

Generalization of operator type Shannon inequality and its reverse one

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Abstract. We shall state the following generalization of operator type Shannon inequality and its reverse one as a simple corollary of parametric extensions of Shannon inequality in Hilbert space operators.

Let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H . If $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I$, then

$$\begin{aligned} \sum_{j=1}^n S_2(A_j|B_j) &\geq \left[\sum_{j=1}^n B_j A_j^{-1} B_j \right] \log \left[\sum_{j=1}^n B_j A_j^{-1} B_j \right] \geq \log \left[\sum_{j=1}^n B_j A_j^{-1} B_j \right] \\ &\geq \sum_{j=1}^n S_1(A_j|B_j) \geq 0 \geq \sum_{j=1}^n S(A_j|B_j) \\ &\geq -\log \left[\sum_{j=1}^n A_j B_j^{-1} A_j \right] \geq -\left[\sum_{j=1}^n A_j B_j^{-1} A_j \right] \log \left[\sum_{j=1}^n A_j B_j^{-1} A_j \right] \\ &\geq \sum_{j=1}^n S_{-1}(A_j|B_j) \end{aligned}$$

where $S_q(A|B) = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$ for $A > 0$, $B > 0$ and any real number q and $S(A|B) = S_0(A|B) = A^{\frac{1}{2}} (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$ which is the relative operator entropy of $A > 0$ and $B > 0$.

Our results can be considered as parametric extensions of the following celebrated Shannon inequality ([3],[5] and [233 p ,1]) which is very useful and so famous in information theory. Let $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ be two probability vectors. Then

$$0 \geq \sum_{j=1}^n a_j \log b_j - \sum_{j=1}^n a_j \log a_j \text{ (see inequalities (2.4) of Corollary 2.4).}$$

§1 Introduction

First the Shannon inequality asserts: Let $\{a_1, a_2, \dots, a_n\}$ and $\{b_1, b_2, \dots, b_n\}$ be two probability vectors. Then

$$(1.1) \quad 0 \geq \sum_{j=1}^n a_j \log \frac{b_j}{a_j}.$$

We remark that $0 \geq \sum_{j=1}^n a_j \log \frac{b_j}{a_j}$ in (1.1) is equivalent to $D = \sum_{j=1}^n a_j \log \frac{a_j}{b_j} \geq 0$ which is the original number type Shannon inequality and this D is called “divergence” in [3] and [5].

In this paper we shall state parametric extensions of Shannon inequality and its reverse one in Hilbert space operators.

A bounded linear operator T on a Hilbert space H is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is invertible and positive.

Definition 1.1. $S_q(A|B)$ for $A > 0$, $B > 0$ and any real number q is defined by

$$S_q(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^q(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

We recall that $S_0(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = S(A|B)$ is the relative operator entropy in [2] and $S(A|I) = -A \log A$ is the usual operator entropy in [4].

Definition 1.2. $A\sharp_q B$ for $A > 0$ and $B > 0$ and any real number q is defined by

$$A\sharp_q B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^qA^{\frac{1}{2}}$$

and $A\sharp_p B$ for $p \in [0, 1]$ just coincides with $A\sharp_p B$ which is well known as p -power mean.

We remark that $S_1(A|B) = -S(B|A)$ and moreover $S_q(A|B) = -S_{1-q}(B|A)$ for any q .

Following after Definition 1.1, The original Shannon inequality can be expressed as follows:

$$0 \geq \sum_{j=1}^n a_j \log \frac{b_j}{a_j} = \sum_{j=1}^n a_j^{\frac{1}{2}} (\log a_j^{-\frac{1}{2}} b_j a_j^{-\frac{1}{2}}) a_j^{\frac{1}{2}} = \sum_{j=1}^n S(a_j|b_j).$$

Consequently $0 \geq \sum_{j=1}^n S(a_j|b_j)$ in the original Shannon inequality can be extended to

$0 \geq \sum_{j=1}^n S(A_j|B_j)$ in operator version case (2.4) of Corollary 2.4, so that the form of (1.1)

is convenient for operator type extension. We can summarize the following contrast:

*The original Shannon inequality
and its reverse one*

$$0 \geq \sum_{j=1}^n a_j \log \frac{b_j}{a_j} \geq -\log \sum_{j=1}^n \frac{a_j^2}{b_j}.$$

for $a_j, b_j > 0$ with $1 = \sum_{j=1}^n a_j = \sum_{j=1}^n b_j$.

*The operator version Shannon inequality
and its reverse one*

$$0 \geq \sum_{j=1}^n S(A_j|B_j) \geq -\log \sum_{j=1}^n A_j B_j^{-1} A_j.$$

for $A_j, B_j > 0$ with $I = \sum_{j=1}^n A_j = \sum_{j=1}^n B_j$.

§2 Parametric extensions of operator reverse type Shannon inequality derived from two operator concave functions $f_1(t) = \log t$ and $f_2(t) = -t \log t$

Firstly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function $f(t) = \log t$.

Theorem 2.1. Let $p \in [0, 1]$ and also let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^n A_j \sharp_p B_j \leq I$, where I means the identity operator on H . Then

$$\begin{aligned} (2.1) \quad & \log \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0 \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] - \log t_0 \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \\ & \geq \sum_{j=1}^n S_p(A_j|B_j) \\ & \geq -\log \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + t_0 \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \right] + \log t_0 \left(I - \sum_{j=1}^n A_j \sharp_p B_j \right) \end{aligned}$$

for fixed real number $t_0 > 0$, where $S_p(A|B)$ is defined in Definition 1.1 and $A \natural_q B$ is defined in Definition 1.2.

Secondly we shall state the following parametric extensions of Shannon inequality and its reverse one in Hilbert space operators derived from an operator concave function $f(t) = -t \log t$.

Theorem 2.2. Let $p \in [0, 1]$ and also let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^n A_j \sharp_p B_j \leq I$, where I means the identity operator on H . Then

$$\begin{aligned}
(2.2) \quad & \sum_{j=1}^n S_{p+1}(A_j|B_j) \\
& \geq \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0(I - \sum_{j=1}^n A_j \natural_p B_j) \right] \log \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + t_0(I - \sum_{j=1}^n A_j \natural_p B_j) \right] \\
& \quad - t_0 \log t_0(I - \sum_{j=1}^n A_j \natural_p B_j) \quad \text{for fixed real number } t_0 > 0,
\end{aligned}$$

and

$$\begin{aligned}
(2.2') \quad & \sum_{j=1}^n S_{p-1}(A_j|B_j) \\
& \leq - \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + t_0(I - \sum_{j=1}^n A_j \natural_p B_j) \right] \log \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + t_0(I - \sum_{j=1}^n A_j \natural_p B_j) \right] \\
& \quad + t_0 \log t_0(I - \sum_{j=1}^n A_j \natural_p B_j) \quad \text{for fixed real number } t_0 > 0,
\end{aligned}$$

where $S_q(A|B)$ is defined in Definition 1.1 and $A \natural_q B$ is defined in Definition 1.2.

We shall state the following result which can be shown by combining Theorem 2.1 with Theorem 2.2.

Corollary 2.3. *Let $p \in [0, 1]$ and also let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H such that $\sum_{j=1}^n A_j \natural_p B_j \leq I$, where I means the identity operator on H . Then*

$$\begin{aligned}
(2.3) \quad & \sum_{j=1}^n S_{p+1}(A_j|B_j) \\
& \geq \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + (I - \sum_{j=1}^n A_j \natural_p B_j) \right] \log \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + (I - \sum_{j=1}^n A_j \natural_p B_j) \right] \\
& \geq \log \left[\sum_{j=1}^n (A_j \natural_{p+1} B_j) + (I - \sum_{j=1}^n A_j \natural_p B_j) \right] \\
& \geq \sum_{j=1}^n S_p(A_j|B_j) \\
& \geq - \log \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + (I - \sum_{j=1}^n A_j \natural_p B_j) \right]
\end{aligned}$$

$$\begin{aligned}
&\geq -\left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + (I - \sum_{j=1}^n A_j \sharp_p B_j)\right] \log \left[\sum_{j=1}^n (A_j \natural_{p-1} B_j) + (I - \sum_{j=1}^n A_j \sharp_p B_j)\right] \\
&\geq \sum_{j=1}^n S_{p-1}(A_j|B_j)
\end{aligned}$$

where $S_q(A|B)$ is defined in Definition 1.1 and $A \natural_q B$ is defined in Definition 1.2.

Corollary 2.3 easily implies the following result which can be considered as *operator version of Shannon inequality and its reverse one*.

Corollary 2.4. Let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space H . If $\sum_{j=1}^n A_j = \sum_{j=1}^n B_j = I$, then

$$\begin{aligned}
(2.4) \quad \sum_{j=1}^n S_2(A_j|B_j) &\geq \left[\sum_{j=1}^n B_j A_j^{-1} B_j\right] \log \left[\sum_{j=1}^n B_j A_j^{-1} B_j\right] \geq \log \left[\sum_{j=1}^n B_j A_j^{-1} B_j\right] \\
&\geq \sum_{j=1}^n S_1(A_j|B_j) \geq 0 \geq \sum_{j=1}^n S(A_j|B_j) \\
&\geq -\log \left[\sum_{j=1}^n A_j B_j^{-1} A_j\right] \geq -\left[\sum_{j=1}^n A_j B_j^{-1} A_j\right] \log \left[\sum_{j=1}^n A_j B_j^{-1} A_j\right] \\
&\geq \sum_{j=1}^n S_{-1}(A_j|B_j).
\end{aligned}$$

Remark 2.1. We recall $S_q(A|B)$ for $A > 0$, $B > 0$ and any real number q as follows:

$$S_q(A|B) = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q (\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

By an easy calculation we have

$$\frac{d}{dq} [S_q(A|B)] = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^q [\log A^{-\frac{1}{2}} B A^{-\frac{1}{2}}]^2 A^{\frac{1}{2}} \geq 0,$$

so that $S_q(A|B)$ is an increasing function of q , and it is interesting to point out that the decreasing order of the positions of $\sum_{j=1}^n S_2(A_j|B_j)$, $\sum_{j=1}^n S_1(A_j|B_j)$, $\sum_{j=1}^n S(A_j|B_j)$, and $\sum_{j=1}^n S_{-1}(A_j|B_j)$ in (2.4) of Corollary 2.4 is quite reasonable since $\sum_{j=1}^n S(A_j|B_j) = \sum_{j=1}^n S_0(A_j|B_j)$.

This paper will appear elsewhere with complete proofs.

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