$, $c = -xya + (x^2 - y^2)2b$, $d = (x^2 + y^2)(m/2) - \chi(-1)z^2$. For $\xi_1 = \eta + \eta' + (1 + \sqrt{\ell})/2$ we get $\text{Ind } \xi_1 \equiv 1 \pmod{2}$. Put $\xi_2 = \eta + \eta'$. If $abm \not\equiv 0 \pmod{3}$, then $\text{Ind } \xi_2 \not\equiv 0 \pmod{3}$. We choose $\xi_3 = \eta$, $\xi_4 = \eta + (1 + \sqrt{\ell})/2$. If $a \equiv 0 \pmod{3}$, then from $b \not\equiv 0 \pmod{3}$ we have $\text{Ind } \xi_3 \equiv \pm 3 \pmod{9}$ for $m \not\equiv 0 \pmod{3}$ and $\text{Ind } \xi_4 \not\equiv 0 \pmod{3}$ for $m \equiv 0 \pmod{3}$. If $b \equiv 0 \pmod{3}$, then from $a \not\equiv 0 \pmod{3}$ we obtain $\text{Ind } \xi_1 \not\equiv 0 \pmod{3}$ for $m \not\equiv 0 \pmod{3}$ and $\text{Ind } \xi_1 \equiv \pm 3 \pmod{9}$ for $m \equiv 0 \pmod{3}$. Thus we have $e = 0$ and $e' \leq 1$.

ii) The non-cyclic cases. Without loss of generality we can set $K = Q(\sqrt{\ell m_1}, \sqrt{\ell m_2})$, where $\ell m_1 m_2$ is a square-free integer and $\ell > 0$. For brevity we denote $(1 + \sqrt{\ell m_1})/2$, $(1 + \sqrt{\ell m_2})/2$, $(\sqrt{\ell m_2} + \sqrt{m_1 m_2})/2$ by α , β , γ respectively. ii), If $\ell m_1 \equiv 1$, $\ell m_2 \equiv 2, 3 \pmod{4}$, then $\{1, \alpha, 2\beta - 1, \gamma\}$ is an integral basis

of K and the field discriminant $d(K) = 16\ell^2 m_1^2 m_2^2$ holds. For $\xi_1 = \alpha - (2\beta - 1) + 2\gamma$ we can compute $\text{Ind } \xi_1 \equiv \ell m_1^2 \equiv 1 \pmod{2}$. If $(\ell - m_2)m_1 \not\equiv 0 \pmod{3}$, then $\text{Ind } \xi_1 \not\equiv 0 \pmod{3}$ follows. If $m_1 \equiv 0 \pmod{3}$, then for $\xi_2 = \alpha + (2\beta - 1)$ we have $\text{Ind } \xi_2 \equiv \ell^2 m_2 \not\equiv 0 \pmod{3}$. If $\ell - m_2 \equiv 0 \pmod{3}$, then $\ell m_2 \not\equiv 0 \pmod{3}$. We can restrict $m_1 \not\equiv 0 \pmod{3}$. In the case of $\ell - m_2 \equiv 0 \pmod{9}$, we have $\ell - 4m_2 \not\equiv 0 \pmod{9}$. Then $\text{Ind } \xi_1 \not\equiv 0 \pmod{9}$. In the case of $\ell - m_2 \equiv \pm 3 \pmod{9}$, for $\xi_3 = \alpha + (2\beta - 1) + \gamma$ we get $\text{Ind } \xi_3 \equiv |(\ell - m_2)m_1^2| \equiv \pm 3 \pmod{9}$. Thus we obtain $e = 0$ and $e' \leq 1$.

ii)₂ if $\ell m_1 \equiv 3$, $\ell m_2 \equiv 2 \pmod{4}$, then $\{1, 2\alpha - 1, 2\beta - 1, \gamma\}$ is an integral basis and $d(K) = 64\ell^2 m_1^2 m_2^2$. By $\ell \equiv 1 \pmod{2}$, $\ell - m_1 \equiv 2 \pmod{4}$ for $\xi_1 = \gamma$ we have $\text{Ind } \xi_1 \equiv 1 \pmod{2}$. Next if $(\ell - m_1)m_2 \not\equiv 0 \pmod{3}$, then $\text{Ind } \xi_1 \not\equiv 0 \pmod{3}$ holds. If $m_2 \equiv 0 \pmod{3}$, then for $\xi_2 = (2\alpha - 1) + (2\beta - 1)$ we have $\text{Ind } \xi_2 = |(4\ell)^2 (4m_2 - m_1)| \not\equiv 0 \pmod{3}$. If $\ell - m_1 \equiv 0 \pmod{9}$, then $\ell m_1 \not\equiv 0 \pmod{3}$. We can restrict $m_2 \not\equiv 0 \pmod{3}$. For $\xi_3 = 2(2\beta - 1) + \gamma$ $\text{Ind } \xi_3 \equiv |(-m_2)(25\ell - m_1)(25m_2)| \equiv \pm 3 \pmod{9}$ holds. If $\ell - m_1 \equiv \pm 3 \pmod{9}$, then $\text{Ind } \xi_1 \equiv \pm 3 \pmod{9}$. Therefore we have $e = 0$ and $e' \leq 1$.

ii)₃ If $\ell m_1 \equiv \ell m_2 \equiv 1 \pmod{4}$, then $\{1, \alpha, \beta, \alpha\beta \pm ((\ell - 1)/4)(2\gamma - 2\beta + 1)\}$ for $\ell \equiv m_1 \equiv m_2 \equiv 1 \pmod{4}$ and $\{1, \alpha, \beta, \alpha\beta + (1/2) \mp ((\ell - 1)/4)(2\gamma - 2\beta + 1)\}$ for $\ell \equiv m_1 \equiv m_2 \equiv 3 \pmod{4}$ are integral bases, where the sign is positive if and only if $m_1 < 0$ and $m_2 < 0$. For any integer ξ in K we have $\text{Ind } \xi \equiv 0 \pmod{2}$. Moreover in the case of $m_1 - m_2 \equiv 4 \pmod{8}$ (resp. $\equiv 0 \pmod{8}$), for $\xi_1 = \alpha + \beta$ (resp. $2\alpha + \beta$) we get $\text{Ind } \xi_1 \equiv 4 \pmod{8}$. If $\ell(m_1 - m_2) \not\equiv 0 \pmod{3}$, then $\text{Ind } \xi_1 \not\equiv 0 \pmod{3}$. We denote by δ the fourth numbers of the integral bases. If $\ell \equiv 0$ and $m_1 - m_2 \not\equiv 0 \pmod{3}$, then for $\xi_2 = \alpha + \beta + 2\delta$

we have $\text{Ind } \xi_2 \equiv |m_1 m_2(m_1 - m_2)| \not\equiv 0 \pmod{3}$. If $\ell \not\equiv 0$ and $m_1 - m_2 \equiv 0 \pmod{9}$, then $m_1 m_2 \not\equiv 0 \pmod{3}$ holds. If for $\xi_3 = 2\alpha + \beta$ $\text{Ind } \xi_3 = |4\ell^2(m_2 - 4m_1)| \equiv 0 \pmod{9}$, then we have $3m_1 \equiv 0 \pmod{9}$. This is a contradiction. If $\ell \not\equiv 0 \pmod{3}$ and $m_1 - m_2 \equiv \pm 3 \pmod{9}$, then $\text{Ind } \xi_1 \equiv \pm 3 \pmod{9}$. Next if $\ell \equiv 0 \pmod{3}$ and $m_1 - m_2 \equiv \pm 3 \pmod{9}$, then $\text{Ind } \xi_2 \equiv \pm 3 \pmod{9}$. Finally if $\ell \equiv 0 \pmod{3}$ and $m_1 - m_2 \equiv 0 \pmod{9}$, then for $\xi_4 = \beta + 2\delta$ we have $\text{Ind } \xi_4 \equiv \pm 3 \pmod{9}$. The estimates of $ii)_{1 \sim 3}$ imply $1 \leq e \leq 2$ and $e' \leq 1$. Therefore we have proved Lemma 1.

3. Results. Works related to the problem of Hasse are found in [1], [4], [5] and the references mentioned in [7]. From [6] and [7] we have

Theorem 1. There exist infinitely many non-cyclic but abelian (resp. exist cyclic) biquadratic fields over $\mathbb{Q}$ whose integer rings have a power basis.

In our case by Lemma 1 the index $m(K)$ is not larger than 12. In fact it follows

Theorem 2. There exist infinitely many such abelian biquadratic fields K over $\mathbb{Q}$ that the index is equal to 12 (resp. 6) and that neither $\{1, \alpha, \alpha^2, \beta\}$ nor $\{1, \alpha, \beta, \alpha^3\}$ (resp. $\{1, \alpha, \beta, \alpha^3\}$) for any α, β in K forms (resp. does not form) an integral basis of K .

The method of a proof of this theorem is the same as in [6].

i) The cyclic case. Let n be the conductor of the field K .

We choose $n = a^2 + 72^2$, $a \equiv 5 \pmod{12}$ (resp. $n = a^2 + 12^2$, $a \equiv 1 \pmod{12}$). Since the set $\{1, \gamma, \gamma', \gamma''\}$ with the Gauss period γ makes an integral basis of K , we may put $\xi = x\gamma + y\gamma' + z\gamma''$ for any integer ξ in K . Then we obtain $\text{Ind } \xi = |cN\alpha_n|$ where α_n is the same number of $Q(\sqrt{n})$ as in the previous section. By virtue of $\lambda = a + 72i$ and

$$N\alpha_n \equiv \begin{cases} (x-z)^2 y^2 - (xy + yz + zx)^2 \pmod{3} \\ 2(x+y+z)(x+z)y - (x-y)(z-y)xz \pmod{4} \end{cases} \quad (\text{resp.}$$

$$\lambda = a + 12i \text{ and } N\alpha_n \equiv \begin{cases} (x-z)^2 y^2 - (xy + yz + zx)^2 \pmod{3} \\ 0 \pmod{2} \end{cases})$$

we have $\text{Ind } \xi \equiv 0 \pmod{12}$ (resp. $\text{Ind } \xi \equiv 0 \pmod{6}$ and $\text{Ind } \gamma \equiv 2 \pmod{4}$). Then by Lemma 1 we get $m(K) = 12$ (resp. $m(K) = 6$).

Moreover by $\chi(2) = 1$ (resp. $\begin{cases} \chi(2) = -1 \\ \chi(3) = 1 \end{cases}$) we can see

$$\sigma^j(\gamma)^2 \equiv \sigma^j(\gamma) \pmod{2} \quad (\text{resp. } \begin{cases} \sigma^j(\gamma)^2 \equiv \sigma^{j+2}(\gamma) \pmod{2} \\ \sigma^j(\gamma)^3 \equiv \sigma^j(\gamma) \pmod{3} \end{cases}). \text{ Since}$$

$\text{Ind } \xi$ is equal to the absolute value of the determinant of the transformation matrix for $\{1, \xi, \xi^2, \xi^3\}$ with respect to an integral basis $\{1, \gamma, \gamma', \gamma''\}$, we can see that any three rows in the matrix are linearly dependent modulo 2 (resp. the second and the fourth rows are so modulo 3). Then none of $\{1, \alpha, \alpha^2, \beta\}$ nor $\{1, \alpha, \beta, \alpha^3\}$ (resp. $\{1, \alpha, \beta, \alpha^3\}$) for all integers α, β can make (resp. can not make) a $\mathbb{Z}$ -basis of $\mathcal{O}_K$. Finally our parametrization satisfies the next lemma.

Lemma 2[6]. For $a > 0, b, c \in \mathbb{Z}$, $a \equiv b, c \equiv 1 \pmod{2}$, set

$$n(t) = at^2 + bt + c.$$

Let the congruences $n(t) \equiv 0 \pmod{q^2}$ have at most two solutions

for every prime q within $1 \leq t \leq q^2$. Then the number $n(t)$ is square-free for infinitely many $t \in \mathbb{Z}$.

ii) The non-cyclic case. For a field $K = \mathbb{Q}(\sqrt{\ell m_1}, \sqrt{\ell m_2})$ assume $\ell m_1 \equiv \ell m_2 \equiv 1 \pmod{24}$. Then using Lemma 1 we have $m(K) = 12$. Next we choose $\ell m_1 \equiv \ell m_2 \equiv 1 \pmod{3}$, $\ell \equiv 5 \pmod{8}$ and $m_1 \equiv m_2 \equiv 1 \pmod{16}$. Then we can see $\text{Ind} \xi \equiv 0 \pmod{6}$ for any integer ξ in K . Also for $\xi_0 = ((1 + \sqrt{\ell m_1})/2) + ((1 + \sqrt{\ell m_1}) \times (1 + \sqrt{\ell m_2})/4) + ((\ell - 1)/4)\sqrt{m_1 m_2}$ it follows $\text{Ind} \xi_0 \equiv 2 \pmod{4}$. Thus we obtain $m(K) = 6$. Under this parametrization we can perform the same argument as in the case i). Therefore we obtain Theorem 2.

Remark 1. Among the fields K with even conductor there does not exist any K which satisfies the properties in Theorem 2.

Theorem 3. There exist infinitely many non-cyclic but abelian biquadratic fields K which have the index 1 and still whose minimum indices are greater than N for any given integer N . Consequently the integer rings $\mathcal{O}_K$ have not a power basis.

Proof. We consider the field $K_\ell = \mathbb{Q}(\sqrt{\ell m_1}, \sqrt{\ell m_2})$ with $\ell m_1 \equiv 1, \ell m_2 \equiv -1 \pmod{12}$. Then from Lemma 1 the index $m(K_\ell)$ is odd. Under the same notations as in the proof ii), of Lemma 1 for a number $\xi = x\alpha + y\beta + z\gamma$ we obtain

$$\text{Ind} \xi = |(x^2 \ell - z^2 m_2)(z^2 m_1 - (2y + z)^2 \ell)(x^2 m_1 - (2y + z)^2 m_2)|/4.$$

Thus it holds that $\text{Ind}(\alpha + \beta) \not\equiv 0 \pmod{3}$. Then $m(K_\ell) = 1$ holds. In an imaginary case we select $0 > m_1 \equiv 1, 0 < -m_2 \equiv 1, 0 < \ell \equiv 1 \pmod{12}$. Then $\text{Ind} \xi > \ell$ holds for any primitive element ξ in $\mathcal{O}_{K_\ell}$.

In a real case set $0 < \ell \equiv -1 \pmod{12}$. We estimate the factor $I = x^2 m_1 - (2y + z)^2 m_2$ of $\text{Ind } \xi$. For any integer $N > 0$ we can find the following primes $p_i \equiv -1, q_i \equiv 1 \pmod{12}$ and $p_i \neq \ell$ for $1 \leq i \leq N$. Put $m_1 = p_1$ and $m_2 = q_1$ such that

$$\left(\frac{x^2 p_1}{q_1}\right) = \left(\frac{p_1}{q_1}\right) \neq \left(\frac{1}{q_1}\right), \text{ where } \left(\frac{*}{p}\right) \text{ denotes the Legendre symbol.}$$

Then $I \neq \pm 1$. Next for a prime $q_2 > q_1$, there exists an integer

$$a_2 \text{ with } \left(\frac{a_2}{q_2}\right) \neq \left(\frac{2}{q_2}\right). \text{ We select } p_2 \text{ such that } p_2 \equiv \begin{cases} p_1 \pmod{q_1} \\ a_2 \pmod{q_2} \end{cases}.$$

Reset $m_1 = p_2, m_2 = q_1 q_2$, then $I = \pm 1, \pm 2$. Successively we

$$\text{can choose primes } p_N, q_N \text{ such that } p_N \equiv \begin{cases} p_{N-1} \pmod{q_1 \dots q_{N-1}} \\ a_N \pmod{q_N} \end{cases}$$

$$\text{with } q_N > q_{N-1} \text{ and } \left(\frac{a_N}{q_N}\right) \neq \left(\frac{N}{q_N}\right). \text{ For } m_1 = p_N, m_2 = q_1 \dots q_N$$

define the biquadratic field $K_N = Q(\sqrt{\ell m_1}, \sqrt{\ell m_2})$, then it holds that $\mathfrak{M}(K_N) > N$. Therefore we have proved Theorem 3.

References

- [1] K. Györy, On discriminants and indices of integers of an algebraic number field, J. Reine Angew. Math. 324(1981), 114 - 126.
- [2] M. Hall, Indices in cubic fields, Bull. Amer. Math. Soc. 43 (1937), 104 - 108.
- [3] H. Hasse, Arithmetische Bestimmung von Grundeinheit und Klassenzahl in zyklischen und biquadratischen Zahlkörpern, Abh. Deutsch. Akad. Wiss. Berlin, Math. -Nat. Kl. 2(1950), 3 - 95 (= Math. Abhandlungen Bd. 3, 289 - 379, Berlin - New York, 1975).

- [4] I. Kátai; B. Kovács, Kanonische Zahlssysteme in der Theorie der quadratischen algebraischen Zahlen, Acta Sci. Math. (Szeged), 42(1980), 99 - 107.
- [5] B. Kovács, Canonical number systems in algebraic number fields, Acta Math. Acad. Sci. Hungar. 37(1981), 159 - 164.
- [6] T. Nakahara, On cyclic biquadratic fields related to a problem of Hasse, Mh. Math. 94(1982), 125 - 132.
- [7] T. Nakahara, On a problem of Hasse, RIMS Kōkyūroku, 456(1982), 152 - 157.
- [8] E. von Zyliński, Zur Theorie der ausserwesentlichen Diskriminantenteiler algebraischer Körper, Math. Ann. 73(1913), 273 - 274.

Department of Mathematics
Faculty of Science and Engineering
Saga University
Saga 840, Japan