

On the cone conjecture for log Calabi-Yau mirrors of Fano 3-folds

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Kawamata-Morrison cone conjecture (1993)

X : Calabi-Yau 3-fold

Then (i) $\text{Aut}(X) \curvearrowright \text{Nef}^e(X)$ with a rational polyhedral fundamental domain (R.P.F.D.);

(ii) $\text{PsAut}(X) \curvearrowright \text{Mov}^e(X)$ with an R.P.F.D.

Important idea

$f: \underbrace{Y}_{\substack{\text{sm proj} \\ \text{3 fold}}} \longrightarrow \mathbb{P}^1$ a K3 fibration s.t. $-K_Y = f^*\mathcal{O}(1)$

Z : flop of Y

Then the extremal rays of $\overline{\text{Cuv}}(Z)$ in the $K_Z < 0$ region is either of (numbering from Mori's Classification of Extremal Rays of $\overline{\text{Cuv}}(Y)$, see Kollár-Mori)

- Type (1): blowup of a smooth curve T
- Type (6): conic bundle

Thm [L.]

Y : sm. proj. 3-fold with K3 fibration $f: Y \rightarrow \mathbb{P}^1$ s.t. $-K_Y = f^*(\mathcal{O}(1))$

Then (i) $\text{PsAut}(Y) \curvearrowright \{ \text{Type (6) codim. 1 faces of } \text{Mov}^e(Y) \}$ with finitely many orbits;

(ii) $\text{PsAut}(Y) \curvearrowright \{ \text{Type (1) codim. 1 faces of } \text{Mov}^e(Y) \}$ with finitely many orbits if $H^3(Y, \mathbb{C}) = 0$.

Rem If $H^3(Y, \mathbb{C}) = 0$, then in the case of Type (1) : genus $g(T) = 0$.

Conj $\text{PsAut}(Y) \curvearrowright \{ \text{Type (1) codim. 1 faces of } \text{Mov}^e(Y) \}$ with finitely many orbits $\forall g \geq 0$.

By a result of Kawamata, we know

Thm $\text{PsAut}(Y) \curvearrowright \{ \text{codim 1 faces of } \text{Mov}^e(Y) \text{ containing } -K_Y \}$ w/ finitely many orbits.

Rem Conj + Thm (Kawamata) $\Rightarrow \text{PsAut}(Y) \curvearrowright \{ \text{all codim 1 faces of } \text{Mov}^e(Y) \}$ w/ finitely many orbits.

Pf of Main Thm

Kawamata-Morrison-Totaro Cone Conjecture (Proved by Totaro when $\dim Y = 2$)

(Y, Δ) : Klt and $K_Y + \Delta = 0$

Then (i) $\text{Aut}(Y, \Delta) \curvearrowright \text{Nef}^e(Y)$ w/ RPF;D;

(ii) $\text{BAut}(Y, \Delta) \curvearrowright \text{Mov}^e(Y)$ w/ RPF;D.

Consider Y CY3 with K3 fibration $f: Y \rightarrow \mathbb{P}^1$ s.t. $-K_Y = f^*(\mathcal{O}(1))$

F : general fiber of f

$F_1, F_2 \sim F$: sm. fibers

$$\Delta = \frac{1}{2} F_1 + \frac{1}{2} F_2$$

Then (Y, Δ) : Klt CY.

Rem Under our assumptions, such Y arise naturally as mirrors to sm. Fano 3-folds.

105 deformation types

[Birkar-Cascini-Hacon-McKernan] $\Rightarrow \text{Mov}(Y) = \bigcup_{\substack{Y \dashrightarrow Z \\ \text{SQM}}} \text{Nef}(Z)$ is locally rational polyhedral in $\text{Int}(\text{Mov}(Y))$.

Under our assumptions, the only SQMs are flops / compositions of flops.

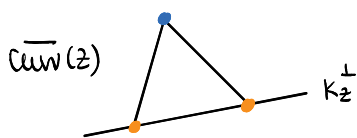
Let $S := \bigcup_{\substack{Y \rightarrow Z \\ \text{SQMS}}} \{ \text{codim. 1 faces of } \text{Nef}(Z) \text{ in the boundary of } \text{Mov}(Y) \} \rightarrow \{ \text{codim. 1 faces of } \text{Mov}^e(Y) \}.$


$$\left. \begin{array}{l} \bigcup_{Y \rightarrow Z} \{ \text{extremal rays of } \overline{\text{Cov}}(Y) \} \\ \text{flops} \qquad \qquad \qquad \text{Nef}(Z)^* \end{array} \right] \text{ Strategy: classify these extremal rays}$$

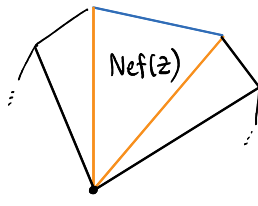
Mois's Cone Theorem

2. sm. proj. var.

(Simplified picture)



↔
dual



$$\text{Mod}(Y) = \bigcup_{\substack{Y \twoheadrightarrow Z \\ \text{SOM}}} \text{Nef}(Z)$$

- Blue extremal rays in $K_Z < 0$ region: either Type (1) or Type (6)
- Orange extremal rays on $K_Z^\perp$: use Kawamata's Theorem

Type (6) Conic bundle $g: Z \rightarrow S$

$$\begin{array}{ccc} Y & \dashrightarrow & Z \xrightarrow{g} S \\ \text{flops} \downarrow & & \downarrow \text{K3 fibration} \\ & & \mathbb{P}^1 \end{array}$$

• $h := g|_F : F \rightarrow S$ is finite, deg 2
generic fiber

• $\exists$ involution i corresponding to h :

$$\begin{array}{c} F \xrightarrow{i} F \\ \downarrow \\ S : \text{smooth} \end{array}$$

$\Rightarrow \begin{cases} (1) \text{Fix}(i) = \emptyset \Rightarrow S \text{ is rat'l surf.} \\ (2) \text{Fix}(i) \neq \emptyset \Rightarrow S : \text{Enriques surf. (cone conj holds (done))} \end{cases}$

Suppose S : rat'l surf. Then run MMP on S .

$$\pi : S \rightarrow \bar{S}$$

$\mathbb{P}^2 \text{ or } \mathbb{F}_n \ (0 \leq n \leq 4, n \neq 1)$

Now have $F \xrightarrow{h} S \xrightarrow{\pi} \bar{S} \xrightarrow{\phi} \hat{S}$

$\theta := \pi \circ h$

Let $L := \theta^* M$, where

If $\bar{S} = \mathbb{P}^2$: take $\phi = \text{id}$ and $M = H$ a hyperplane class
 $\Rightarrow L^2 = 2$

If $\bar{S} = \mathbb{F}_n$: take $\phi : \mathbb{F}_n \rightarrow \underbrace{\mathbb{P}(1,1,n)}_{\hat{S}}$ the contraction of the neg. section of $\mathbb{F}_n$ and M the positive section of $\mathbb{F}_n$.
 $\Rightarrow L^2 = 2M^2 = 2n \leq 8$

Thm [Stein]

$F: K3$ surf.

L : nef line bundle, $L^2 = 2k$ for some fixed $k \in \mathbb{N}$

Then $\text{Aut}(F) \curvearrowright \{ \text{all such } L \}$ w/ finitely many orbits.

$\leadsto \text{Aut}(F) \curvearrowright \{ \text{Type (6) faces of } \text{Mov}^e(Y) \}$ w/ finitely many orbits.
generic fiber

and $\text{Aut}(F_1) \subset \text{PsAut}(Y)$.

Type (1) B.V. of sm. curve T or contraction of a ruled surf. E .

$\mathbb{P}^1$ -bdl $g: E \rightarrow T$ (trivial bdl)

$\Rightarrow \forall$ general fiber F of $K3$ fibr. f., $T \subset F$

Assume $H^3(Y, \mathbb{C}) = 0$. Then $H^1(T) = 0$ so $T \cong \mathbb{P}^1$.

By Adjunction Formula, $T^2 = -2$.

Thm [Stein, Kawamata]

$S: K3$ surf / k (field of char 0)

Then $\text{Aut}(S) \curvearrowright \text{Nef}^e(S)$ w/ a R.P.F.D.

In part., $\text{Aut}(S) \curvearrowright \{ \text{all } (-2)\text{-curves } C \subset S \}$ w/ finitely many orbits.

Let Y_2 : the generic fiber of K3 fib. $f: Y \rightarrow \mathbb{P}^1$

Then,

$\text{Aut}(Y_2) \curvearrowright \{(-2)\text{-curves } C \subset Y_2\}$ w/ finitely many orbits.

$\Rightarrow \text{PsAut}(Y_2) \curvearrowright \{(-2)\text{-curves } C \subset Y_2\}$ w/ finitely many orbits.

$\Rightarrow \text{PsAut}(Y) \curvearrowright \{(-2)\text{-curves } C \subset Y_2\}$ w/ finitely many orbits.

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