

AN APPLICATION OF ALUTHGE TRANSFORM TO PUTNAM INEQUALITY FOR LOG-HYPONORMAL OPERATORS

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ABSTRACT. In this note, we give a short proof to the Putnam inequality for log-hyponormal operators due to Tanahashi: If T is invertible and log-hyponormal, i.e., $\log T^*T \geq \log TT^*$, then

$$\|\log T^*T - \log TT^*\| \leq \frac{1}{\pi} \iint_{\sigma(T)} r^{-1} dr d\theta,$$

where $\sigma(T)$ is the spectrum of T . It is based on his original idea that the log-hyponormality is regarded as 0-hyponormality.

1. Introduction. After the Furuta inequality was originated by Furuta [12], see also [5,13,16,18], we initiated to study it under the chaotic order $A \gg B$, i.e., $\log A \geq \log B$, for positive invertible operators A and B [11]. We finally characterized $A \gg B$ by a Furuta-type inequality [6] and [7,8]: For $A, B > 0$, $A \gg B$ if and only if

$$(1) \quad (A^r B^p A^r)^{\frac{2r}{p+2r}} \leq A^{2r}$$

holds for all $p, r \geq 0$.

Furthermore we interpolated between the Furuta inequality and Theorem A as follows [9,10]:

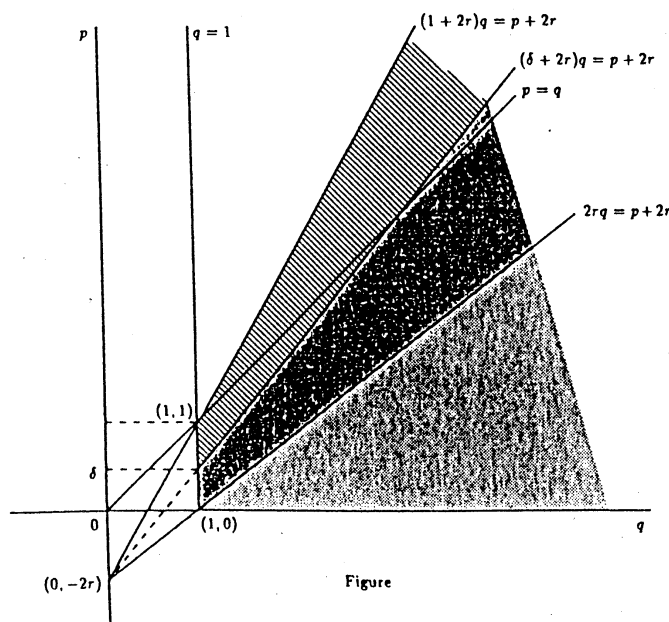
Theorem A. For a fixed $\delta > 0$, $A^\delta \geq B^\delta$ for $A, B \geq 0$ if and only if for each $r \geq 0$

$$(2) \quad A^{\frac{p+2r}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}}$$

holds for $p \geq 0$ and $q \geq 1$ with

$$(3) \quad (\delta + 2r)q \geq p + 2r.$$

We note that (1) is equivalent to the case $\delta = 0$ in Theorem A and the Furuta inequality is just the case $\delta = 1$. The domain given by (3) is explained by the figure below:



From the viewpoint of this, Tanahashi [19] defined the log-hyponormality for invertible operators by $|T| \gg |T^*|$, where $|X|$ is the square root of X^*X , and constructed the Putnam inequality for log-hyponormal operators:

Theorem T. *If T is an invertible log-hyponormal operator, i.e., $\log T^*T - \log TT^* \geq 0$, then*

$$(4) \quad \|\log T^*T - \log TT^*\| \leq \frac{1}{\pi} \iint_{\sigma(T)} r^{-1} dr d\theta,$$

where $\sigma(T)$ is the spectrum of T .

It was conjectured from the Putnam inequality for p -hyponormal operators by Cho and Itoh [3]:

If T is a p -hyponormal operator, i.e., $(T^*T)^p - (TT^*)^p \geq 0$, then

$$(5) \quad \|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\pi} \iint_{\sigma(T)} r^{2p-1} dr d\theta.$$

As a matter of fact, he understood (5) as follows:

$$(6) \quad \left\| \frac{(T^*T)^p - (TT^*)^p}{p} \right\| \leq \frac{1}{\pi} \iint_{\sigma(T)} r^{2p-1} dr d\theta.$$

By taking $p \rightarrow \infty$, he constructed Theorem T and proved it by the idea developed in (5).

The purpose of this note is to continue his consideration directly. That is, we here propose a straight and simple proof of Theorem T which might be along with his intention; Tanahashi might regards the log-hyponormality as the 0-hyponormality. Our tool in this note is the Aluthge transform which is grown up the p -hyponormality, see [1,2,4,14,15,20].

2. Preliminary.

For the sake of convenience, we cite the following characterization of chaotic order which implies (1) by the help of the Furuta inequality [7], see also [8] and [9].

Theorem B. *For $A, B > 0$, $A \gg B$, i.e., $\log A \geq \log B$, if and only if for any $\delta \in (0, 1]$ there exists an $\alpha = \alpha_\delta > 0$ such that*

$$(e^\delta A)^\alpha > B^\alpha.$$

The essential part of Theorem B is as follows: If A and B are selfadjoint and $A > B$, then there exists an $\alpha \in (0, 1]$ such that

$$(*) \quad e^{\alpha A} > e^{\alpha B}.$$

It has the following simple proof: The assumption $A > B$ means that $A - B \geq \epsilon > 0$ for some ϵ . We here take $0 < \alpha < \epsilon/(e^{\|A\|} + e^{\|B\|})$ and $\alpha \leq 1$. Then we have

$$\begin{aligned} e^{\alpha A} - e^{\alpha B} &= \alpha(A - B) + \sum_{n=2}^{\infty} \frac{\alpha^n}{n!} (A^n - B^n) \\ &\geq \alpha\epsilon + \alpha^2 \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!} (A^n - B^n) \\ &\geq \alpha\epsilon - \alpha^2 \left\| \sum_{n=2}^{\infty} \frac{\alpha^{n-2}}{n!} (A^n - B^n) \right\| \\ &\geq \alpha\epsilon - \alpha^2 \sum_{n=2}^{\infty} \frac{1}{n!} (\|A\|^n + \|B\|^n) \\ &\geq \alpha(\epsilon - \alpha(e^{\|A\|} + e^{\|B\|})) > 0. \end{aligned}$$

Here we should note an interesting characterization of chaotic order recently obtained by Yamazaki and Yanagida [22], which is associated with Kantorovich inequality and consequently Specht's ratio, see [23].

3. Proof.

We begin with the Putnam inequality for p -hyponormal operators; we turn it into the following lemma via the Löwner-Heinz inequality:

Lemma 1. *If T is a q -hyponormal operator, then T satisfies (6) for all $0 < p \leq q$. Consequently (4) holds for any q -hyponormal operators T .*

The second half is ensured by the fact that $\lim_{t \rightarrow 0} \frac{A^t - 1}{t} = \log A$ for a positive invertible operator A .

Thus Lemma 1 suggests us to find a family $\{T_q; q > 0\}$ of q -hyponormal operators such that $\|T_q - T\| \rightarrow 0$ as $q \rightarrow 0$ for a given log-hyponormal operator T . In this situation, the Aluthge transform completely responds to our demand. As a matter of fact, Tanahashi prepared the following result in [20]:

Lemma 2. *If T is an invertible log-hyponormal operator with the polar decomposition $T = U|T|$, then the Aluthge transform $\tilde{T} = |T|^q U |T|^{1-q}$ is q -hyponormal for $0 < q \leq \frac{1}{2}$.*

To prove Theorem T, we take $T_q = |T|^q U |T|^{1-q}$ for $0 < q < \frac{1}{2}$. Since $\sigma(T_q) = \sigma(T)$ for all q , it is complete.

We finally give a short proof to Lemma 2 via Theorem A

Putting $p = 2q$ and $r = 1 - q$ in (1), we have

$$\begin{aligned} (T_q^* T_q)^{1-q} &= (|T|^{1-q} U^* |T|^{2q} U |T|^{1-q})^{1-q} \\ &= U^* (U |T|^{1-q} U^* |T|^{2q} U |T|^{1-q} U^*)^{1-q} U \\ &= U^* (|T^*|^{1-q} |T|^{2q} |T^*|^{1-q})^{1-q} U \\ &\geq U^* |T^*|^{2(1-q)} U \\ &= |T|^{2(1-q)}, \end{aligned}$$

so that $(T_q^* T_q)^{2q} \geq |T|^{2q}$ via the Löwner-Heinz inequality by $1 - q \geq q$. On the other hand, we have also

$$\begin{aligned} (T_q T_q^*)^q &= (|T|^q U |T|^{2(1-q)} U^* |T|^q)^q \\ &= (|T|^q |T^*|^{2(1-q)} |T|^q)^q \\ &\leq |T|^{2q}. \end{aligned}$$

Therefore it follows that

$$(T_q^* T_q)^q \geq |T|^{2q} \geq (T_q T_q^*)^q,$$

as desired.

Remark. (1) (5) is obtained by Putnam [17] for $p = 1$, Xia [21] for $\frac{1}{2} \leq p < 1$ and Cho-Itoh [3] for $0 < p < \frac{1}{2}$.

(2) The proof of Lemma 2 is available to show Tanahashi's result [20; Theorem 4].

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