

# Information Transmission in Quantum Open Systems

東京理科大学理工学部情報科学科  
大矢雅則 ・ 須鎗弘樹

(Extended Abstract)

開放系の数理モデルは、統計物理学において、対象とする系の非可逆性や平衡状態への推移の議論に非常に多く用いられており、その応用範囲は極めて広い。

本論文では、筆者の一人が提唱している情報力学の複雑量を、量子光学における典型的な開放系のモデルに用いて、系の非可逆性と平衡状態への推移の議論を行う。つまり、量子光学でよく用いられる開放系のモデルを量子マルコフ過程に適用し、その上で、相互エントロピーを具体的に計算する。

情報理論において、相互エントロピーは、初期状態の情報量が状態変換を表すチャネルによって、終状態にどれだけ正しく伝達されたかを表す量であるから、この複雑量を用いて、対象とする系の非可逆性及び平衡状態への推移を議論することができる。この相互エントロピーを計算する際に重要なのが、状態変換を表すチャネルである。チャネルは、量子マルコフ過程の定義に用いられている推移期待値と密接な関係がある。つまり、推移期待値の共役写像は、リフティングと呼ばれる写像であり、これは、状態空間からその状態空間を含むより大きな状態空間への写像で、系の間の相互作用を表している。そして、リフティングから上述のチャネルは、部分トレースを取ることによって容易に得ることができる。つまり、チャネルと推移期待値の概念は、リフティングの概念を通して、関係付けられる。

これらの相互関係から、従来、量子情報理論において用いられてきた量子相互エントロピーを量子マルコフ過程上で開放系のモデルに適用し、具体的に計算することによって、系の非可逆性を論じることは、情報理論の手法を統計物理学に応用するという意味においても、非常に有効と思われる。実際、この相互エントロピー等の複雑量を用いて、多種多様な量子マルコフ過程を分類、特徴付けることは、系を特徴付ける意味においても有用と思われる。

本論文では、量子光学でよく用いられる開放系のモデルとして、具体的にハミルトニアンを与え、チャネルを定めた。その上で、ハミルトニアンの特徴から、チャネルの Stinespring 表現を厳密に導くことに成功した。このことによって Stinespring 表現が、物理的具體例において初めて構成されたことになった。この Stinespring 表現を用いて、相互エントロピーを何等近似を用いずに厳密に計算し、その結果から、系の非可逆性を議論した。具体的には、熱浴系の状態として真空状態とギブス状態を用意し、いずれの場合も、観測系と熱浴系の相互作用の回数の増加に伴って、相互エントロピーが単調に減少することを示し、さらに、この熱浴系の2つの状態を比べると、相互エントロピーはギブス状態の方がより大きく減少することを数値計算によって示した。

このように、本論文は、チャネル・リフティング・推移期待値・相互エントロピー等の量子情報理論や量子マルコフ過程論における様々な数学的・物理的概念から出発して、具体的なチャネルや複雑量の導出、その数値計算及びそれらの結果から得られる系の非可逆性までを、簡潔に示したものである。

# Information Transmission in Quantum Open Systems

Masanori Ohya    and    Hiroki Suyari

Department of Information Sciences

Science University of Tokyo, Japan

## Abstract

There are several damping phenomena in quantum optics. Such phenomena have been usually explained by open systems. In statistical physics, open system dynamics have been used to study the irreversibility and the approach to equilibrium.

In this paper the dynamical change of the mutual entropy is discussed for a model of quantum Markov chain. The concrete Stinespring's expression for this model is obtained and applied to the derivation of the mutual entropy, and some computational results are presented.

## 1 Introduction

The purpose of this paper is to show the use of the mutual entropy in quantum Markov chain (QMC for short) for a study of the irreversible process.

In section 2, a well known model in quantum optics is presented. This model is often called "damping", and its dynamical change with a unitary evolution is popular not only in quantum optics but also in statistical physics. In section 3, we review some concepts for the mutual entropy and a generalized QMC, introduced in [O1] and [A1, A2], respectively. We here stand on a concept "lifting" [A3] to study the above concepts in a unified manner. In section 4, we give a construction of QMC on the model of section 2. The Stinespring-Kraus expression of a completely positive map has mathematically been established, but its concrete expression is missing. In section 5, we show its concrete expression for our model. In section 6, we apply this expression to the computation of the mutual entropy in QMC. In section 7, some computational results are presented.

## 2 Model for Open Systems with Harmonic Oscillators

In statistical mechanics, a state of interacting system is changed under some effects from the outside of the system (reservoir), that is, the interaction between two systems is considered and the reduced state after interaction is studied. Mathematical expression of this process is given as follows.

Let a system  $\Sigma_0$  be described by a Hilbert space  $\mathcal{H}_0$ , which interacts with an external system  $\Sigma_1$  described by another Hilbert space  $\mathcal{H}_1$ , and let the initial states of  $\Sigma_0$ ,  $\Sigma_1$  be  $\rho \in \mathfrak{S}_0$ ,  $\omega \in \mathfrak{S}_1$  ( $\mathfrak{S}_i$  is the set of all states on  $\mathcal{H}_i$  ( $i = 0, 1$ )), respectively. Then the combined state  $\tilde{\rho} \in \mathfrak{S}_0 \otimes \mathfrak{S}_1$  after the interaction between the two systems is given by

$$\tilde{\rho} = U_t(\rho \otimes \omega) U_t^* \quad (2.1)$$

where  $U_t = \exp(-itH)$  with a total Hamiltonian  $H$  on  $\mathcal{H}_0 \otimes \mathcal{H}_1$ . Here we took  $\hbar = 1$ .

The above total Hamiltonian  $H$  for two weakly coupled oscillators in quantum optics is given by [L1]:

$$H = H_0 + H_1 + H_{01} \quad (2.2)$$

$$H_0 = a^*a, \quad H_1 = \sum_j b_j^*b_j, \quad H_{01} = \sum_j (\varepsilon_j b_j a^* + \varepsilon_j^* b_j^* a) \quad (2.3)$$

where  $H_0$ ,  $H_1$ ,  $H_{01}$  are the Hamiltonians for the observed system  $\Sigma_0$ , the external system (reservoir)  $\Sigma_1$  and an interaction between  $\Sigma_0$  and  $\Sigma_1$ , respectively.  $a, a^*$  on  $\mathcal{H}_0$  and  $b_j, b_j^*$  on  $\mathcal{H}_1$  are pairs of annihilation and creation operators, respectively, and  $\varepsilon_j$  ( $j \in N$ ) are the coupling constants.

In this paper we assume, for simplicity, that the reservoir has a single mode, that is, the total hamiltonian  $H$  is given by

$$\begin{aligned} H &= H_0 + H_1 + H_{01} \\ &= a^*a + b^*b + \varepsilon(a^*b + ab^*) \end{aligned} \quad (2.4)$$

In the model, the evolution of the initial state  $\rho$  after the interaction is mathematically expressed as

$$\rho \mapsto \bar{\rho} = \text{tr}_{\mathcal{H}_1} U_t(\rho \otimes \omega) U_t^*. \quad (2.5)$$

### 3 Mutual Entropy in Quantum Markov Chain

In this section the mutual entropy is discussed in quantum Markov chain (QMC) of our model.

For the sake of the formulation of mutual entropy in QMC, we review some definitions and fundamental results :

**Definition 3.1** [O1]: A completely positive map  $\Lambda^* : \mathfrak{S}(\mathcal{H}) \rightarrow \mathfrak{S}(\mathcal{K})$  is called a quantum mechanical channel. Here  $\mathfrak{S}(\mathcal{H})$  is the set of states on a Hilbert space  $\mathcal{H}$ .

The quantum mechanical channel  $\Lambda^* : \mathfrak{S}_0 \rightarrow \mathfrak{S}_0$  in our model can be written as :

$$\Lambda^* \rho = \text{tr}_{\mathcal{H}_1} U_t(\rho \otimes \omega) U_t^* \quad (3.1)$$

The concept of channel is related to that of “lifting” introduced in [A3].

**Definition 3.2** [A3]: Let  $\mathfrak{S}_0, \mathfrak{S}_1$  be the set of all states on  $\mathcal{H}_0, \mathcal{H}_1$  respectively. A lifting from  $\mathfrak{S}_0$  to  $\mathfrak{S}_0 \otimes \mathfrak{S}_1$  is a continuous map

$$\mathcal{E}^* : \mathfrak{S}_0 \rightarrow \mathfrak{S}_0 \otimes \mathfrak{S}_1. \quad (3.2)$$

For example, the correspondence  $\rho \mapsto \tilde{\rho}$  in (2.1) is a lifting ;  $\mathcal{E}^*\rho = U_t(\rho \otimes \omega)U_t^*$  is a continuous map from  $\mathfrak{S}_0$  to  $\mathfrak{S}_0 \otimes \mathfrak{S}_1$ . The correspondence between a quantum mechanical channel  $\Lambda^*$  and a lifting  $\mathcal{E}^*$  is explained in [A3].

Let the Schatten decomposition of  $\rho \in \mathfrak{S}(\mathcal{H})$  be

$$\rho = \sum_k \lambda_k E_k, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots, \quad E_i \perp E_j \quad (i \neq j) \quad (3.3)$$

where  $\lambda_k$  is an eigenvalue of  $\rho$  and  $E_k$  is the associated onedimensional projection. The decomposition is not unique unless every  $\lambda_k$  is not degenerated.

The relation between an initial state  $\rho$  and the final state  $\Lambda^*\rho$  is expressed by the compound states introduced in [O1, O2] such that

$$\mathcal{E}_E^*\rho \equiv \sum_k \lambda_k E_k \otimes \Lambda^* E_k \quad (3.4)$$

The definition of mutual entropy  $I(\rho; \Lambda^*)$  is given as follows :

**Definition 3.3** [O1] : Using the above notations (3.3) and (3.4), the mutual entropy  $I(\rho; \Lambda^*)$  is defined by

$$I(\rho; \Lambda^*) = \sup_E \{S(\mathcal{E}_E^*, \mathcal{E}_0^*) ; E = \{E_k\}\} \quad (3.5)$$

where  $\mathcal{E}_0^* = \rho \otimes \Lambda^*\rho$  and  $S(\rho, \sigma)$  is the relative entropy defined by

$$S(\rho, \sigma) = \text{tr} \rho (\log \rho - \log \sigma) \quad (3.6)$$

In (3.5), we have to take “sup” over all Schatten decompositions when some eigenvalues are degenerated. We have the following fundamental inequality of Shannon’s type.

**Theorem 3.4** [O1] : For an input state  $\rho$  and a channel  $\Lambda^*$ , the following inequalities hold.

$$0 \leq I(\rho; \Lambda^*) \leq \min \{S(\rho), S(\Lambda^*\rho)\} \quad (3.7)$$

This theorem means that the information correctly transmitted through a channel  $\Lambda^*$  is always less than the initial information. In other words, the mutual entropy is useful as an efficiency of communication processes. [W1]. Other properties of the mutual entropy are discussed in [O3, O4].

Then, before reviewing QMC, we need the concept of a transition expectation :

**Definition 3.5** [A4] : Let  $\mathcal{B}_0, \mathcal{B}_1$  be the algebras of all bounded operators on Hilbert spaces  $\mathcal{H}_0, \mathcal{H}_1$  respectively, and let  $\mathcal{B}_0 \otimes \mathcal{B}_1$  be a fixed tensor product of  $\mathcal{B}_0$  and  $\mathcal{B}_1$ . A transition expectation from  $\mathcal{B}_0 \otimes \mathcal{B}_1$  to  $\mathcal{B}_0$  is a completely positive linear map  $\mathcal{E} : \mathcal{B}_0 \otimes \mathcal{B}_1 \rightarrow \mathcal{B}_0$  satisfying

$$\mathcal{E}(1 \otimes 1) = 1 \quad (3.8)$$

When  $\mathcal{B} = \mathcal{B}_0 = \mathcal{B}_1$  we say that  $\mathcal{E}$  is a transition expectation on  $\mathcal{B}$ .

By using the transition expectations, a generalized QMC is defined as follows :

**Definition 3.6** [A4] : Let  $\{\mathcal{E}_n\}_{n \geq 0}$  be a sequence of transition expectations on  $\mathcal{B}$ . Then there exists a unique completely positive identity preserving map  $E_0] : \otimes^n \mathcal{B} \rightarrow \mathcal{B}$  such that for each integer  $n$  and for each  $A_0, A_1, \dots, A_n \in \mathcal{B}$ , one has

$$E_0](A_0 \otimes A_1 \otimes \dots \otimes A_n \otimes 1 \otimes \dots) = \mathcal{E}_0(A_0 \otimes \mathcal{E}_1(A_1 \otimes \dots \otimes \mathcal{E}_n(A_n \otimes 1) \dots)) \quad (3.9)$$

Let  $\varphi_0$  be a state on  $\mathcal{B}$  and define a state  $\varphi$  on  $\otimes_N \mathcal{B}$  by

$$\varphi = \varphi_0 \circ E_0] \quad (3.10)$$

Then  $\varphi$  satisfies

$$\varphi(A_0 \otimes A_1 \otimes \dots \otimes A_n \otimes 1 \otimes \dots) = \varphi_0(\mathcal{E}_0(A_0 \otimes \mathcal{E}_1(A_1 \otimes \dots \otimes \mathcal{E}_n(A_n \otimes 1) \dots))) \quad (3.11)$$

The state  $\varphi$ , characterized by (3.10) is called the generalized Markov chain associated to the pair  $(\varphi_0, \{\mathcal{E}_n\})$ . If for each  $n$

$$\mathcal{E} \equiv \mathcal{E}_n = \mathcal{E}_0, \quad (3.12)$$

then the state  $\varphi$  is called homogeneous.

Clearly the dual of a “linear” lifting is a transition expectation, therefore to any linear lifting one can associate a QMC in the standard way as explained in [A3]. If the lifting is of convex product type, then we can take an advantage of the special structure to extend the construction of QMC to the case of a not necessarily linear lifting.

## 4 A model of Quantum Markov Chain in Quantum Optics

In order to construct a model in QMC, a model by Kümmerer [K2] is useful and its physical meanings are discussed in this section. We start to review some technical terms to define the QMC :

**Definition 4.1** [U1, T1] : Let  $\mathcal{A}$  be a  $*$ -algebra and  $\mathcal{A}_0 \subseteq \mathcal{A}$  a  $*$ -subalgebra. A conditional expectation from  $\mathcal{A}$  to  $\mathcal{A}_0$  is a linear map  $E : \mathcal{A} \rightarrow \mathcal{A}_0$  such that

$$a \geq 0 \Rightarrow E(a) \geq 0 ; a \in \mathcal{A} \quad (4.1)$$

$$E(a_0 a) = a_0 E(a) ; a_0 \in \mathcal{A}_0, a \in \mathcal{A} \quad (4.2)$$

$$E(1) = 1 \quad (4.3)$$

$$E(a)^* E(a) \leq E(a^* a) ; a \in \mathcal{A} \quad (4.4)$$

**Definition 4.2** [A4] : Let  $\mathcal{B} \subseteq \mathcal{A}$  be  $C^*$ -algebras and let  $E : \mathcal{A} \rightarrow \mathcal{B}$  be a conditional expectation. An  $E$ -conditional density amplitude is an operator  $K \in \mathcal{A}$  such that

$$E(K^*K) = 1 \quad (4.5)$$

**Definition 4.3** [A4] : A conditional amplitude from  $\mathcal{B}_0 \otimes \mathcal{B}_1$  to  $\mathcal{B}_0$  is an operator  $K \in \mathcal{B}_0 \otimes \mathcal{B}_1$  such that, for every density operator  $\rho \in \mathcal{B}_0$ ,

$$K(\rho \otimes 1)K^* \quad (4.6)$$

is a density operator in  $\mathcal{B}_0 \otimes \mathcal{B}_1$ . If  $\mathcal{B} = \mathcal{B}_0 = \mathcal{B}_1$ , then  $K$  is called a conditional amplitude on  $\mathcal{B}$ .

A two parameter family

$$K = (K(m, n)) \quad (4.7)$$

$m, n \in N$ , ( $m < n$ ) of operators in  $\mathcal{A} \equiv \otimes_N \mathcal{B}$  is called a right multiplicative functional (with respect to the localization  $\mathcal{A}_{[m, n]} \equiv \otimes_{i=m}^n \mathcal{B}_i$ ) if :

$$K(m, n)K(n, p) = K(m, p) \quad ; \quad m < n < p \quad (4.8)$$

$$K(m, n) \in \mathcal{A}_{[m, n]} \quad (4.9)$$

If each  $(K(m, n))$  is a conditional amplitude on  $\mathcal{A}_{[m, n]} \cong \otimes^{n-m+1} \mathcal{B}$ , then the multiplicative functional  $K = (K(m, n))$  is said to be normalized. It is clear that a normalized multiplicative functional  $K = (K(m, n))$  has the form

$$K(0, n) = K(0, 1) \cdot K(1, 2) \cdot \dots \cdot K(n-1, n) \quad (4.10)$$

There is a conditional density amplitude  $K_n$  on  $\mathcal{B}$  for each  $n$  such that

$$K(n-1, n) = (j_{n-1} \otimes j_n)(K_{n-1}) \quad (4.11)$$

where  $j_n : \mathcal{B} \rightarrow \otimes_N \mathcal{B}$  is the natural embedding onto the  $n$ -th factor:

$$j_n : b \in \mathcal{B} \rightarrow \underbrace{1 \otimes 1 \otimes \dots \otimes 1}_{n-1} \otimes \underbrace{b}_n \otimes 1 \otimes \dots \in \otimes_N \mathcal{B}$$

**Remark 4.4** : Let  $\psi_0$  be a state on  $\mathcal{B}$  and let  $\phi$  be the product state on  $\mathcal{A} = \otimes_N \mathcal{B}$ . Let  $E_0 : \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$  denote the conditional expectation characterized by

$$E_0(a \otimes b) = \psi_0(b)a \otimes 1 \quad ; \quad a, b \in \mathcal{B} \quad (4.12)$$

and let  $K \in \mathcal{B} \otimes \mathcal{B}$  be an  $E_0$ -conditional density amplitude and define  $K_{n-1, n}$  as in (3.11). Then there exists a unique state  $\phi_K$  on  $\mathcal{A} = \otimes_N \mathcal{B}$  such that, for each  $k \in N$  and  $a \in \mathcal{A}_{[0, k]}$ , one has

$$\phi_K = \lim_{n \rightarrow \infty} \phi(K_{n-1, n}^* \cdots K_{0, 1}^* \cdot a \cdot K_{0, 1} \cdots K_{n-1, n}) \quad (4.13)$$

According to the above remark, the model of QMC [K2] can be constructed and the mutual entropy is computed for that model.

As in section 2, take  $\mathfrak{S} = \mathfrak{S}_0 \otimes \mathfrak{S}_1$  and  $\mathcal{B} = \mathcal{B}_0 \otimes \mathcal{B}_1$ . Let  $\varphi_0$  and  $\varphi_1$  be states on  $\mathcal{B}_0$  and  $\mathcal{B}_1$  respectively and let  $u_0$  be the partial shift on  $\otimes_N (\mathcal{B}_0 \otimes \mathcal{B}_1) \cong \otimes_N \mathcal{B} \cong (\otimes_N \mathcal{B}_0) \otimes (\otimes_N \mathcal{B}_1) \equiv \mathcal{A}$ , that is,  $u_0$  acts trivially on  $\otimes_N \mathcal{B}_0$  and does as the usual shift on  $\otimes_N \mathcal{B}_1$ . Let  $U \in \mathcal{B}_0 \otimes \mathcal{B}_1$  be a unitary operator, and define

$$U(n-1, n) = u_0^{n-1}(U) ; n \geq 1 \quad (4.14)$$

If  $\varphi = \otimes_N (\varphi_0 \otimes \varphi_1)$  is a product state on  $\mathcal{A}$ , and if  $\varphi_U$  is the quantum Markov chain constructed as in the above Remark 3.4, then the restriction of  $\varphi_U$  to the algebra  $\mathcal{B}_0 \otimes (\otimes_N \mathcal{B}_1)$  ( $\mathcal{B}_0$  is the “time zero algebra”) gives a quantum Markov chain in the sense of Kümmerer [K2].

## 5 A Concrete Example of Stinespring Expression in Our Model

In this section the Stinespring expression is concretely obtained in our model [A6].

The Stinespring theorem is well known in the following form :

**Theorem 5.1 [S1] :** *Let  $\mathcal{A}$  be a  $C^*$ -algebra with a unit and  $\mathcal{B}(\mathcal{H})$  be a set of the bounded operators on a Hilbert space  $\mathcal{H}$ . For any completely positive map  $\Lambda : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$ , there exist a  $*$ -representation  $\{\mathcal{K}, \pi\}$  of  $\mathcal{A}$  and a bounded linear map  $V : \mathcal{H} \rightarrow \mathcal{K}$  such that*

$$\Lambda(A) = V^* \pi(A) V \quad \forall A \in \mathcal{A} \quad (5.1)$$

In particular, if  $\mathcal{A} = \mathcal{B}(\mathcal{H})$  in the above Theorem 4.1, then we have the following lemma due to Kraus.

**Lemma 5.2 [K1] :** *Let  $\mathcal{H}$  be a Hilbert space. A linear map  $\Lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is a completely positive if and only if it has the form*

$$\Lambda(A) = \sum_{i=1}^n V_i^* A V_i \quad A \in \mathcal{A} \quad (5.2)$$

where  $V_i : \mathcal{H} \rightarrow \mathcal{H}$  are partial isometries and  $n \leq \dim(\mathcal{H})$ .

We shall give a concrete expression for partial isometries  $V_i$  in our model.

From the form (2.4) of the Hamiltonian  $H$  of the total system  $\mathcal{H}_0 \otimes \mathcal{H}_1$ , it is easy to see that

$$[H_0 + H_1, H_{01}] = 0. \quad (5.3)$$

Hence the time evolution of the system comes from only the interaction Hamiltonian  $H_{01}$ :

$$U_t = \exp(-itH_{01}). \quad (5.4)$$

Clearly the spectra of  $H_0 + H_1$  are natural numbers, so that let  $\mathcal{K}_n$  be the eigenspace of  $H_0 + H_1$  for an eigenvalue  $n \in N$ , then  $\mathcal{K}_n$  is a subspace of  $\mathcal{H}_0 \otimes \mathcal{H}_1$  spanned by  $\{|j \otimes n - j\rangle ; j = 0, \dots, n\}$  and  $H_{01}\mathcal{K}_n \subseteq \mathcal{K}_n$ . Therefore we define  $H_{01}^{(n)}$  by :

$$H_{01}^{(n)} \equiv H_{01} | \mathcal{K}_n \text{ (i.e., the restriction of } H_{01} \text{ to } \mathcal{K}_n) \quad (5.5)$$

where  $H_{01}^{(n)}$  is a finite dimensional self-adjoint operator  $C^{n+1} \rightarrow C^{n+1}$  and has  $(n+1)$  eigenvalues  $\lambda_0^{(n)}, \dots, \lambda_n^{(n)}$  and the corresponding eigenvectors  $\psi_k^{(n)}$  associated to  $\lambda_k^{(n)}$  ( $k = 0, \dots, n$ ); namely,

$$H_{01}^{(n)} \psi_j^{(n)} = \lambda_j^{(n)} \psi_j^{(n)} \quad (j = 0, \dots, n) \quad (5.6)$$

$$\psi_j^{(n)} = \sum_{\alpha=0}^n C_{\alpha}^{(n,j)} |\alpha \otimes n - \alpha\rangle \in \mathcal{K}_n \quad (5.7)$$

where  $C_{\alpha}^{(n,j)}$  is the coefficient determined by  $\langle \psi_i^{(n)}, \psi_j^{(n)} \rangle = \delta_{ij} \delta_{mn}$  and the completeness of  $\{\psi_j^{(n)}\}$ . (5.6) implies

$$H_{01}^{(n)} = \sum_{j=0}^n \lambda_j^{(n)} |\psi_j^{(n)}\rangle \langle \psi_j^{(n)}| \quad (5.8)$$

$$H_{01} = \sum_{n=0}^{\infty} H_{01}^{(n)} = \sum_{n=0}^{\infty} \sum_{j=0}^n \lambda_j^{(n)} |\psi_j^{(n)}\rangle \langle \psi_j^{(n)}| \quad (5.9)$$

Using the above expressions, a unitary operator  $U_t = \exp(-itH)$  on  $\mathcal{H}_0 \otimes \mathcal{H}_1$  can be written as

$$\begin{aligned} U_t &= \exp(-itH) = \exp(-it(H_0 + H_1)) \exp(-itH_{01}) \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^n \sum_{\alpha, \beta=0}^n e^{-it(n+\lambda_j^{(n)})} C_{\alpha}^{(n,j)} \overline{C_{\beta}^{(n,j)}} |\alpha\rangle \langle \beta| \otimes |n - \alpha\rangle \langle n - \beta| \end{aligned} \quad (5.10)$$

Therefore the channel  $\Lambda^* \rho \in \mathfrak{S}_0$  is written by

$$\begin{aligned} \Lambda^* \rho &= \text{Tr}_2 U_t (\rho \otimes \omega) U_t^* \\ &= \sum_{m,n=0}^{\infty} \sum_{\alpha, \beta=0}^n \sum_{\alpha', \beta'=0}^m d_{\alpha, \beta}^{(n)} \overline{d_{\alpha', \beta'}^{(m)}} \langle n - \beta | \omega | m - \beta' \rangle \langle \beta | \rho | \beta' \rangle \delta_{m-\alpha', n-\alpha} |\alpha\rangle \langle \alpha'| \end{aligned} \quad (5.11)$$

where

$$d_{\alpha, \beta}^{(n)} \equiv \sum_{j=0}^n e^{-ite\lambda_j^{(n)}} C_{\alpha}^{(n,j)} \overline{C_{\beta}^{(n,j)}} \quad (5.12)$$

After some calculations [A6], we get the following formula :

$$\Lambda^* \rho = \sum_{\nu=0}^{\infty} T_{\nu} (\rho \otimes \omega) T_{\nu}^* \quad (5.13)$$

where

$$T_{\nu} \equiv \sum_{n=\nu}^{\infty} |n - \nu\rangle \langle \varphi_{n-\nu}^{(n)}|, \quad \langle \varphi_{n-\nu}^{(n)}| \equiv \sum_{\beta=0}^n d_{n-\nu, \beta}^{(n)} \langle \beta| \otimes \langle n - \beta| \in \mathcal{K}_n \quad (5.14)$$

Since  $\{|\varphi_{n-\nu}^{(n)}\rangle ; n \geq \nu, n, \nu \in N\}$  is orthogonal shown in [A6], the above  $T_{\nu}$  is a partial isometry. Therefore (5.13) and (5.14) give a concrete expression for Stinespring's. These formulations are useful for the computation of several entropies. In the following section, we rigourously compute the mutual entropy.

## 6 Computation of Mutual Entropy in Our Model

From (5.13) and (5.14), we have

$$\Lambda^* \rho = \sum_{\nu} \sum_{m, n \geq \nu} \langle \varphi_{n-\nu}^{(n)} | \rho \otimes \omega | \varphi_{m-\nu}^{(m)} \rangle |n-\nu\rangle \langle m-\nu| \quad (6.1)$$

Here, in the calculation below, we assume the following Schatten decomposition :

$$\rho = \sum_l \lambda_l^\rho |l\rangle \langle l|, \omega = \sum_\gamma \mu_\gamma^\omega |\gamma\rangle \langle \gamma|. \quad (6.2)$$

Then (6.2) implies

$$\Lambda^* \rho = \sum_i \left( \sum_{\substack{l, \nu \\ (\nu-l)+i \geq 0}} \lambda_l^\rho \mu_{(\nu-l)+i}^\omega |d_{i,l}^{(\nu+i)}|^2 \right) |i\rangle \langle i|. \quad (6.3)$$

Clearly (6.3) is also a Schatten decomposition. Suppose that the decomposition (6.2) of  $\rho$  is unique.

From (6.3) we find the mutual entropy is

$$I(\rho; \Lambda^*) = \sum_{k,t} \lambda_k^\rho \sum_{\substack{\nu \\ (\nu-k)+t \geq 0}} \mu_{(\nu-k)+t}^\omega |d_{t,k}^{(\nu+t)}|^2 \log \frac{\sum_{\substack{\nu_1 \\ (\nu_1-k)+t \geq 0}} \mu_{(\nu_1-k)+t}^\omega |d_{t,k}^{(\nu_1+t)}|^2}{\sum_{\substack{l, \nu \\ (\nu-l)+t \geq 0}} \lambda_l^\rho \mu_{(\nu-l)+t}^\omega |d_{t,l}^{(\nu+t)}|^2} \quad (6.4)$$

The typical situations are shown in the followings :

(i) The state  $\omega \in \mathfrak{S}_1$  is a vacuum state :  $\omega = |0\rangle \langle 0|$ . Then

$$I(\rho; \Lambda^*) = \sum_{k,t} \lambda_k^\rho |d_{t,k}^{(k)}|^2 \log \frac{|d_{t,k}^{(k)}|^2}{\sum_l \lambda_l^\rho |d_{t,l}^{(l)}|^2} \quad (6.5)$$

(ii) The state  $\omega \in \mathfrak{S}_1$  is a Gibbs state :

$$\omega_\beta = \frac{e^{-\beta H_1}}{\text{tr} e^{-\beta H_1}} = \sum_n \frac{e^{-\beta n}}{\text{tr} e^{-\beta H_1}} |n\rangle \langle n| = (1 - e^{-\beta}) \sum_n e^{-\beta n} |n\rangle \langle n|$$

Then

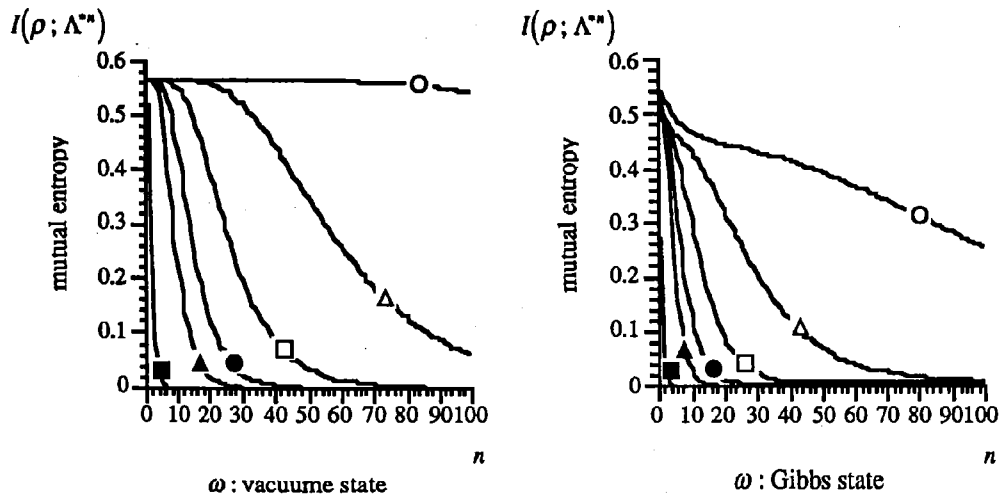
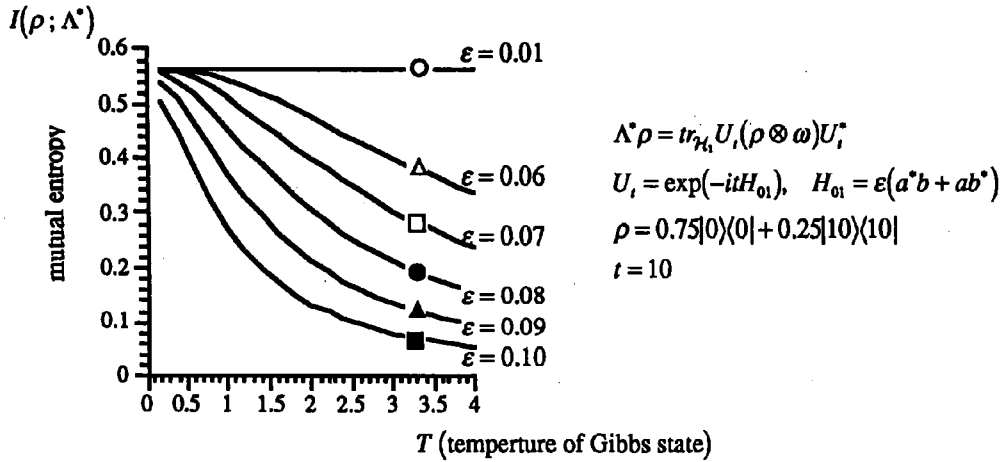
$$I(\rho; \Lambda^*) = (1 - e^{-\beta}) \sum_{k,t} \lambda_k^\rho D_{k,t}^\beta \log \frac{D_{k,t}^\beta}{\sum_l \lambda_l^\rho D_{l,t}^\beta} \quad (6.6)$$

where

$$D_{k,t}^\beta = \sum_{\substack{\nu \\ (\nu-k)+t \geq 0}} e^{-\beta((\nu-k)+t)} |d_{t,k}^{(\nu+t)}|^2 \quad (6.7)$$

## 7 Computer Experiments

Under the above computations we show some numerical results the change of the mutual entropy  $I(\rho; \Lambda^*)$  w.r.t.  $T$ , the temperature of Gibbs state in the reservoir, and  $I(\rho; \Lambda^{*n})$  w.r.t.  $n$ , the times of the channel, respectively.



As expected, the mutual entropy decreases with respect to  $n$ , and the Gibbs state plays as a noise and its effect is obviously larger than that of the vacuume state. These numerical experiments show that our model of open system gives a dissipative change of a state, although the interaction Hamiltonian is symmetric in two modes  $a$  and  $b$ .

## 8 Conclusion

In this paper we first established the Stinespring expression in an open system damping model, and we rigorously compute the mutual entropy through a typical model of QMC.

Once we have the concrete expression of the mutual entropy, it will be useful to investigate the dynamical change of the mutual entropy in analysis of communication

processes and nonequilibrium behavior in statistical mechanics. The detail discussions for the change of mutual entropy will be reported in the forthcoming paper [A7].

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