

POINCARÉ SERIES CONSTRUCTED FROM A WHITTAKER FUNCTION ON $Sp(2; \mathbb{R})$

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1. WHITTAKER FUNCTION

1.1. Structure of Lie group and Lie algebra. Let G be the symplectic group $Sp(2; \mathbb{R})$ realized as

$$G = \{g \in SL_4(\mathbb{R}) \mid {}^t g J g = J\}, \quad \text{with } J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix} \in M_4(\mathbb{R}),$$

where ${}^t g$ denotes the transpose of a matrix g and 1_2 denotes a unit matrix of size 2.

Let $O(4)$ be the orthogonal group of degree 2. Take a maximal compact subgroup $K = G \cap O(4)$. We denote by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebra of G , K , respectively. Let $\theta(X) = -{}^t X$ be a Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$.

We set $\mathfrak{a} = \mathbb{R}H_1 + \mathbb{R}H_2$ with $H_1 = \text{diag}(1, 0, -1, 0)$, $H_2 = \text{diag}(0, 1, 0, -1)$. Then $\mathfrak{a}$ is a maximally Cartan subalgebra of $\mathfrak{g}$ and the restricted root system $\Delta = \Delta(\mathfrak{g}; \mathfrak{a})$ is expressed as $\Delta = \Delta(\mathfrak{g}; \mathfrak{a}) = \{\pm\lambda_1 \pm \lambda_2, \pm 2\lambda_1, \pm 2\lambda_2\}$, where λ_j is the dual of H_j . We choose a positive root system Δ^+ as $\Delta^+ = \{\lambda_1 \pm \lambda_2, 2\lambda_1, 2\lambda_2\}$.

We also denote the corresponding nilpotent subalgebra by $\mathfrak{n} = \sum_{\beta \in \Delta^+} \mathfrak{g}_\beta$. Here $\mathfrak{g}_\beta$ is the root subspace of $\mathfrak{g}$ corresponding to $\beta \in \Delta^+$. Then one obtains an Iwasawa decomposition of $\mathfrak{g}$ and G ; $\mathfrak{g} = \mathfrak{n} + \mathfrak{a} + \mathfrak{k}$, $G = NAK$ with $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$.

1.2. Representation of the maximal compact subgroup. Firstly, we review the parametrization of the finite-dimensional irreducible representations of $SL_2(\mathbb{C})$. Let $\{f_1, f_2\}$ be the standard basis of the vector space $V = V_1 = \mathbb{C} \oplus \mathbb{C}$. Then $GL_2(\mathbb{C})$ acts on V by matrix multiplication. We denote the symmetric tensor space of 2 dimension by $V_d = S^d(V)$. Here $V_0 = \mathbb{C}$. We consider V_d as a $SL_2(\mathbb{C})$ -module by

$$\text{sym}^d(g)(v_1 \otimes v_2 \otimes \cdots \otimes v_d) = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_d.$$

It is well known that all the finite-dimensional irreducible (polynomial) representations of $SL_2(\mathbb{C})$ can be obtained in this way. By Weyl's unitary trick, all irreducible unitary representations of $SU(2)$ are obtained by restriction of sym^d ($d \geq 0$).

The maximal compact subgroup K is isomorphic to the unitary group $U(2)$ of degree 2 by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \rightarrow A + \sqrt{-1}B, \quad \text{for } \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K.$$

For $d, m \in \mathbb{Z}, d \geq 0$, we define a holomorphic representation $(\sigma_{d,m}, V_d)$ of $GL_2(\mathbb{C})$ by $\sigma_{d,m}(g) = \text{sym}^d(g) \otimes \det(g)^m$. Then we know $U(2) = \{\sigma_{d,m}|_{U(2)} \mid d, m \in \mathbb{Z}, d \geq 0\}$. We set $\lambda = (\lambda_1, \lambda_2) = (m + d, m)$ and $\tau_\lambda = \sigma_{d,m}|_{U(2)}$. By the isomorphism between K and $U(2)$, we obtain

$\hat{K} = \{(\tau_\lambda, V_\lambda) \mid \lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}, \lambda_1 \geq \lambda_2\}$. We choose the basis of V_λ as

$$V_\lambda = \left\{ v_k = \frac{n!}{k!(n-k)!} f_1^{\otimes k} \otimes f_2^{\otimes(n-k)} \text{ (symmetric tensor)} \mid 0 \leq k \leq n \right\}_{\mathbb{C}}.$$

1.3. Characters of the unipotent radical. The commutator subgroup $[N, N]$ of N is given by

$$[N, N] = \left\{ \left(\begin{array}{cc|cc} 1 & 0 & n_1 & n_2 \\ 0 & 1 & n_2 & 0 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right) \mid n_1, n_2 \in \mathbb{R} \right\}.$$

Hence a unitary character η of N is written for some constant $\eta_0, \eta_3 \in \mathbb{R}$ as

$$\left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array} \right) \left(\begin{array}{cc|cc} 1 & 0 & n_1 & n_2 \\ 0 & 1 & n_2 & n_3 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right) \mapsto \exp\{\sqrt{-1}(\eta_0 n_0 + \eta_3 n_3)\} \in \mathbb{C}^\times.$$

A unitary character η of N is said to be non-degenerate if $\eta_0 \eta_3 \neq 0$.

1.4. Parametrization of the discrete series. Let us now parametrize the discrete series of $Sp(2; \mathbb{R})$. Take a compact Cartan subalgebra $\mathfrak{h}$ defined by $\mathfrak{h} = \mathbb{R}h_1 \oplus \mathbb{R}h_2$ with $h_1 = X_{13} - X_{31}$, $h_2 = X_{24} - X_{42}$, where the X'_{ij} s are elementary matrices given by $X_{ij} = (\delta_{ip}\delta_{jq})_{1 \leq p, q \leq 4}$, with Kronecker's delta $\delta_{i,p}$, and let $\mathfrak{h}_{\mathbb{C}}$ be its complexification. Then the absolute root system is expressed as

$$\tilde{\Delta} = \Delta(\mathfrak{g}; \mathfrak{h}) = \{\pm(2, 0), \pm(0, 2), \pm(1, 1), \pm(1, -1)\},$$

where by $\beta = (r, s)$, we mean $r = \beta(-\sqrt{-1}h_1)$, $s = \beta(-\sqrt{-1}h_2)$. Let

$$\tilde{\Delta}^+ = \{(2, 0), (0, 2), (1, 1), (1, -1)\}.$$

We write the set of compact positive roots by $\tilde{\Delta}_c^+ = \{(1, -1)\}$. Then there are 4 sets of positive roots $\tilde{\Delta}_J^+$ ($J = I, II, III, IV$) of $(\mathfrak{g}, \mathfrak{h})$ containing $\Delta_c^+(\mathfrak{g}; \mathfrak{h})$ as follows:

$$\tilde{\Delta}_I^+ = \{(2, 0), (1, 1), (0, 2), (1, -1)\}, \quad \tilde{\Delta}_{II}^+ = \{(1, 1), (2, 0), (1, -1), (0, -2)\},$$

$$\tilde{\Delta}_{III}^+ = \{(2, 0), (1, -1), (0, -2), (-1, -1)\}, \quad \tilde{\Delta}_{IV}^+ = \{(1, -1), (0, -2), (-1, -1), (-2, 0)\}.$$

We put $\delta_{G,J} = 2^{-1} \sum_{\beta \in \tilde{\Delta}_J^+} \beta$ (resp. $\delta_K = 2^{-1} \sum_{\beta \in \tilde{\Delta}_c^+} \beta$), the half sum of positive roots (resp. the half sum of compact positive roots). By definition, the space of Harish-Chandra parameters Ξ_c is given by

$$\Xi_c = \{\Lambda \in \mathfrak{h}_{\mathbb{C}}^* \mid \Lambda + \delta_{G,I} \text{ is analytically integral and} \\ \Lambda \text{ is regular and } \tilde{\Delta}^+ \text{-dominant}\}.$$

For each $J = I, II, III, IV$, we set $\Xi_J = \{\Lambda \in \Xi_c \mid \langle \Lambda, \alpha \rangle > 0 \ (\alpha \in \tilde{\Delta}_J^+)\}$. Then Ξ_c is written as a disjoint union $\Xi_c = \coprod_{J=I}^N \Xi_J$.

It is well-known that there exists a bijection from Ξ_c to the set of equivalence classes of discrete series representations of G . Let π_{Λ} be the discrete series representation associated to Λ in Ξ_J , then τ_{λ} ($\lambda = \Lambda + \delta_{G,J} - 2\delta_K$) is the unique minimal K -type of π_{Λ} . We note that for each Λ in Ξ_c , $\lambda = \Lambda + \delta_{G,J} - 2\delta_K$ is called the Blattner parameter. An easy computation implies

$$\Xi_c = \{(\Lambda_1, \Lambda_2) \in \mathbb{Z} \oplus \mathbb{Z} \mid \Lambda_1 \neq 0, \Lambda_2 \neq 0, \Lambda_2 < \Lambda_1, \Lambda_1 + \Lambda_2 \neq 0\}.$$

We note that Ξ_I (resp. Ξ_N) corresponds to the holomorphic (resp. anti-holomorphic) discrete series, and Ξ_{II} and Ξ_{III} corresponds to the large discrete series in the sense of Vogan, [V].

1.5. Characterization of the minimal K -type of a discrete series representation. Let η be a unitary character of N . Then we set

$$C_{\eta}^{\infty}(N \setminus G) = \{\phi : G \rightarrow \mathbb{C}, C^{\infty}\text{-class} \mid \phi(ng) = \eta(n)\phi(g), (n, g) \in N \times G\}.$$

By the right regular action of G , $C_{\eta}^{\infty}(N \setminus G)$ has a structure of smooth G -module. For any finite dimensional K -module (τ, V) , we set

$$C_{\eta, \tau}^{\infty}(N \setminus G/K) = \\ \{F : G \rightarrow V, C^{\infty}\text{-class} \mid F(n g k^{-1}) = \eta(n)\tau(k)F(g), (n, g, k) \in N \times G \times K\}.$$

Let (π_{Λ}, H) be the discrete series representation of G with Harish-Chandra parameter Λ in Ξ_J , ($J = I, II, III, IV$), and denote its associated $(\mathfrak{g}_{\mathbb{C}}, K)$ -module by the same symbol. For W in $Hom_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi_{\Lambda}^*, C_{\eta}^{\infty}(N \setminus G))$, we define F_W in $C_{\eta, \tau_{\lambda}}^{\infty}(N \setminus G/K)$ by

$$W(v^*)(g) = \langle v^*, F_W(g) \rangle, \quad (v^* \in V_{\lambda}^*, g \in G).$$

Here $(\tau_{\lambda}, V_{\lambda})$ denotes the minimal K -type of π_{Λ} and $\langle *, * \rangle$ denotes the canonical pairing on $V_{\lambda}^* \times V_{\lambda}$.

Now let us recall the definition of the Schmid-operator. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$ and $Ad = Ad_{\mathfrak{p}_{\mathbb{C}}}$ be the adjoint representation of K on $\mathfrak{p}_{\mathbb{C}}$. Then we can define a differential operator $\nabla_{\eta, \lambda}$ from $C_{\eta, \tau_{\lambda}}^{\infty}(N \setminus G/K)$ to $C_{\eta, \tau_{\lambda} \otimes Ad}^{\infty}(N \setminus G/K)$

as $\nabla_{\eta,\lambda}F = \sum_i R_{X_i}F(\cdot) \otimes X_i$. Here the set $\{X_i\}_i$ is any fixed orthonormal basis of $\mathfrak{p}$ with respect to the Killing form on $\mathfrak{g}$ and R_XF denotes the right differential of the function F by X in $\mathfrak{g}$ i.e. $R_XF(g) = \frac{d}{dt}F(g \cdot \exp tX)|_{t=0}$. This operator $\nabla_{\eta,\lambda}$ is called the Schmid operator.

Let $(\tau_\lambda^-, V_\lambda^-)$ be the sum of irreducible K -submodules of $V_\lambda \otimes p_{\mathbb{C}}$ with heighest weight of the form $\lambda - \beta$ ($\beta \in \tilde{\Delta}_{J,n}^+$, $J = I, II, III, IV$). Let P_λ be the projection from $V_\lambda \otimes p_{\mathbb{C}}$ to V_λ^- . We define a differential operator from $C_{\eta,\tau_\lambda}^\infty(N \setminus G/K)$ to $C_{\eta,\tau_\lambda^-}^\infty(N \setminus G/K)$ by $\mathcal{D}_{\eta,\lambda}F(g) = P_\lambda(\nabla_{\eta,\lambda}F(g))$ for $F \in C_{\eta,\tau_\lambda}^\infty(N \setminus G/K)$, $g \in G$. Yamashita obtain the following result.

Proposition 1.1 ([Y1] H.Yamashita, Proposition(2.1)). *Let π_Λ be a representation of discrete series with Harish-Chandra parameter $\Lambda \in \Xi_J$ of $Sp(2; \mathbb{R})$. Set $\lambda = \Lambda + \delta_G - 2\delta_K$. Then the linear map*

$$W \in Hom_{\mathfrak{g}_{\mathbb{C}}, K}(\pi_\Lambda^*, C_\eta^\infty(N \setminus G)) \rightarrow F_W \in Ker(\mathcal{D}_{\eta,\lambda})$$

is injective, and if Λ is far from the walls of the Weyl chambers, it is bijective.

1.6. A basis on the Whittaker space on $Sp(2; \mathbb{R})$. By the result of Kostant [Ko], and Vogan [V], if η is non-degenerate, we obtain the expression

$$\dim_{\mathbb{C}} Hom_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi_\Lambda, C_\eta^\infty(N \setminus G)) = \begin{cases} 4, & \text{if } \Lambda \in \Xi_{II} \cup \Xi_{III}, \\ 0, & \text{if } \Lambda \in \Xi_I \cup \Xi_{IV}. \end{cases}$$

Oda obtain the following result.

Theorem 1.1 ([O] Oda).

Let us assume that η is non-degenerate and $\Lambda \in \Xi_{II}$. We choose the basis $V_\lambda = \{v_k \mid 0 \leq k \leq d\}_{\mathbb{C}}$ defined in §4.2. Here $d = \lambda_1 - \lambda_2$. Then

(1) *$F \in Ker \mathcal{D}_{\eta,\lambda}$ if and only if F satisfies the conditions*

$$(1.1) \quad \begin{aligned} (\partial_1 - k)h_{d-k} + \sqrt{-1}\eta_0 h_{d-k-1} &= 0, & \text{for } 0 \leq k \leq d-1, \\ \{\partial_1 \partial_2 + (a_1/a_2)^2 \eta_0^2\} h_d &= 0, \end{aligned}$$

$$(1.2) \quad \{(\partial_1 + \partial_2)^2 + 2(\lambda_2 - 1)(\partial_1 + \partial_2) - 2\lambda_2 + 1 + 4\eta_3 a_2^2 \partial_2\} h_d = 0.$$

Here $\partial_i = \frac{\partial}{\partial a_i}$, $i = 1, 2$ and $\{h_k \mid 0 \leq k \leq d\}$ is determined by

$$F|_A(a) = \sum_{k=0}^d c_{i,k}^{(1)}(a) v_k,$$

$$c_k(a) = a_1^{\lambda_2+1} a_2^{\lambda_1} \left(\frac{a_1}{a_2}\right)^k \exp(\eta_3 a_2^2) h_k(a), \quad \text{for } a \in A, 0 \leq k \leq d.$$

(2) If $\eta_3 < 0$, $\text{Ker } \mathcal{D}_{\eta, \lambda}$ contains the function F such that $h_d(a)$ has the integral representation for $a = \text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \in A$

$$h_d(a) = \int_0^\infty t^{-\lambda_2 + \frac{1}{2}} W_{0, -\lambda_2}(t) \exp\left(\frac{t^2}{32\eta_3 a_2^2} + \frac{8\eta_0^2 \eta_3 a_1^2}{t^2}\right) \frac{dt}{t}.$$

By Theorem 1.1, Oda showed that if $\Lambda \in \Xi_{II} \cup \Xi_{III}$ and η is non-degenerate,

$$\text{Hom}_{(\mathfrak{g}_{\mathbb{C}}, K)}(\pi_\Lambda^*, \mathcal{A}_\eta(N \setminus G)) \cong \begin{cases} \mathbb{C}, & \eta_3 < 0, \\ 0, & \eta_3 > 0. \end{cases}$$

Here we put

$$\mathcal{A}_\eta(N \setminus G) = \{F \in C_\eta^\infty(N \setminus G) \mid K\text{-finite and for any } X \in U(\mathfrak{g}_{\mathbb{C}}) \text{ there exists a constant } C_X > 0 \text{ such that } |F(g)| \leq C_X \text{tr}(^t g g), g \in G\}$$

and $U(\mathfrak{g}_{\mathbb{C}})$ denotes the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

We set for $t \in \mathbb{C}$, $|\arg t| < \pi$,

$$k_{i, \nu}(t) = \begin{cases} I_\nu(\sqrt{t}/2), & \text{if } i = 1, 2, \\ K_\nu(\sqrt{t}/2), & \text{if } i = 3, 4. \end{cases}$$

and for $F \in C_\eta^\infty(N \setminus G/K)$, set $h_k, c_k \in C^\infty(A)$ ($0 \leq k \leq d$) as in Theorem 1.1. Then we obtain the following results.

Theorem 1.2. *Let us assume that η is non-degenerate and $\Lambda \in \Xi_{II}$.*

(1) *$\text{Ker } \mathcal{D}_{\eta, \lambda}$ has the basis $\{F_i \mid 1 \leq i \leq 4\}$ such that $h_{i, d}(a)$ ($1 \leq i \leq 4$) have the integral representations for $a = \text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \in A$*

$$h_{i, d}(a) = \int_{\tilde{C}_i} t^{\frac{1}{2}(1-\lambda_2)} k_{i, -\lambda_2}(t) \exp\left(\frac{t}{32\eta_3 a_2^2} + \frac{8\eta_0^2 \eta_3 a_1^2}{t}\right) \frac{dt}{t}.$$

Here we set the contours $\tilde{C}_i$ ($1 \leq i \leq 4$)

$$\int_{\tilde{C}_i} dt = \begin{cases} \int_C dt, & \text{if } i = 1, 3, \\ \int_0^\infty dt, & \text{if } i = 2, 4, \end{cases}$$

where $\int_C dt$ is the contour integral on C given in Theorem 2.1-(2) and $\int_0^\infty dt$ is the usual integral on $\mathbb{R}_{>0}$.

(2) We set

$$X = \left\{ (k, j) \in \mathbb{Z} \times \mathbb{Z} \mid 0 \leq k \leq \left\lfloor \frac{d-1}{2} \right\rfloor, j = 1, 2, 0 \leq 2k + j \leq d \right\}$$

For $\eta_3 \in \mathbb{R}$ and for any $r, \epsilon_1, \epsilon_2 > 0$, there exist constants $b_{(3)}^{i,k} > 0$ ($1 \leq i \leq 4$, $(k, j) \in X$) such that

$$|c_{i,d-(2k+j)}(a)| \leq b_{(3)}^{i,k} a_1^{1+\lambda_1} a_2^{1-l_i \lambda_2} \left(\frac{a_1}{a_2} \right)^{\alpha_{i,j,k}^{(2)}} \times \exp \left\{ ((-1)^{i+1} |\eta_0| + \epsilon_1) \frac{a_1}{a_2} + (2|\eta_3| + \eta_3 + \epsilon_2) a_2^2 \right\},$$

for $r \geq a_1 > 0$, $a_2 > 0$,

where we set for $1 \leq i \leq 4$ and $(k, j) \in X$

$$\alpha_{i,j,k}^{(2)} = \begin{cases} 1 - (2k + j), & \text{if } i = 2, 4 \text{ and } 1 \leq 2k + j \leq d, \\ 0, & \text{otherwise.} \end{cases}$$

(3) If $\eta_3 < 0$, then for any $r, \epsilon_1, \epsilon_2 > 0$, there exist constants $b_{(4)}^{i,j,k}$ such that

$$|c_{i,d-(2k+j)}(a)| \leq b_{(4)}^{i,j,k} a_1^{1+\lambda_1} a_2^{1-l_i \lambda_2} \left(\frac{a_1}{a_2} \right)^{\beta_{i,j,k}^{(2)}(\epsilon_1)} \times \exp \left\{ ((-1)^{i+1} |\eta_0| + \epsilon_2) \frac{a_1}{a_2} - l_i \eta_3 a_2^2 \right\},$$

for $r \geq a_1 > 0$, $a_2 > 0$,

where we set for each $1 \leq i \leq 4$ and $0 \leq k \leq d$ and any fixed $\epsilon > 0$

$$\beta_{i,j,k}^{(2)}(\epsilon_1) = \begin{cases} 1 - (2k + j) - \epsilon_1, & \text{if } i = 2, 4 \text{ and } 1 \leq 2k + j \leq d, \\ 1 - 2\lambda_2 - (2k + j) - \epsilon_1, & \text{if } i = 1, 3 \text{ and } -2\lambda_2 \leq 2k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1. Firstly F_i is defined as a linear combination of the series solutoin.

Then from Theorem1.2 we obtain the folloing result.

Corollary 1.1. If $i = 1, 3$, for any $r, \epsilon_1, \epsilon_2 > 0$, there exist constants b_i such that

$$|c_{i,k}(a)| \leq b_i a_1^{1+\lambda_1} a_2^{1-\lambda_2} \exp \left\{ (|\eta_0| + \epsilon_1) \frac{a_1}{a_2} + (2|\eta_3| + \eta_3 + \epsilon_2) a_2^2 \right\}$$

for $r \geq a_1 > 0$, $a_2 > 0$.

2. POINCARÉ SERIES

We assume $\eta_3 < 0$.

2.1. The Convergence of the Poincaré series. We denote by α_1, α_2 the functions

$$\alpha_i(g) = a_i, \quad (i = 1, 2),$$

$$\text{for } g = n \cdot \text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \cdot k, \quad n \in N, k \in K, a_1, a_2 > 0$$

and denote by Γ the group $Sp(2; \mathbb{Z})$.

Then we know the following result.

Lemma 2.1 ([?] B.Diehl). *If $\Re(s_2) > 2$ and $\Re(s_1) > \Re(s_2) + 2$, then the sum.*

$$\sum_{N \cap \Gamma \backslash \Gamma} \alpha_1(\gamma g)^{s_1} \alpha_2(\gamma g)^{s_2}$$

is absolutely convergent.

We set $\eta'_i = \frac{\eta_i}{2\pi}$ $i = 0, 3$. Then we define the following functions.

Definition 2.1. For $s_1, s_2 \in \mathbb{C}$, we define the function $f_i^{s_1, s_2}$ ($1 \leq i \leq 4$) by

$$f_i^{s_1, s_2}(g) = \exp \left\{ - \left(s_1 |\eta_0| \frac{a_1}{a_2} + s_2 |\eta_3| a_2^2 \right) \right\} F_i(g), \quad \text{for } g \in G$$

For $\eta'_0, \eta'_3 \in \mathbb{Z}$, we define the Poincaré series $P_{s_1, s_2}(g)$ by

$$P_{s_1, s_2}(g) = \sum_{P \cap \Gamma \backslash \Gamma} f_1^{s_1, s_2}(\gamma g)$$

Then we obtain the following result from Lemma 2.1.

Theorem 2.1. *For $\Lambda_2 < -1, \Lambda_1 + \Lambda_2 > 1$ and $\Re(s_i) > 1$ ($i = 1, 2$), the Poincaré series $P_{s_1, s_2}(g)$ is absolutely convergent.*

2.2. The Fourier coefficients. We investigate the Fourier coefficients of $P_{s_1, s_2}(g)$ with respect to N . Here we consider only in the case of unitary character of N .

Let W be the Weyl group of G ,

$$W = \{w_i \mid 0 \leq i \leq 7\},$$

where we put

$$w_1 = \left(\begin{array}{c|c} & 1 \\ \hline 1 & \\ -1 & \\ \hline & 1 \end{array} \right), w_2 = \left(\begin{array}{c|c} 1 & \\ \hline & 1 \\ -1 & \\ \hline & \end{array} \right), w_3 = \left(\begin{array}{c|c} & 1 \\ \hline -1 & \\ & \\ \hline & 1 \\ -1 & \end{array} \right),$$

$$w_4 = w_1 w_3, w_5 = w_2 w_3, w_6 = w_1 w_2 w_3, w_7 = w_1 w_2.$$

We set $M = Z_K(\mathfrak{a})$, $\Gamma(w) = \Gamma \cap MNAwN$ for $w \in W$. Let for $t_0, t_3 \in \mathbb{R}$,

$$\eta_{t_0, t_3} : n \in N \rightarrow \exp\{2\pi\sqrt{-1}(t_0n_0 + t_3n_3)\} \in \mathbb{C}^\times$$

be a unitary character of N . For $(g, \gamma) \in G \times \Gamma$ and $l_0, l_3 \in \mathbb{Z}$, we set

$$\begin{aligned}\Phi^\gamma(g; l_0, l_3) &= \int_{N \cap P\gamma\Gamma \backslash N} F_{s_1, s_2}(\gamma ng) \eta_{l_0, l_3}(n^{-1}) dn, \\ \Phi_w(g; l_0, l_3) &= \sum_{(P \cap \Gamma) \backslash \Gamma(w) / (N \cap \Gamma)} \Phi^\gamma(g; l_0, l_3), \\ b_{l_0, l_3}(g) &= \int_{N \cap \Gamma \backslash N} P_{s_1, s_2}(ng) \eta_{l_0, l_3}(n^{-1}) dn.\end{aligned}$$

where we put $P^\gamma = \gamma^{-1}P\gamma$.

Then we obtain the result.

Theorem 2.2. *We assume that $\eta_3 < 0$ and $\eta'_i = \frac{\eta_i}{2\pi} \in \mathbb{Z}$ ($i = 0, 3$). Then we obtain the following results.*

(1) *For $l_0, l_3 \in \mathbb{Z}$, we have*

$$b_{l_0, l_3}(g) = \sum_{0 \leq i \leq 7} \Phi_{w_i}(g; l_0, l_3).$$

(2) *If $i = 2, 3, 4, 5$, we have*

$$\Phi_{w_i}(g; l_0, l_3) = 0 \quad \text{for } g \in G, l_0, l_3 \in \mathbb{Z}.$$

If $i = 0$, we have

$$\Phi_{w_0}(g; l_0, l_3) = \begin{cases} 0, & \text{if } (l_0, l_3) \neq (\eta'_0, \eta'_3), \\ f_1^{s_1, s_2}(g), & \text{if } (l_0, l_3) = (\eta'_0, \eta'_3). \end{cases}$$

If $i = 1$, there exist constants $C_1(l_0, a(\gamma))$ such that

$$\Phi_{w_1}(g; l_0, l_3) = \begin{cases} 0, & \text{if } \eta'_3\beta^2 - l_3 \in \mathbb{Z} \setminus \{0\}, \\ C_1(l_0, a(\gamma)) \exp(\eta'_0n_0 + \eta'_3n_3 + \eta'_3\beta^2u_3 + l_0u_0) f_2^{s_1, s_2}(g; l_0, \eta'_3\beta(\gamma)^2) \\ \quad \times \begin{cases} 1, & \text{if } \eta'_3\beta^2 - l_3 = 0, \\ \frac{\exp(\eta'_3\beta^2 - l_3) - 1}{\eta'_3\beta^2 - m_3}, & \text{if } \eta'_3\beta^2 - l_3 \in \mathbb{Q} \setminus \mathbb{Z}. \end{cases} \end{cases}$$

If $i = 6$, there exist constants $C_2(l_3, a(\gamma))$ such that

$$\Phi_{w_6}(g; l_0, l_3) = \begin{cases} 0, & \text{if } \eta'_0 \frac{\alpha}{\beta} - l_0 \in \mathbb{Z} \setminus \{0\}, \\ C_2(l_3, a(\gamma)) \exp \left(\eta'_0 n_0 + \eta'_3 n_3 + \eta'_0 \frac{\alpha}{\beta} u_0 + l_3 u_3 \right) f_3^{s_1, s_2} \left(g; \eta'_0 \frac{\alpha}{\beta}, l_3 \right) \\ \times \begin{cases} 1, & \text{if } \eta'_0 \frac{\alpha}{\beta} - l_0 = 0, \\ \frac{\exp \left(\eta'_0 \frac{\alpha}{\beta} - l_0 \right) - 1}{\eta'_0 \frac{\alpha}{\beta} - l_0}, & \text{if } \eta'_0 \frac{\alpha}{\beta} - l_0 \in \mathbb{Q} \setminus \mathbb{Z}. \end{cases} \end{cases}$$

If $i = 7$, there exist constants $C_3(l_0, l_3, a(\gamma))$ such that

$$\Phi_{w_7}(g; l_0, l_3) = C_3(l_0, l_3, a(\gamma)) \exp(\eta'_0 n_0 + \eta'_3 n_3 + l_0 u_0 + l_3 u_3) f_4^{s_1, s_2}(g; l_0, l_3).$$

Where we know $\gamma \in \Gamma$ has a unique decomposition $\gamma = mna$ ($m \in M, a \in A, n, u \in N$) and put

$$a(\gamma) = a = \text{diag}(\alpha, \beta, \alpha^{-1}, \beta^{-1}), \quad u = \left(\begin{array}{cc|cc} 1 & u_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -u_0 & 1 \end{array} \right) \left(\begin{array}{cc|cc} 1 & 0 & u_1 & u_2 \\ 0 & 1 & u_2 & u_3 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right),$$

$$n = \left(\begin{array}{cc|cc} 1 & n_0 & & \\ & 1 & & \\ \hline & & 1 & \\ & & -n_0 & 1 \end{array} \right) \left(\begin{array}{cc|cc} 1 & 0 & n_1 & n_2 \\ 0 & 1 & n_2 & n_3 \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array} \right).$$

We denote by $f_i^{s_1, s_2}(\cdot; t_0, t_3)$ the function $f_i^{s_1, s_2}(\cdot)$ for the unitary character η_{t_0, t_3} ($t_0, t_3 \in \mathbb{Q}$) of N .

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