

ニューマークのベータ法の安定性について (On the Stability of Newmark's β method)

CHIBA, Fumihiko* and KAKO, Takashi†
(千葉文浩) (加古 孝)

Abstract

For the second order evolution equation in time, we consider Newmark's β method without imposing the assumption of the Rayleigh damping for the dissipation term. We derive the trinomial recurrence relation of Newmark's method which is due to Chaix-Leleux, and give a proof of stability of the scheme for the homogeneous equation by an energy method.

1. The second order evolution equation and Newmark's method

In a finite dimensional real Hilbert space $\mathcal{H}$, we consider the following second order differential equation in time t :

$$\frac{d^2}{dt^2}u(t) + C\frac{d}{dt}u(t) + Ku(t) = f(t), \quad u(t) \in \mathcal{H}, \quad (1)$$

where C and K are non-negative linear operators on $\mathcal{H}$ and f is a given function: $f: [0, \infty) \rightarrow \mathcal{H}$.

Let τ be a time step, $U(t)$ be a difference approximation of $u(t)$, $V(t)$ be a difference approximation of $\frac{d}{dt}u(t)$, $A(t)$ be a difference approximation of $\frac{d^2}{dt^2}u(t)$, and β and γ be fixed real numbers. Then we can write Newmark's method[2] as follows:

$$\begin{cases} A(t) + CV(t) + KU(t) = f(t) \\ U(t + \tau) = U(t) + \tau V(t) + \frac{1}{2}\tau^2 A(t) + \beta\tau^2(A(t + \tau) - A(t)) \\ V(t + \tau) = V(t) + \tau A(t) + \gamma\tau(A(t + \tau) - A(t)). \end{cases} \quad (2)$$

The case $\gamma = \frac{1}{2}$ is the standard Newmark's β method.

2. The iteration scheme of Newmark's method

The iteration scheme of Newmark's method (2) for the equation (1) is written as follows:

- I. Compute $A(t)$ from initial data $U(t)$ and $V(t)$ by using (1):

$$A(t) = f(t) - (C V(t) + K U(t)).$$

- II. Compute $A(t + \tau)$ from $f(t + \tau)$, $U(t)$, $V(t)$ and $A(t)$:

$$\begin{aligned} A(t + \tau) &= (I + \gamma\tau C + \beta\tau^2 K)^{-1} \\ &\quad \times \{-KU(t) - (C + \tau K)V(t) \\ &\quad + (-\tau C + \gamma\tau C - \frac{1}{2}\tau^2 K + \beta\tau^2 K)A(t) + f(t + \tau)\}, \end{aligned}$$

where I is the identity operator.

*Doctor Course Student, Dep. Computer Science and Information Mathematics, The University of Electro-Communications, chiba@im.uec.ac.jp

†Dep. Computer Science and Information Mathematics, The University of Electro-Communications, kako@im.uec.ac.jp

- III. Compute $V(t + \tau)$ from $V(t)$, $A(t)$ and $A(t + \tau)$:

$$V(t + \tau) = V(t) + \tau A(t) + \gamma \tau (A(t + \tau) - A(t)).$$

- IV. Compute $U(t + \tau)$ from $U(t)$, $V(t)$, $A(t)$ and $A(t + \tau)$:

$$U(t + \tau) = U(t) + \tau V(t) + \frac{1}{2} \tau^2 A(t) + \beta \tau^2 (A(t + \tau) - A(t)).$$

- V. Replace t by $t + \tau$, and return to II.

3. The trinomial recurrence relation of Newmark's method

We derive a trinomial recurrence relation for $U(t - \tau)$, $U(t)$ and $U(t + \tau)$ from the following system of equations:

$$\begin{cases} A(t) + CV(t) + KU(t) = f(t) \\ A(t + \tau) + CV(t + \tau) + KU(t + \tau) = f(t + \tau) \\ U(t + \tau) = U(t) + \tau V(t) + \frac{1}{2} \tau^2 A(t) + \beta \tau^2 (A(t + \tau) - A(t)) \\ V(t + \tau) = V(t) + \tau A(t) + \gamma \tau (A(t + \tau) - A(t)). \end{cases} \quad (3)$$

3.1 Derivation of the trinomial recurrence relation of Newmark's method

We eliminate $A(t)$, $A(t + \tau)$ and $V(t + \tau)$ from (3) and get an equation for $U(t)$, $U(t + \tau)$ and $V(t)$. Next we eliminate $A(t)$, $A(t + \tau)$ and $V(t)$ from (3) and substitute $t - \tau$ for t , and get another equation for $U(t - \tau)$, $U(t)$ and $V(t)$. Lastly we obtain the following equation eliminating $V(t)$ from these two equations:

$$\begin{aligned} & (I + \gamma \tau C + \beta \tau^2 K)U(t + \tau) + \{-2I + \tau(1 - 2\gamma)C + \frac{1}{2} \tau^2(1 - 4\beta + 2\gamma)K\}U(t) \\ & + \{I + \tau(-1 + \gamma)C + \frac{1}{2} \tau^2(1 + 2\beta - 2\gamma)K\}U(t - \tau) \\ & = \beta \tau^2 f(t + \tau) + \frac{1}{2} \tau^2(1 - 4\beta + 2\gamma)f(t) + \frac{1}{2} \tau^2(1 + 2\beta - 2\gamma)f(t - \tau). \end{aligned} \quad (4)$$

In this calculation, we must take care of the non-commutativity between C and K . In the case $\gamma = \frac{1}{2}$, we get a recurrence relation for the standard Newmark's β method:

$$\begin{aligned} & (I + \frac{1}{2} \tau C + \beta \tau^2 K)U(t + \tau) + \{-2I + \tau^2(1 - 2\beta)K\}U(t) + (I - \frac{1}{2} \tau C + \beta \tau^2 K)U(t - \tau) \\ & = \beta \tau^2 f(t + \tau) + \tau^2(1 - 2\beta)f(t) + \beta \tau^2 f(t - \tau). \end{aligned} \quad (5)$$

3.2 Representation by difference operators

We define difference operators with time step τ as follows:

$$\begin{aligned} D_\tau U(t) & \equiv \frac{1}{\tau}(U(t + \tau) - U(t)) \sim \frac{d}{dt}u(t + \tau/2), \\ D_{\bar{\tau}} U(t) & \equiv \frac{1}{\tau}(U(t) - U(t - \tau)) \sim \frac{d}{dt}u(t - \tau/2), \\ D_{\tau\bar{\tau}} U(t) & \equiv \frac{1}{\tau^2}(U(t + \tau) - 2U(t) + U(t - \tau)) \sim \frac{d^2}{dt^2}u(t), \\ \frac{1}{2}(D_\tau + D_{\bar{\tau}})U(t) & \equiv \frac{1}{2\tau}(U(t + \tau) - U(t - \tau)) \sim \frac{d}{dt}u(t). \end{aligned}$$

Using these definitions, we obtain the trinomial recurrence relation for $U(t - \tau)$, $U(t)$ and $U(t + \tau)$ as follows:

$$\begin{aligned} & (I + \beta\tau^2 K)D_{\tau\bar{\tau}}U(t) + \gamma CD_{\tau}U(t) + \{(1 - \gamma)C + \tau(\gamma - \frac{1}{2})K\}D_{\bar{\tau}}U(t) + KU(t) \\ &= \{I + \tau(\gamma - \frac{1}{2})D_{\bar{\tau}} + \beta\tau^2 D_{\tau\bar{\tau}}\}f(t). \end{aligned} \quad (6)$$

Especially, in the case $\gamma = \frac{1}{2}$, we have (see [1],[3] for the case $C \equiv 0$):

$$(I + \beta\tau^2 K)D_{\tau\bar{\tau}}U(t) + \frac{1}{2}C(D_{\tau} + D_{\bar{\tau}})U(t) + KU(t) = (I + \beta\tau^2 D_{\tau\bar{\tau}})f(t). \quad (7)$$

4. Stability analysis by energy method

We consider Newmark's β method for the homogeneous equation: $f(t) \equiv 0$ in (1), and derive a stability estimate for the approximate solution of (7) by means of an 'energy method'.

We take an inner-product between (7) and $\frac{1}{2}(D_{\tau} + D_{\bar{\tau}})U(t)$:

$$\begin{aligned} & ((I + \beta\tau^2 K)D_{\tau\bar{\tau}}U(t), \frac{1}{2}(D_{\tau} + D_{\bar{\tau}})U(t)) + (\frac{1}{2}C(D_{\tau} + D_{\bar{\tau}})U(t), \frac{1}{2}(D_{\tau} + D_{\bar{\tau}})U(t)) \\ & \quad + (KU(t), \frac{1}{2}(D_{\tau} + D_{\bar{\tau}})U(t)) = 0. \end{aligned} \quad (8)$$

Since $C \geq 0$, the second term in the left-hand side of (8) is non-negative. Moving this term to the right-hand side, we have

$$\begin{aligned} & ((I + \beta\tau^2 K)D_{\tau\bar{\tau}}U(t), \frac{1}{2}(D_{\tau} + D_{\bar{\tau}})U(t)) + (KU(t), \frac{1}{2}(D_{\tau} + D_{\bar{\tau}})U(t)) \\ & \quad = -(\frac{1}{2}C(D_{\tau} + D_{\bar{\tau}})U(t), \frac{1}{2}(D_{\tau} + D_{\bar{\tau}})U(t)) \leq 0. \end{aligned}$$

Hence, we get the inequality:

$$((I + \beta\tau^2 K)D_{\tau\bar{\tau}}U(t), \frac{1}{2}(D_{\tau} + D_{\bar{\tau}})U(t)) + (KU(t), \frac{1}{2}(D_{\tau} + D_{\bar{\tau}})U(t)) \leq 0. \quad (9)$$

Multiplying both sides of (9) by $2\tau^3$, we have

$$\begin{aligned} & ((I + \beta\tau^2 K)(U(t + \tau) - 2U(t) + U(t - \tau)), U(t + \tau) - U(t - \tau)) \\ & \quad + (\tau^2 KU(t), U(t + \tau) - U(t - \tau)) \leq 0. \end{aligned}$$

Inserting $U(t) - U(t) = 0$ in the inner-product of the first term in the left-hand side, we get

$$\begin{aligned} & ((I + \beta\tau^2 K)(U(t + \tau) - U(t)), U(t + \tau) - U(t)) \\ & + ((I + \beta\tau^2 K)(U(t + \tau) - U(t)), U(t) - U(t - \tau)) \\ & - ((I + \beta\tau^2 K)(U(t) - U(t - \tau)), U(t + \tau) - U(t)) \\ & - ((I + \beta\tau^2 K)(U(t) - U(t - \tau)), U(t) - U(t - \tau)) \\ & \quad + (\tau^2 KU(t), U(t + \tau) - U(t - \tau)) \leq 0. \end{aligned}$$

Arranging this formula, we obtain the following inequality:

$$\begin{aligned} & ((I + \beta\tau^2 K)(U(t + \tau) - U(t)), U(t + \tau) - U(t)) + (\tau^2 KU(t + \tau), U(t)) \\ & \leq ((I + \beta\tau^2 K)(U(t) - U(t - \tau)), U(t) - U(t - \tau)) + (\tau^2 KU(t), U(t - \tau)). \end{aligned}$$

Dividing both sides of this inequality by τ^2 , we have

$$\begin{aligned} & ((I + \beta\tau^2 K)D_\tau U(t), D_\tau U(t)) + (KU(t + \tau), U(t)) \\ & \leq ((I + \beta\tau^2 K)D_\tau U(t - \tau), D_\tau U(t - \tau)) + (KU(t), U(t - \tau)) \\ & \leq ((I + \beta\tau^2 K)D_\tau U(0), D_\tau U(0)) + (KU(\tau), U(0)). \end{aligned}$$

Using this inequality and the fact that

$$(KU(t + \tau), U(t)) = (KU(t), U(t)) + \tau(KD_\tau U(t), U(t))$$

and $K \geq 0$, we get

$$\|D_\tau U(t)\|^2 + \beta\tau^2 \|K^{1/2} D_\tau U(t)\|^2 + \|K^{1/2} U(t)\|^2 + \tau(K^{1/2} D_\tau U(t), K^{1/2} U(t)) \leq C_0, \quad (10)$$

where

$$\begin{aligned} C_0 &= ((I + \beta\tau^2 K)D_\tau U(0), D_\tau U(0)) + (KU(\tau), U(0)) \\ &= ((I + \beta\tau^2 K)D_\tau U(0), D_\tau U(0)) + (KU(0), U(0)) + \tau(KD_\tau U(0), U(0)) \\ &= \|D_\tau U(0)\|^2 + \beta\tau^2 \|K^{1/2} D_\tau U(0)\|^2 + \|K^{1/2} U(0)\|^2 + \tau(K^{1/2} D_\tau U(0), K^{1/2} U(0)). \end{aligned}$$

If α is a positive real number, from Schwarz's inequality, we get

$$\begin{aligned} |\tau(K^{1/2} D_\tau U(t), K^{1/2} U(t))| &\leq \|\tau K^{1/2} D_\tau U(t)\| \|K^{1/2} U(t)\| \\ &= \alpha \|\tau K^{1/2} D_\tau U(t)\| \times \frac{1}{\alpha} \|K^{1/2} U(t)\| \\ &\leq \frac{1}{2}\alpha^2 \tau^2 \|K^{1/2} D_\tau U(t)\|^2 + \frac{1}{2\alpha^2} \|K^{1/2} U(t)\|^2. \end{aligned} \quad (11)$$

Moving the forth term in the left-hand side of (10) to the right-hand side and using (11), we have

$$\begin{aligned} \|D_\tau U(t)\|^2 + \beta\tau^2 \|K^{1/2} D_\tau U(t)\|^2 + \|K^{1/2} U(t)\|^2 &\leq C_0 - \tau(K^{1/2} D_\tau U(t), K^{1/2} U(t)) \\ &\leq C_0 + |\tau(K^{1/2} D_\tau U(t), K^{1/2} U(t))| \\ &\leq C_0 + \frac{1}{2}\alpha^2 \tau^2 \|K^{1/2} D_\tau U(t)\|^2 + \frac{1}{2\alpha^2} \|K^{1/2} U(t)\|^2. \end{aligned} \quad (12)$$

Finally moving the second and the third terms in the last formula of (12) to the left-hand side, we obtain an energy inequality:

$$\|D_\tau U(t)\|^2 + \tau^2 \left(\beta - \frac{\alpha^2}{2}\right) \|K^{1/2} D_\tau U(t)\|^2 + \left(1 - \frac{1}{2\alpha^2}\right) \|K^{1/2} U(t)\|^2 \leq C_0. \quad (13)$$

Using this inequality, we have the following results.

Theorem 1 In the case $\beta \geq \frac{1}{4}$, we have the stability estimate, with positive constants C_1 and C_2 ,

$$\|U(t)\| \leq C_1 + C_2 t,$$

and in the case $0 \leq \beta < \frac{1}{4}$, if we choose τ such that

$$\tau < \sqrt{\frac{1}{(\frac{1}{4} - \beta) \|K^{1/2}\|^2}},$$

then we have, with positive constants C_3 and C_4 ,

$$\|U(t)\| \leq C_3 + C_4 t,$$

From now on, we show the proof of this theorem. First, we consider the case $\beta \geq \frac{1}{4}$. If we put $\alpha = \sqrt{2\beta}$ in (13), then we have, for $\beta > \frac{1}{4}$, that

$$\|D_\tau U(t)\|^2 + (1 - \frac{1}{4\beta})\|K^{1/2}U(t)\|^2 \leq C_0$$

and

$$\|D_\tau U(t)\|, \|K^{1/2}U(t)\| \leq C_\beta = (1 - \frac{1}{4\beta})^{-1}C_0 < \infty,$$

where C_β is a constant independent of t . Hence, we get

$$\beta > \frac{1}{4} \implies \|D_\tau U(t)\|, \|K^{1/2}U(t)\| \leq C_\beta.$$

And we also obtain that

$$\beta \geq \frac{1}{4} \implies \|D_\tau U(t)\| \leq \sqrt{C_0}.$$

Then recalling the definition:

$$D_\tau U(t) = \frac{1}{\tau}(U(t+\tau) - U(t)),$$

we get

$$\|U(t+\tau) - U(t)\| \leq \sqrt{C_0}\tau,$$

and

$$\|U(t+\tau)\| \leq \|U(t)\| + \sqrt{C_0}\tau \leq \dots \leq \|U(0)\| + \sqrt{C_0}(t+\tau).$$

Putting $C_1 = \|U(0)\|$ and $C_2 = \sqrt{C_0}$, where C_1 is constant independent of τ , we can conclude that

$$\beta \geq \frac{1}{4} \implies \|U(t)\| \leq C_1 + C_2 t. \quad (14)$$

Next, we consider the case $0 \leq \beta < \frac{1}{4}$. Put $\alpha^2 = \frac{1}{2}$ in (13). Then we have

$$\|D_\tau U(t)\|^2 + \tau^2(\beta - \frac{1}{4})\|K^{1/2}D_\tau U(t)\|^2 \leq C_0$$

and

$$\|D_\tau U(t)\|^2 \leq C_0 + \tau^2(\frac{1}{4} - \beta)\|K^{1/2}D_\tau U(t)\|^2. \quad (15)$$

Let $y \in \mathcal{H}$ and $\|K^{1/2}\|$ be the operator norm of $K^{1/2}$, then we have $\|K^{1/2}y\| \leq \|K^{1/2}\|\|y\|$. Applying this inequality to (15), we get

$$\|D_\tau U(t)\|^2 \leq C_0 + \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2\|D_\tau U(t)\|^2$$

and

$$(1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2)\|D_\tau U(t)\|^2 \leq C_0.$$

Noticing the fact that, for $\tau > 0$,

$$0 < 1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2 \iff \tau < \sqrt{\frac{1}{(\frac{1}{4} - \beta)\|K^{1/2}\|^2}},$$

we obtain

$$\tau < \sqrt{\frac{1}{(\frac{1}{4} - \beta)\|K^{1/2}\|^2}} \implies \|D_\tau U(t)\| \leq \sqrt{\frac{C_0}{1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2}},$$

and we obtain:

$$\|U(t)\| \leq C_3 + C_4 t,$$

where

$$C_3 = \|U(0)\|, C_4 = \sqrt{\frac{C_0}{1 - \tau^2(\frac{1}{4} - \beta)\|K^{1/2}\|^2}}.$$

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