

Multicriteria Multipliers of Banach-valued Functions on Locally Compact Abelian Group*

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Abstract

Let G be a *locally compact Abelian (LCA) group*, A a commutative *Banach algebra*, “ X ” and “ Y ” denote the *Banach spaces* of A -module. $L^1(G, A)$ stands for the space of all A -valued commutative *Banach algebra* with convolution product. $L^p(G, X)$, $1 \leq p \leq \infty$, for each p , is a *Banach space*. In this note, we study the multipliers of $L^1(G, A)$ and the representation of the homomorphism $L^1(G, A)$ module multipliers of $L^1(G, A)$ to $L^p(G, Y)$ which can be identified by $L^1(G, A) \otimes L^q(G, Y^*)^*$ under reasonable conditions, where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. The multipliers of $L^1(G, A)$ to $C_0(G, X)$ is also subscribed.

Key words and phrases: locally compact Abelian (LCA) group, separable Banach space, Radon Nikodym property, multipliers, invariant operator, projective tensor product space.

1 Introduction and preliminaries

Let G be a *locally compact Abelian (= LCA) group* with *Haar measure* dt and dual group $\widehat{G}$. Let A be a commutative *Banach algebra* with a bounded approximate identity. A continuous linear map $T \in \mathfrak{L}(A) \cong \mathfrak{L}(A, A)$ is called a **multiplier** of A if

$$T(a \cdot b) = a \cdot Tb = (Ta) \cdot b \text{ for all } a, b \in A.$$

Denote by $\mathfrak{M}(A)$ the space of all multipliers for A .

Clearly, $\mathfrak{M}(A)$ is a *Banach subalgebra* of $\mathfrak{L}(A)$. In particular, if $A = L^1(G)$, a

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commutative group algebra under convolution product, then the multiplier algebra $\mathfrak{M}(L^1(G))$ has the following equivalent statements (i)~(iv). (See Larsen [7], cf also Lai, Lee and Liu [1]):

Theorem 1. *Let $T \in \mathfrak{L}(L^1(G))$. Then the following statements are equivalent.*

(i) *T commutes with convolution (call T a **multiplier**)*

$$T(f * g) = Tf * g = f * T(g), \text{ for all } f, g \in L^1(G)$$

(ii) *T commutes with translation operator τ_a ($a \in G$). (call T an **invariant operator**)*

$$T\tau_a = \tau_a T, \tau_a f(t) = f(t - a), \text{ for all } a \in G,$$

(iii) $\exists! \mu \in M_b(G)$, space of all bounded regular Borel measures such that,
 $Tf = \mu * f$, for all $f \in L^1(G)$.

(iv) *there exists a bounded function ϕ on $\hat{G}$ such that*

$$\widehat{Tf} = \phi \hat{f} \text{ or } \phi = \hat{\mu} \in \widehat{M_b(G)} \subsetneq C^b(\hat{G}).$$

It is remarkable that

(a) the Fourier transforms $\widehat{L^1(\hat{G})} = A(\hat{G}) \subsetneq C_0(\hat{G})$ is dense of 1st category in $C_0(\hat{G})$, the continuous function on $\hat{G}$, vanishing at infinite.

(b) Similarly, it is known that the Fourier – Stieltjes transforms :

$\hat{\mu} \in \widehat{M_b(G)} \subsetneq C^b(\hat{G})$, the space of all bounded continuous functions on $\hat{G}$.

By Theorem 1, we see that the definition of multipliers is in various types. Actually the concept of multiplier comes from *Fourier Series* of a function f by using a bounded sequence $\phi(n)$ multiply the *Fourier coefficient* c_n of f , it still approve as a *Fourier coefficient* of another function of g . This ideal leads to study for multipliers in harmonic analysis on *locally compact Abelian group* G .

In this Note, we would like to extend the multipliers of $L^1(G)$ to the multipliers of $L^1(G, A)$ as well as multipliers of $L^1(G, X)$ to $L^1(G, Y)$ under module homomorphism of *Banach vector – valued functions* defined on *LCA group* G , and compare

the Banach algebras $L^1(G, A)$ and $L^1(G)$, do have the same properties as in the Theorem 1? Actually, the invariant operator T in $\mathfrak{L}(L^1(G, A))$ can not be a multiplier of $L^1(G, A)$ provided $\dim A > 1$. (See Tewari, Dutta and Vaidya [9]). That is, in Theorem 1, (ii) $\Rightarrow$ (i) is false, the other implications are true.

2 Multipliers of Banach algebra .

Let A be a commutative Banach algebra, we say that a Banach space X is A – module if

$$AX \subset X, \text{ and } \|a \cdot x\|_X \leq \|a\|_A \|x\|_X \text{ for each } a \in A, x \in X.$$

and X is said to be an essential A – module if

$$AX = X, \text{ and } \|ax\|_X \leq \|a\|_A \|x\|_X, \text{ for each } a \in A, x \in X.$$

For convenience, we give following Theorem to check that an A – module Banach space to be essential.

Theorem 2. *Let A be a commutative Banach algebra with uniform bounded approximate identity. Then any A – module Banach space is essential.*

For example, the group algebra $L^1(G)$ has bounded approximate identity: $\{e_\alpha\}$, where e_α is $e_\alpha = \frac{\chi_{V_\alpha}}{|V_\alpha|}$, where $\{V_\alpha\}$ is defined by an open neighborhood system of the identity $\theta \in G$ with ordered by $\alpha \prec \beta$ if $V_\beta \subset V_\alpha$, then $\|e_\alpha\|_1 = \int_G \frac{\chi_{V_\alpha}}{|V_\alpha|} dt = 1$. Thus by Theorem 2, directly we get easily that

$$L^1(G) * L^p(G) = L^p(G), \text{ if } 1 < p < \infty$$

if $p = \infty$, we choose $C_0(G)$, the space of continuous functions vanishing at infinite on G , we also have

$$L^1(G) * C_0(G) = C_0(G)$$

Remark1 It is remarkable that not every Banach algebra has a bounded approximate identity. For example, the space

$$A^p(G) = \{f \in L^1(G) \mid \hat{f} \in L^p(\hat{G}), 1 < p \leq \infty\} \quad (\subset L^1(G))$$

with norm defined by $\|f\|_{A^p} = \|f\|_1 + \|\hat{f}\|_p$ is a *commutative Banach algebra* for each p , $1 \leq p < \infty$. But there is an approximate identity $\{e_\alpha\}$ in $L^1(G)$ with *Fourier transform* $\widehat{e_\alpha}$ having compact support in $\hat{G}$ for each α , then $\widehat{e_\alpha} \in L^p(\hat{G})$ shows that $\{e_\alpha\}$ is also an approximate identity of $A^p(G)$, but this system $\{e_\alpha\}$ of approximate identity is not uniform bounded in $A^p(G)$. (cf. Lai [2, p254])

3 Multipliers of Banach Module Homomorphism .

Let A be a *commutative Banach algebra* and X, Y A – *module Banach spaces*. A bounded linear operator $T \in \mathcal{L}(X, Y)$ satisfying

$$(3.1) \quad T(ax) = a(Tx) \text{ for all } a \in A, x \in X,$$

is called a **multiplier** of X to Y under A -module. The space of such multipliers is A – *module homomorphisms* from X to Y and is denoted by

$$(3.2) \quad \mathfrak{M}_A(X, Y) = \text{Hom}_A(X, Y) = \{T \in \mathcal{L}(X, Y) \mid T(ax) = a(Tx), a \in A, x \in X\}.$$

It is a closed subalgebra of $\mathcal{L}(X, Y)$, the space of all bounded linear mappings of X into Y . In particular, if $A = X = Y = L^1(G)$, then the multiplier space $\mathfrak{M}(L^1(G))$ coincides with the expression of isometrically isomorphic relations “ $\cong$ ” as follows.

$$(3.3) \quad \mathfrak{M}(L^1(G)) = \text{Hom}_{L^1(G)}(L^1(G), L^1(G)) \cong (L^1(G), L^1(G)) \cong M_b(G).$$

where $(E(G), F(G))$ stands for the space of all invariant operators commute with translation operator τ_a on the function spaces of $E(G)$ to $F(G)$.

In general, the multiplier space $\text{Hom}_A(X, Y^*)$ was characterized by Rieffel [8] as the following dual space of the module tensor product $X \otimes_A Y$:

$$(3.4) \quad \text{Hom}_A(X, Y^*) \cong (X \otimes_A Y)^*,$$

where $\otimes_A$ denotes the A – *module tensor product* defined by $X \otimes_A Y = X \hat{\otimes}_\gamma Y / K$. K is the closed linear subspace of the complete projective tensor product space $X \hat{\otimes}_\gamma Y$ generating by elements: $ax \otimes y - x \otimes ay$, for $a \in A, x \in X, y \in Y$. Here $\hat{\otimes}_\gamma$ is the completion of the algebra tensor $X \otimes Y$ under the largest reasonable cross norm γ , and

$$X \otimes Y = \{u = \sum_i x_i \otimes y_i \mid \sum_i \|x_i\|_X \|y_i\|_Y < \infty\}$$

with norm $\gamma(u) \equiv |||u||| = \inf_u \sum_i \|x_i \otimes y_i\| = \inf_u \sum_i \|x_i\|_X \|y_i\|_Y$, $\inf_u$ means that the infimum is taken by all representations of $u = \sum_i x_i \otimes y_i$ in $X \otimes Y$.

The *reasonable crossnorm* means that

$$u \in X \otimes Y, u = x \otimes y \text{ implies } \|u\| = \|x \otimes y\| = \|x\|_X \|y\|_Y;$$

$$\text{and } u = \sum_i x_i \otimes y_i, \|u\| = \inf \sum_i \|x_i\|_X \|y_i\|_Y.$$

Note that a bounded linear operator $T \in \text{Hom}_A(X, Y^*)$ in (3.4) corresponding a continuous linear functional ψ on $X \otimes_A Y$ is given by

$$(Tx)(y) = \psi(x \otimes y) \text{ for all } x \in X, y \in Y.$$

Here $\text{Hom}_A(X, Y^*) = \mathcal{M}_A(X, Y^*)$ is the space of all A -module homomorphisms from X to Y^* , the topological dual of Y , that is, each $T \in \text{Hom}_A(X, Y^*)$ satisfies

$$T(ax) = a(Tx) \text{ for all } a \in A, x \in X, Tx \in Y^*.$$

where T is a bounded linear operator from X to Y^* ; $X \otimes_A Y$ denotes the A -module tensor product space of X and Y .

There are some known results in scalar-valued function space of $L^1(G)$ -module by convolution. We state three typical $L^1(G)$ -module multiplier problems as follows.

Theorem 3. (i) $\text{Hom}_G(L^1(G), L^1(G)) \cong M_b(G)$, (by Theorem 1, (iii) $\iff$ (i))
where $\text{Hom}_G = \text{Hom}_{L^1(G)}$, and $M_b(G)$ is the space of all bounded regular Borel measures on G .

(ii) $\text{Hom}_G(L^1(G), L^p(G)) \cong (L^1(G) \otimes_G L^q(G))^* = (L^q(G))^* = L^p(G)$,
for $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ where $\otimes_G = \otimes_{L^1(G)}$.

(iii) $\text{Hom}_G(L^p(G), L^p(G)) \cong (L^p(G) \otimes_G L^q(G))^* \cong S_p(G)^*$,
where $S_p(G)$ is a Banach algebra generated by

$$\{u = \sum_i f_i g_i : f_i \in L^p(G), g_i \in L^q(G), \sum_i \|f_i\|_p \|g_i\|_q < \infty\}$$

under pointwise product and the norm is defined by (cf. Larsen [7])

$$|||u||| = \inf \left\{ \sum_i \|f_i\|_p \|g_i\|_q : u = \sum_i f_i \cdot g_i \in S_p(G) \right\}.$$

4 Multipliers of Banach-valued Functions on G .

Let A be a *commutative semi-simple Banach algebra* with bounded approximate identity. Assume X is on A -*module Banach space*. It is not hard to prove that $L^1(G, A) = L^1(G) \widehat{\otimes}_\gamma A$. Since both $L^1(G)$ and A have bounded approximate identity, thus $L^1(G, A)$ is a *commutative Banach algebra* with bounded approximate identity. By Theorem 2

$$L^1(G, A) * L^p(G, X) = L^p(G, X), \quad 1 < p < \infty$$

Denote by

$$L^1(G, A) = \{f : G \longrightarrow A \mid f \text{ is measurable and is Bochner integrable on } G\}$$

Then $L^1(G, A)$ is a *commutative Banach algebra*, under convolution.

Actually

$$\|f * g(t)\|_A \leq \int_G \|f(s-t)\|_A \|g(s)\|_A ds = \|g\|_1 \int_G \|f(s-t)\|_A ds = \|g\|_1 \|f\|_1,$$

$$\|f * g\|_1 = \int_G \|f * g(t)\|_A dt \leq \|g\|_1 \int_G \|f(s-t)\|_A dt \leq \|g\|_1 \|f\|_1.$$

Denote by

$$L_X^p = \{f : G \longrightarrow X \mid f \text{ is measurable and } \|f(\cdot)\|_X \in L^p(G)\}, \quad 1 \leq p < \infty,$$

$$\|f\|_p = \left(\int_G \|f(t)\|_X^p dt \right)^{\frac{1}{p}}, \quad \text{for } f \in L_X^p, \quad 1 \leq p < \infty \quad (2.1)$$

$$\text{and for } p = \infty, \quad \|f\|_\infty = \operatorname{esssup}_{t \in G} \|f(t)\|_X \quad \text{for } f \in L_X^\infty \quad (2.2)$$

Show that $L_X^p, 1 \leq p \leq \infty$ are *Banach spaces* with the norm $\|f\|_p, 1 \leq p < \infty$, as (2.1) and if $p = \infty$, the norm is taken $\|\cdot\|_\infty$ as (2.2). If $X = \mathbb{C}$, the complex numbers, then

$$L_X^p = L^p = L^p(G), \quad 1 \leq p \leq \infty.$$

If X and Y are A -*module Banach space*, the multiplier space of X to Y is given by

$$\operatorname{Hom}_A(X, Y) = \{T \in \mathfrak{L}(X, Y) \mid T(ax) = aT(x), \quad a \in A, \quad x \in X\}.$$

Recall [8], Rieffel characterized the homomorphism module multiplier is represented by the dual space of module tensor product as the following form:

$$Hom_A(X, Y^*) \cong (X \otimes_A Y)^* \text{ or } Hom_A(X, Y) \cong (X \otimes_A Y^*)^* \text{ (if } Y \text{ is reflexive)}$$

where $\otimes_A$ is namely module tensor product of X into Y^* or of X into Y and Z^* denotes the dual space of the Banach space Z . The space $\otimes_A$ is the complete projective tensor product $X \widehat{\otimes}_\gamma Y^*$ quotients by K , that is, $X \otimes_A Y = X \widehat{\otimes}_\gamma Y / K$.

Here K is the closed linear subspace of the projective tensor product space $X \widehat{\otimes}_\gamma Y$ generated by the elements $ax \otimes y - x \otimes ay$; for $a \in A, x \in X, y \in Y$

and $X \otimes_\gamma Y$ is the completion of the algebra tensor $x \otimes y$ under the γ -norm, and

$$X \otimes Y = \{u = \sum_i x_i \otimes y_i \mid x_i \in X, y_i \in Y, \sum_i \|x_i\| \|y_i\| < \infty\}$$

$$\begin{aligned} \gamma(u) &= \inf_u \left\{ \sum_i \|x_i\| \|y_i\| \mid u = \sum_i x_i \otimes y_i \in Y \right\} \\ &= \|u\| = \inf_u \sum_i \|x_i \otimes y_i\|, x_i \in X, y_i \in Y = \inf_u \sum_i \|x_i\|_X \|y_i\|_Y \end{aligned}$$

where $\inf_u$ means that the infimum is taken by all representations of $u = \sum_i x_i \otimes y_i$ in $X \otimes Y$, and the tensor norm. We state the following Theorem for the characterization of the invariant operators. For detail, we consult Lai [3,4] and [6] cf. also Lai [5].

Theorem 4. *Let X and Y be Banach spaces. Then the following two statements are equivalent.*

- (a) $T \in (L^1(G, Y), L^1(G, X))$ is an invariant operator.
- (b) There exists a unique continuous linear map $L \in \mathcal{L}(Y, M_b(G, X))$ such that $T(f \otimes y) = f * L_y$ for all $f \in L^1(G)$, $y \in Y$.

Moreover, $(L^1(G, Y), L^1(G, X)) \cong \mathcal{L}(Y, M_b(G, X))$.

Theorem 5. *Let A be a commutative semi-simple Banach algebra (not necessarily with identity) and X a Banach A -module. Then*

$$(5.1) \quad Hom_{L^1(G, A)}(L^1(G, A), L^1(G, X)) \cong Hom_A(A, M_b(G, X)).$$

In Lai [6], he showed that an invariant operator is also a multiplier if and only if the A in $L^1(G, A)$ must be scalar space $\mathbb{C}$.

Theorem 6. *Let A be a commutative Banach algebra with identity of norm 1. X be a unit linked, order-free, Banach-module and ρ a faithful representation on X , then each invariant operator $T : L^1(G, A) \rightarrow F(G, X)$ is a multiplier if and only if $A \cong \mathbb{C}$. Here $F(G, X) = L^p(G, X)$ for each p , $1 \leq p \leq \infty$, or $F(G, X) = C_0(G, X)$.*

References

- [1] H.C.Lai, Jin-Chirng Lee and Cheng-Te Liu, "Multipliers of Banach-valued Function Spaces On LCA group", J. Nonlinear Convex Analysis.(May,2015)To appear.
- [2] H.C. Lai, "On some properties of $A^p(G)$ -algebra", *Proc. Japan Acad.*, 45: 572-576, 1969.
- [3] H.C. Lai., "Multipliers for some spaces of Banach algebra-valued functions", *Rocky Mountain J. Math.*, 15(1): 157-166, 1985.
- [4] H.C. Lai, "Multipliers of Banach-valued function spaces", *J. Austral. Math. Soc.*, 39: 51-62, 1985.
- [5] H.C. Lai, "Duality of Banach-valued function spaces and the Radon-Nikodym property", *Acta Math. Hung.*, 47: 45-52, 1986.
- [6] H.C. Lai and T. K. Chang, "Multipliers and translation invariant operators", *Tohoku Math. J.*, 41: 31-41, 1989.
- [7] R. Larsen, *An introduction to the theory of multipliers*, Springer Verlag, Heidelberg, New York, 1971.
- [8] M.A. Rieffel, "Multipliers and tensor product on L^p -spaces of locally compact group", *Studia Math.*, 33: 71-82, 1969.
- [9] U.B. Tewari and M. Dutta and D.P. Vaidya, "Multipliers of group algebras of vector-value function", *Proc. Amer. Math. Soc.*, 81(2): 223-229, 1981.