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Outer conjugacy problem of orbit preserving transformations

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We report here on the outer conjugacy problem of orbit preserving transformations, which is very closely related to von Neuman algebra theory. Details will be published in [3].

In 1936 F. Murray and J. von Neuman raised a problem of classifying factors which are von Neuman algebras with the trivial center. As they showed, an example of factors can be constructed from an ergodic automorphism on a Lebesgue space, which is so called a cross product von Neuman algebra. A measurable and invertible mapping ϕ from a σ-finite Lebesgue space $(\Omega,\mathcal{B},m)$ onto a σ-finite Lebesgue space $(\Omega',\mathcal{B}',m')$ is called an isomorphism if $m'(\phi(E)) = 0$ if and only if $m(E) = 0$. An isomorphism of Ω onto itself is called an automorphism. Let T be an automorphism of $(\Omega,\mathcal{B}, m)$. The cross product von Neuman algebra $L^\infty(\Omega)\underset{T}{\otimes}Z$ is the weak closure of the linear hull of the sets of operators U and L_f for $f\in L^\infty(\Omega)$ acting on the Hilbert space $L^2(\Omega)\otimes\ell^2(Z)$, defined by the following: for $\xi(\omega,n)\in L^2(\Omega)\otimes\ell^2(Z)$

$$U\xi(\omega,n) = \xi(T^{-1}\omega,n-1)((dmT^{-1}/dm)(\omega))^{1/2}$$

$$L_f(\omega,n) = f(\omega)\xi(\omega,n).$$

In ergodic theory isomorphism problems for automorphisms have been studied. On the other hand operator algebrists consider an isomorphism problem for $*$-automorphisms of a von Neuman algebra. Let M be a von Neuman algebra and α and α'

be $*$-automorphisms of M. It is natural to ask when $*$-automorphisms α and α' are conjugate, i.e. $\beta\alpha\beta^{-1} = \alpha'$ for some $*$-automorphism β of M, or when they are outer conjugate, i.e. $\beta\alpha\beta^{-1} = \alpha'\gamma$ for an inner $*$-automorphism γ of M and a $*$-automorphism β of M. This is called an isomorphism problem in non commutative ergodic theory.

We discuss about this problem on the cross product von Neumann algebra $L^\infty(\Omega) \otimes_T Z$, which is not abelian. For this we consider orbit preserving transformations (o.p.t.) R of T. They are automorphisms of Ω satisfying

$$\{RT^i\omega : i \in Z\} = \{T^i R\omega : i \in Z\} \text{ a.e.}\omega.$$

If $R\omega$ is in $\{T^i\omega : i \in Z\}$ a.e.ω then it is said to be inner. We write

$$N[T] = \{\text{o.p.t.'s of } T\}$$

$$[T] = \{\text{inner o.p.t.'s of } T\}$$

and call them the normalizer group and the full group of T. Every o.p.t. R induces a $*$-automorphism R of the cross product von Neuman algebra $L^\infty(\Omega) \otimes_T Z$ as follows: Let $RTR^{-1}\omega = T^n\omega$ $\omega \in A_n$, where $\{A_n\}_{-\infty<n<\infty}$ is a partition of Ω, then $*$-automorphism R is defined by

$$R : U \longmapsto \sum_{-\infty<n<\infty} U^n L_{\chi_{A_n}},$$

and for $f \in L^\infty(\Omega)$

$$R : L_{f(\omega)} \longmapsto L_{f(R\omega)}.$$

If R is an inner automorphism of T then the $*$-automorphism R is inner. Because, since $R\omega = T^n\omega$, $\omega \in B_n$ for a partition

$\{B_n\}_{-\infty<n<\infty}$ of Ω , we have $R = V \cdot V^{-1}$, where V is the unitary element in $L^\infty(\Omega)\underset{T}{\otimes}Z$ defined by

$$V = \sum_{-\infty<n<\infty} U^n L_{\chi_{B_n}} .$$

What we are going to discuss is the following

Outer conjugacy problem of O.P.T.'s.

We assume that an automorphism T of $(\Omega,\mathcal{B},m)$ is ergodic. R and R' in $N[T]$ are said to be outer conjugate if there is a ϕ in $N[T]$ such that

$$\phi R\phi^{-1} \in R'[T],$$

or equivalently if the cosets $R[T]$ and $R'[T]$ are conjugate in $N[T]/[T]$. We remark that in this case R and R' are outer conjugate as a $*$-automorphism of $L^\infty(\Omega)\underset{T}{\otimes}Z$.

As an invariant for outer conjugacy one can consider the outer period $p_0(R)$ of R in $N[T]$. It is the least positive integer p such that R^p is in $[T]$ if it exists. If otherwise, we define $p_0(R) = 0$ and say that such R is outer aperiodic. Then it is obvious that the outer period $p_0(R)$ is an invariant for the outer conjugacy.

When T has a σ-finite invariant measure μ equivalent to m (in this case we say T is of type II), for R in $N[T]$ the Radon-Nikodym density $(d\mu R/d\mu)(\omega)$ is constant a.e.ω, whih we denote by modR. Of course if the measure μ is finite (in this case we say T is of type II_1), then modR is always 1. If T is of type II, the couple of $p_0(R)$ and modR is a complete invariant for the outer conjugacy, which was proved by A. Connes and W. Krieger[1].

When T has no σ-finite invariant measures equivalent to m (in this case we say T is of type III), a complete invariant for outer conjugacy is still unknown. So we think about the conjugate classes of the quotient group $N[T]/[T]^-$, which has a close connection with the group $N[T]/[T]$, where $[T]^-$ is the closure of $[T]$ with respect to the topology defined by the following: For R in $N[T]$ the open base of R is the family of the sets $\{\phi \in N[T] : \|R\circ f_i - \phi\circ f_i\|_{L^1(m)} < \varepsilon,\ i = 1,2,\ldots n, \text{ and } m(\omega : RT^iR^{-1}\omega \neq \phi T^i\phi^{-1}\omega) < \varepsilon\ \ i = 0,\pm 1,\ldots,\pm n\}$, where $f_i \in L^1(\Omega), \varepsilon > 0$, $R\circ f(\omega) = f(R^{-1}\omega)(dmR^{-1}/dm)(\omega)$, $f \in L^1(\Omega)$. We note that $N[T]$ is a polish group with respect to this topology.

<u>Theorem 1.</u> Let T be an ergodic automorphism of $(\Omega,\mathcal{B},m)$.

(1) If T has a finite invariant measure equivalent to m, then $N[T] = [T]^-$.

(2) If T has a σ-finite infinite invariant measure equivalent to m, or if T does not admit a σ-finite invariant measure then $N[T]/[T]^-$ is topologically isomorphic to the centralizer $c((F_t))$ of the flow $(F_t)_{t\in R}$ which determines the weak equivalence class of T.

Here $C((F_t))$ is the set of all automorphisms commuting with the flow (F_t) and the topology of the centralizer is the relative topology of the weak topology on the set of all automorphisms: Let $(X,\mathcal{F},\mu)$ be the Lebesgue space on which (F_t) acts. For an automorphism U of X, the open base of U is the family of the sets $\{S: S$ an automorphism of X such that $\| U\circ f_i - S\circ f_i \|_{L^1(X)} < \varepsilon \quad i=1,2,\dots,n\}$ $n=1,2,\dots,\varepsilon > 0$, $f_i \in L^1(X)$.

Let us explain about the flow $(F_t)_{t\in R}$. W. Krieger[4] and T. Hamachi-Y. Oka-M. Oshikawa[2] introduced the flow (F_t) associated with a given ergodic automorphism T satisfying that if T on $(\Omega, \mathcal{B}, m)$ and T' on $(\Omega', \mathcal{B}', m')$ are weakly equivalent, i.e. if there exists an isomorphism ψ from Ω onto Ω' such that $\psi[T]\psi^{-1} = [T']$, then the flows (F_t) and (F_t') are isomorphic. Moreover, Krieger proved that this mapping is a one to one and onto mapping from the weak equivalence class of an ergodic automorphism without σ-finite invariant measure to the isomorphism class of an ergodic conservative flow of automorphisms of a Lebesgue space.

It is known that an ergodic automorphism T has a σ-finite invariant measure if and only if the flow (F_t) is the translation, $u \longmapsto u+t$ on R. We note that in this case the isomorphism between the groups $N[T]/[T]^-$ and $C((F_t))$ is given by

$$R \in N[T] \longrightarrow u \longmapsto u+\log(\mathrm{mod} R) \in C((F_t)),$$

where the kernel is $[T]^- = \{R \in N[T]: \mathrm{mod} R = 1\}$.

Thus by this theorem there is a one to one and onto map from the conjugate classes of $N[T]/|T]^-$ to the conjugate classes of $c((F_t))$. This is a partial answer to our problem at the moment.

Next, which group appears as the quotient group $N[T]/[T]^-$? For instance we have

Theorem 2. Let T be an ergodic automorphism without σ-finite invariant measure and $(F_t)_{t\in R}$ be the associated flow. Then $N[T]/[T]^-$ is compact if and only if $(F_t)_{t\in R}$ is measure preserving and has pure point spectrum. In this case $N[T]/[T]^-$ is isomorphic to the character group of the T-set, which is the set of real numbers t such that the cocycle $\exp(it\log(dmT/dm)(\omega))$ is a coboundary for T, i.e. there is a measurable function $\exp(i\xi_t(\omega))$ such that

$$\exp(it\log(dmT/dm)(\omega)) = \exp(i\xi_t(T\omega))/\exp(i\xi_t(\omega))$$

a.e. ω.

Finally it seems to me that the following question is affirmative: Is the couple of outer period and the conjugate class of the centralizer of $(F_t)_{t\in R}$ a complete invariant for the outer conjugacy of T ?

REFERENCES

[1] A. Connes and W. Krieger, Measure space automorphisms, the normalizers of their full groups, and approximate finiteness, J. Functional Analysis 24(1977), 336-352.

[2] T. Hamachi, Y. Oka and M. Osikawa, Flows associated with ergodic non-singular transformation groups, Publ. RIMS, Kyoto Univ., 11(1975), 31-50.

[3] T. Hamachi, The normalizer group of an ergodic automorphism of type III and the commutant of an ergodic flow, to appear

in J. Functional Analysis 40(1981).

[4] W. Krieger, On ergodic flows and isomorphism of factors, Math. Ann. 223(1976),19-70.

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