

Sufficient conditions for starlikeness

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Abstract. The object of the present paper is to consider a sufficient condition for analytic functions in the open unit disk to be starlike.

1 Introduction.

Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. A function $f(z)$ in A is said to be starlike of order α in U if it satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U).$$

We denote by $S^*(\alpha)$ the subclass of A consisting of all functions $f(z)$ which are starlike of order α in U . We denote by $S^*(0) \equiv S^*$.

Lewandowski, Miller and Zlotkiewicz [1] have shown

Theorem A. If $f(z) \in A$ satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U),$$

then $f(z) \in S^*$.

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Recently, Ramesha, Kumar and Padmanabhan [5] have given

Theorem B. *If $f(z) \in A$ satisfies*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U)$$

for some $\alpha (\alpha \geq 0)$, then $f(z) \in S^$.*

On the other hand, Obradović [4] has proved

Theorem C. *If $f(z) \in A$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < \frac{1}{6} \quad (z \in U),$$

then $f(z) \in S^$.*

Further, more recently, Li and Owa [2] have derived

Theorem D. *If $f(z) \in A$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < \frac{3}{2} \quad (z \in U),$$

then $f(z) \in S^$.*

To derive our theorems, we need the following lemma due to Miller and Mocanu [3].

Lemma. *Let Ω be a set in the complex plane $\mathbb{C}$. Suppose that Φ is a mapping from $\mathbb{C}^2 \times U$ to $\mathbb{C}$ which satisfies $\Phi(ix, y; z) \notin \Omega$ for $z \in U$, and for all real x, y such that $y \leq -(1+x^2)/2$. If the function $p(z)$ is analytic in U with $p(0) = 1$ and $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in U$, then $\operatorname{Re}(p(z)) > 0$ ($z \in U$).*

2 Conditions for starlikeness

In this section, we derive some sufficient conditions for starlikeness, which are the improvements of the previous theorems. Our first result is contained in

Theorem 1. *If $f(z) \in A$ satisfies*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha}{2} \quad (z \in U) \quad (2.1)$$

for some $\alpha (\alpha \geq 0)$, then $f(z) \in S^$.*

Proof. Let us define the analytic function $p(z)$ in U by

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + p_1z + p_2z^2 + \dots \quad (2.2)$$

Making use of the logarithmic differentiations of both sides in (2.2), we know that

$$\frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) = \alpha zp'(z) + \alpha p(z)^2 + (1 - \alpha)p(z). \quad (2.3)$$

Let $\Omega = \{w \in \mathbb{C} : \operatorname{Re}(w) > -\alpha/2\}$ and

$$\Phi(z_1, z_2; z) = \alpha z_2 + \alpha z_1^2 + (1 - \alpha)z_1.$$

Then from (2.1) and (2.3), we have $\Phi(p(z), zp'(z); z) \in \Omega$ for all $z \in U$. Further, we have

$$\begin{aligned} \operatorname{Re} \{ \Phi(ix, y; z) \} &= \alpha y - \alpha x^2 \\ &\leq -\frac{\alpha}{2} - \frac{3}{2}\alpha x^2 \\ &\leq -\frac{\alpha}{2}. \end{aligned}$$

This shows that $\Phi(ix, y; z) \in \Omega$. Therefore, by virtue of Lemma, we conclude that $f(z) \in S^*$.

Letting $\alpha = 1$ in Theorem 1, we have

Corollary 1. If $f(z) \in A$ satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{1}{2} \quad (z \in U),$$

then $f(z) \in S^*$.

Next we derive

Theorem 2. If $f(z) \in A$ satisfies

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha^2}{4}(1-\alpha) \quad (z \in U) \quad (2.4)$$

for some $\alpha (0 \leq \alpha < 2)$, then $f(z) \in S^*(\alpha/2)$.

Proof Define the function $p(z)$ by

$$\frac{zf'(z)}{f(z)} = \left(1 - \frac{\alpha}{2}\right)p(z) + \frac{\alpha}{2} \quad (z \in U). \quad (2.5)$$

Then $p(z)$ is analytic in U and $p(z) = 1 + p_1z + p_2z^2 + \dots$. Differentiating (2.6) logarithmically, we see that

$$\begin{aligned} \frac{zf'(z)}{f(z)} \left(\alpha \frac{zf''(z)}{f'(z)} + 1 \right) &= \alpha \left(1 - \frac{\alpha}{2}\right) zp'(z) + \alpha \left(1 - \frac{\alpha}{2}\right)^2 p(z)^2 \\ &+ \left(1 - \frac{\alpha}{2}\right) (\alpha^2 + 1 - \alpha)p(z) + \frac{\alpha^3}{4} + \frac{\alpha}{2}(1 - \alpha). \end{aligned} \quad (2.6)$$

Let us define

$$\Omega = \left\{ w \in \mathbb{C} : \operatorname{Re}(w) > -\frac{(1-\alpha)\alpha^2}{4} \right\}$$

and

$$\Phi(z_1, z_2; z) = \alpha \left(1 - \frac{\alpha}{2}\right) z_2 + \alpha \left(1 - \frac{\alpha}{2}\right)^2 z_1^2 + \left(1 - \frac{\alpha}{2}\right) (\alpha^2 - \alpha + 1) z_1 + \frac{\alpha^3}{4} + \frac{\alpha}{2}(1 - \alpha).$$

Then by (2.4) and (2.6), we know that $\Phi(p(z), zp'(z); z) \in \Omega$. Further, for all $z \in U$ and for all real x, y such that $y \leq -(1+x^2)/2$, we have

$$\operatorname{Re} \{\Phi(ix, y; z)\} = \alpha \left(1 - \frac{\alpha}{2}\right) y - \alpha \left(1 - \frac{\alpha}{2}\right)^2 x^2 + \frac{\alpha^3}{4} + \frac{\alpha}{2}(1 - \alpha)$$

$$\begin{aligned} &\leq \frac{\alpha^2}{4}(\alpha - 1) - \frac{\alpha}{2}\left(1 - \frac{\alpha}{2}\right)(3 - \alpha)x^2 \\ &\leq -\frac{\alpha^2}{4}(1 - \alpha). \end{aligned}$$

Thus, by applying Lemma, we have $\operatorname{Re}(p(z)) > 0$ for $z \in U$, which, in view of (2.5), is equivalent to $f(z) \in S^*(\alpha/2)$.

If we take $\alpha = 1$ in Theorem 2, then we have

Corollary 2. *If $f(z)$ in A satisfies*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \left(\frac{zf''(z)}{f'(z)} + 1 \right) \right\} > 0 \quad (z \in U),$$

then $f(z) \in S^(1/2)$.*

Finally, we consider

Theorem 3. *If $f(z) \in A$ satisfies*

$$\left| \frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right| < \rho \quad (z \in U),$$

where

$$\rho = \left(\frac{827 + 73\sqrt{73}}{288} \right)^{\frac{1}{2}} = 2.2443697\ldots,$$

then $f(z) \in S^$.*

Proof. Let the function $p(z)$ be defined by (2.2). Then it follows that

$$\frac{zf''(z)}{f'(z)} \left(\frac{zf'(z)}{f(z)} - 1 \right) = (p(z) - 1) \left(\frac{zp'(z)}{p(z)} + p(z) - 1 \right). \quad (2.7)$$

Letting $\Omega = \{w \in \mathbb{C} : |w| < \rho\}$ and

$$\Phi(z_1, z_2 : z) = (z_1 - 1) \left(\frac{z_2}{z_1} + z_1 - 1 \right),$$

we have $\Phi(p(z), zp'(z); z) \in \Omega$. Further, for all $z \in U$, and for all real x, y with $y \leq -(1+x^2)/2$, $\Phi(p(z), zp'(z); z)$ satisfies

$$|\Phi(ix, y; z)| = \sqrt{(1+x^2) \left(1 + \frac{(x^2-y)^2}{x^2}\right)} \equiv \sqrt{g(x^2, y)}, \quad (2.8)$$

where $t = x^2 > 0$ and $y \leq -(1+t)/2$. Since

$$\frac{\partial g(t, y)}{\partial y} = 2 \frac{1+t}{t} (y-t) < 0,$$

we have

$$g(t, y) \geq g\left(t, -\frac{1+t}{t}\right) = \frac{(t+1)^2(9t+1)}{4t} \equiv h(t). \quad (2.9)$$

Further, since

$$h'(t) = \frac{(t+1) \left(t + \frac{\sqrt{73}+1}{36}\right) \left(t - \frac{\sqrt{73}-1}{36}\right)}{4t^2},$$

we obtain

$$\min_{t>0} h(t) = h\left(\frac{\sqrt{73}-1}{36}\right) = \frac{827+73\sqrt{73}}{288} = \rho^2. \quad (2.10)$$

This implies that $|\Phi(ix, y; z)| \geq \rho$. It follows from (2.8), (2.9) and (2.10) that $\Phi(ix, y; z) \notin \Omega$. An application of Lemma gives us that $\operatorname{Re}(p(z)) > 0$ for $z \in U$. Thus we conclude that $f(z) \in S^*$.

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