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学位申請論文

芦野隆一

On Nagumo's H^S -stability in Singular Perturbations

Dedicated to Professor Shigetake Matsuura
on the sixtieth anniversary of his birthday

By

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1. Introduction

In [5], Nagumo defined the H^S -stability in singular perturbations. Here $H^S = H^S(R_{x'}^{n-1})$ is the global Sobolev space with the norm

$$\|u(x')\|_S = \left[(2\pi)^{-n+1} \int |\hat{u}(\xi')|^2 (1+|\xi'|^2)^S d\xi' \right]^{1/2}.$$

We shall generalize the notion of H^S -stability in some sense.

Let us consider the following linear partial differential operator with constant coefficients containing small positive parameter ε ($0 \leq \varepsilon < 1$):

$$L_\varepsilon(D) = \varepsilon \cdot P_1(D) + P_2(D)$$

Denote by m the order of $P_1(D)$ with respect to D_1 and by m' that of $P_2(D)$. Put $m'' = m - m'$ and assume that $m > m' > 0$. Then the order of L_0 is less than that of L_ε for $\varepsilon \neq 0$. Such an operator as L_ε is called a singularly perturbed operator.

We shall study the following so-called singularly perturbed Cauchy problem for $L_\varepsilon(D)$:

$$(CP) \quad \begin{cases} L_\varepsilon(D)u(x) = f_\varepsilon(x), \text{ in } [0, T] \times R_{x'}^{n-1}; \\ D_1^{j-1}u(0, x') = \phi_{\varepsilon, j}(x'), \quad j=1, \dots, m, \end{cases}$$

and the following so-called reduced Cauchy problem for (CP):

$$(RCP) \quad \begin{cases} L_0(D)u(x) = f_0(x), \text{ in } [0, T] \times R_{x'}^{n-1}; \\ D_1^{j-1}u(0, x') = \phi_{0, j}(x'), \quad j=1, \dots, m' \end{cases}$$

The following assumption on P_1 and P_2 will be required.

Assumption 1.

(A1): The symbols of $P_1(D)$ and $P_2(D)$ are represented as

$$P_1(\xi) = \sum_{j=0}^m p_{1,j}(\xi') \xi_1^{m-j},$$

$$P_2(\xi) = \sum_{j=0}^{m'} p_{2,j}(\xi') \xi_1^{m'-j},$$

where $p_{1,0}$ and $p_{2,0}$ are non-zero constants.

(A2): ($m=2$ and $p_{2,0}/p_{1,0}$ is negative real number) or

($m=1$ and the imaginary part of $p_{2,0}/p_{1,0}$ is non-positive)

The following assumption on the Cauchy data and on the solvability of (CP) and (RCP) will be required.

Assumption 2.

There exist real numbers s and s' such that (CP) is uniquely solvable in $C([0,T];H^s)$ and (RCP) is uniquely solvable in $C([0,T];H^s)$ for the Cauchy data $\phi_{\epsilon,j}(x')$ and $\phi_{0,j}(x')$ belong to $H^{s'}$ and $f_{\epsilon}(x)$ and $f_0(x)$ belong to $C([0,T];H^{s'})$

Nagumo defined the H^s -stability of (CP) with respect to a particular solution u_0 of (RCP) in [5] as follows:

Definition 1. Let Assumption 2 be satisfied for $s'=s$.

The Cauchy problem (CP) is said to be H^s -stable in $0 \leq x_1 \leq T$ for $\epsilon \downarrow 0$ with respect to a particular solution $u_0(x)$ of the reduced Cauchy problem (RCP) in $C^m([0,T];H^s)$ if

$$(D1) \quad \sup_{0 \leq x_1 \leq T} \|u_\varepsilon(x_1, \cdot) - u_0(x_1, \cdot)\|_S \rightarrow 0$$

whenever $u_\varepsilon(x)$ are solutions of (CP) in $C^m([0, T]; H^S)$ satisfying the following three conditions:

$$(D2) \quad \sup_{0 \leq x_1 \leq T} \|f_\varepsilon(x_1, \cdot) - f_0(x_1, \cdot)\|_S \rightarrow 0;$$

$$(D3) \quad \|\phi_{\varepsilon, j} - \phi_{0, j}\|_S \rightarrow 0, \quad j=1, \dots, m';$$

$$(D4) \quad \|\phi_{\varepsilon, j}(\cdot) - D_1^{j-1} u_0(0, \cdot)\|_S \rightarrow 0, \quad j=m'+1, \dots, m.$$

If $f_0(x)$ belongs to $C^{m-m'}([0, T]; H^{S'})$ then the initial values $D_1^{j-1} u_0(0, x')$, $j=m'+1, \dots, m$ are uniquely determined and represented as a sum of derivatives of $f_0(x)$ and $\phi_{0, j}(x')$, $j=1, \dots, m'$. When (D4) is required, then the Cauchy data $\phi_{\varepsilon, j}(x')$, $j=m'+1, \dots, m$ are very restricted. For example, when $f_0=0$ and $\phi_{0, j}=0$, $j=1, \dots, m'$, (D4) implies that $\phi_{\varepsilon, j} \rightarrow 0$, $j=1, \dots, m$. Hence another definition of the stability whose convergence on the Cauchy data $\phi_{\varepsilon, j}(x')$, $j=m'+1, \dots, m$ are different from Nagumo's is needed.

Definition 2. Let Assumption 2 be satisfied.

The Cauchy problem (CP) is said to be (s, s') -stable in $0 \leq x_1 \leq T$ for $\varepsilon > 0$ with respect to a particular solution $u_0(x)$ of the reduced Cauchy problem (RCP) in $C^m([0, T]; H^{\max\{s, s'\}})$ if

$$(D1) \quad \sup_{0 \leq x_1 \leq T} \|u_\varepsilon(x_1, \cdot) - u_0(x_1, \cdot)\|_S \rightarrow 0.$$

whenever $u_\varepsilon(x)$ are solutions of (CP) in $C^m([0, T]; H^{\max\{s, s'\}})$ satisfying the following three conditions:

$$(D5) \quad \sup_{0 \leq x_1 \leq T} \|f_\varepsilon(x_1, \cdot) - f_0(x_1, \cdot)\|_{s'} \rightarrow 0;$$

$$(D6) \quad \|\phi_{\varepsilon, j} - \phi_{0, j}\|_{s'} \rightarrow 0, \quad j=1, \dots, m';$$

(D7): There exists a positive number M , which may depend on the choice of the initial data $\phi_{\varepsilon, j}$, $\phi_{0, j}$, and f_0 such that

$$\|\phi_{\varepsilon, j}(\cdot) - D_1^{j-1} u_0(0, \cdot)\|_{s'} \leq M, \quad j=m'+1, \dots, m.$$

The Cauchy problem (CP) is said to be $(s, s'+0)$ -stable in $0 \leq x_1 \leq T$ for $\varepsilon \rightarrow 0$ with respect to a particular solution $u_0(x)$ of (RCP) in $C^m([0, T]; H^{\max\{s, s'\}})$ if (D1) whenever $u_\varepsilon(x)$ are solutions of (CP) in $C^m([0, T]; H^{\max\{s, s'\}})$ satisfying (D5), (D6), and

(D8): There exist positive numbers δ and M , which may depend on the choice of the initial data $\phi_{\varepsilon, j}$, $\phi_{0, j}$, and f_0 such that

$$\|\phi_{\varepsilon, j}(\cdot) - D_1^{j-1} u_0(0, \cdot)\|_{s'+\delta} \leq M, \quad j=m'+1, \dots, m.$$

Remark. For every positive number δ , the (s, s') -stability implies the $(s, s'+0)$ -stability, the $(s, s'+0)$ -stability implies the $(s, s'+\delta)$ -stability, and the $(s, s'+\delta)$ -stability implies the $(s-\delta, s')$ -stability

It will be shown that requiring (A2) is natural when we deal with the (s, s') -stability with respect to solutions of (RCP) for various Cauchy data. Following to the definition of the

C-admissibility of (CP) with respect to (RCP) in [4], we shall define the $C([0,T];H^S)$ -admissibility of (CP) with respect to (RCP)

Definition 3. Let Assumption 2 be satisfied. The Cauchy problem (CP) is said to be $C([0,T];H^S)$ -admissible in $[0,T] \times R^{n-1}$ with the Cauchy data space $(H^{S'})^m$ with respect to (RCP) if for every Cauchy datum $(\psi_1, \dots, \psi_m) \in (H^{S'})^m$, the solutions u_ϵ of (CP) with $\phi_{\epsilon,j} = \psi_j$, $j=1, \dots, m$ and $f_\epsilon = 0$ converge in $C([0,T];H^S)$ to the solution u_0 of (RCP) with $\phi_{0,j} = \psi_j$, $j=1, \dots, m'$ and $f_0 = 0$.

By looking into the proof of Theorem in [2] and §2 and §3 in [3], we can prove that (A2) remains a necessary condition for the $C([0,T];H^S)$ -admissibility with the Cauchy data space $(H^\infty)^m$ when P_1 and P_2 satisfy (A1). We do not give the proof in this paper.

In [5], Nagumo gave a necessary and sufficient condition for the H^S -stability for more general system in the form of inequalities which must be satisfied by the solutions of (CP) with the initial conditions:

$$D_1^{j-1} u(0, x') = \delta_{i,j} \cdot \delta(x'), \quad i, j=1, \dots, m,$$

where $\delta_{i,j}$ is Kronecker's delta and $\delta(x')$ is the Dirac measure.

We have succeeded in seeking a necessary and sufficient condition for the $(s, s'+0)$ -stability but a necessary and sufficient condition for the (s, s') -stability is open. Our condition for

the $(s, s'+0)$ -stability which will be found in §2 is Nagumo type.

As a corollary, we can show that Nagumo's H^S -stability implies the $(s, s+0)$ -stability. In [6], Kumano-go applied Nagumo's result to the following operator:

$$\epsilon \cdot D_1^2 + q \cdot D_1 + Q(D'),$$

where q is a complex number and $Q(D')$ is a polynomial of D'

Kumano-go deduced conditions for the H^S -stability on the complex constant q and on the structure of the polynomial $Q(\xi')$. In §3, we shall give another example for the H^S -stability.

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2. The (s, s'+0)-stability

We shall use the notation and the result in Appendix. Denote the roots of $L_\varepsilon(\xi) = 0$ with respect to ξ_1 by $\tau_j(\varepsilon, \xi')$, $j=1, \dots, m$ and those of $L_0(\xi) = P_2(\xi) = 0$ with respect to ξ_1 by $\sigma_j(\xi')$, $j=1, \dots, m'$, respectively. It is well known that $\tau_j(\varepsilon, \xi')$, $j=1, \dots, m$ are continuous in (ε, ξ') for $\varepsilon \neq 0$ and $\sigma_j(\xi')$, $j=1, \dots, m'$ are continuous in ξ' . Put

$$b(\tau) = (\tau^{j-1}; j+1, \dots, m) \text{ and } c_j = (\delta_{j,k}, k+1, \dots, m),$$

where $\delta_{j,k}$ is Kronecker's delta. Other notation can be found in Appendix. Denote by $Y_j(\varepsilon, x_1, \xi')$, $j=1, \dots, m$ the fundamental solutions of the following ordinary differential equation with parameter (ε, ξ') :

$$L_\varepsilon(D_1, \xi') Y(\varepsilon, x_1, \xi') = 0$$

with initial conditions:

$$D_1^{k-1} Y(\varepsilon, 0, \xi') = \delta_{j,k}, \quad j, k=1, \dots, m,$$

Then Cramer's formula implies that if $\tau_i \neq \tau_j$, $1 \leq i < j \leq m$ then

$$\begin{aligned} & Y_j(\varepsilon, x_1, \xi') \\ &= \int_{k=1}^m \exp i \tau_k x_1 \cdot \frac{\det(b(\tau_1), \dots, b(\tau_{k-1}), c_j, b(\tau_{k+1}), \dots, b(\tau_m))}{\det(b(\tau_1), \dots, b(\tau_m))} \\ &= \frac{\det({}^t(t_a(0), t_a(1), \dots, t_a(j-2), t_e, t_a(j), \dots, t_a(m-1)))}{A(0, 1, \dots, m-1)} \\ &= (-1)^{j-1} \cdot D(0, 1, \dots, j-2, j, \dots, m-1)(\tau_1, \dots, \tau_m, x_1), \quad j=1, \dots, m. \end{aligned}$$

But the last representations remain valid without any restriction

on τ_j , $j=1, \dots, m$. Denote by l the maximum of the polynomial orders of the coefficients $p_{1,j}(\xi')$, $j=0, \dots, m$ in the symbol $P_1(\xi)$ and put

$$\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$$

Then we have the following theorem whose proof will be found at the end of this section.

Theorem 1. Let Assumption 1 and 2 be satisfied. Then the following four conditions are equivalent:

(C1) The Cauchy problem (CP) is $(s, s'+0)$ -stable in $[0, T]$ for $\varepsilon \downarrow 0$ with respect to a particular solution $u_0(x)$ of (RCP)

belonging to $C^m([0, T]; H^{\max\{s, s'\}+l})$

(C2) The Cauchy problem (CP) is $(s, s'+0)$ -stable in $[0, T]$ for $\varepsilon \downarrow 0$ with respect to every solution $u_0(x)$ of (RCP) belonging to

$C^m([0, T]; H^{\max\{s, s'\}+l})$

(C3) There exist positive numbers ε_0 and C_0 such that

$$(E1) \quad \sup_{0 < \varepsilon \leq \varepsilon_0, \xi' \in \mathbb{R}^{n-1}} \int_0^T \frac{1}{\varepsilon} |Y_m(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'}| dx_1 \leq C_0,$$

$$(E2) \quad \sup_{1 \leq j \leq m', 0 < \varepsilon \leq \varepsilon_0, 0 \leq x_1 \leq T, \xi' \in \mathbb{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'}| \leq C_0,$$

and for every positive number δ there exist positive numbers ε_δ and C_δ such that

$$(E3) \quad \sup_{m'+1 \leq j \leq m, 0 < \varepsilon \leq \varepsilon_\delta, 0 \leq x_1 \leq T, \xi' \in \mathbb{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'-\delta}| \leq C_\delta$$

(C4) There exist positive numbers ε'_0 , R_0 , and C'_0 such that

$$(E4) \quad \sup_{0 < \varepsilon \leq \varepsilon'_0, R_0 \leq |\xi'|} \int_0^T \frac{1}{\varepsilon} \cdot |Y_m(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'}| dx_1 \leq C'_0,$$

$$(E5) \quad \sup_{1 \leq j \leq m', 0 < \varepsilon \leq \varepsilon'_0, 0 \leq x_1 \leq T, R_0 \leq |\xi'|} |Y_j(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'}| \leq C'_0,$$

and for every positive number δ there exist positive numbers ε'_δ , R_δ , and C'_δ such that

$$(E6) \quad \sup_{m'+1 \leq j \leq m, 0 < \varepsilon \leq \varepsilon'_\delta, 0 \leq x_1 \leq T, R_\delta \leq |\xi'|} |Y_j(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'-\delta}| \leq C'_\delta.$$

Remark. Nagumo studied the H^S -stability in the following general situation:

$$L_\varepsilon = \sum_{j=0}^m L_j(\varepsilon, D') D_1^{m-j},$$

where the symbols $L_j(\varepsilon, \xi')$ are matrices of polynomials in ξ' with constant coefficients which depend continuously on the parameter $\varepsilon \geq 0$. He proved the equivalence between the following two conditions:

(C5) The Cauchy problem (CP) is H^S -stable in $[0, T]$ for $\varepsilon > 0$ with respect to a particular solution $u_0(x)$ of (RCP) belonging to $C^m([0, T]; H^{s+l})$.

(C6) There exist positive numbers ε_0 and C_0 such that

$$(E7) \quad \sup_{0 < \varepsilon \leq \varepsilon_0, \xi' \in \mathbb{R}^{n-1}} \int_0^T \frac{1}{\varepsilon} \cdot |Y_m(\varepsilon, x_1, \xi')| dx_1 \leq C_0;$$

$$(E8) \quad \sup_{1 \leq j \leq m, 0 < \varepsilon \leq \varepsilon_0, 0 \leq x_1 \leq T, \xi' \in \mathbb{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi')| \leq C_0.$$

Corollary 1. Let Assumption 1 and 2 be satisfied and $u_0(x)$ be a solution of (RCP) belonging to $C^m([0,T]; H^{s+l})$. If the Cauchy problem (CP) is H^s -stable in $[0,T]$ for $\epsilon \neq 0$ with respect to a particular solution u_0 , then the Cauchy problem (CP) is $(s, s+0)$ -stable in $[0,T]$ for $\epsilon \neq 0$ with respect to a particular solution u_0 .

Proof. Since Nagumo's theorem can be applied to our problem and obviously (E8) implies (E2) for $s=s'$ and (E3) for $s=s'$
[Q.E.D.]

To prove Theorem 1 we need several steps. For the solution u_0 of the reduced Cauchy problem (RCP), we shall consider the following singularly perturbed Cauchy problem:

$$(CP1) \quad \begin{cases} L_\epsilon(D)u(x) = f_\epsilon(x), \text{ in } [0,T] \times R^{n-1}; \\ D_1^{j-1}u(0, x') = \phi_{\epsilon,j}(x'), \quad j=1, \dots, m'; \\ D_1^{j-1}u(0, x') = D_1^{j-1}u_0(0, x'), \quad j=m'+1, \dots, m. \end{cases}$$

Here the initial values $D_1^{j-1}u(0, x')$, $j=m'+1, \dots, m$ are fixed.

The reduced Cauchy problem for (CP1) is (RCP). Denote by $u_{\epsilon,1}(x)$ the solution of (CP1)

Lemma 1. (due to Nagumo) Let (A1) and Assumption 2 be satisfied. Then the following two conditions are equivalent:

(C7) The Cauchy problem (CP1) is (s, s') -stable in $[0, T]$ for $\varepsilon > 0$ with respect to a particular solution $u_0(x)$ of (RCP) belonging to $C^m([0, T]; H^{\max\{s, s'\} + l})$

(C8) There exist positive numbers ε_0 and C_0 such that

$$(E1) \quad \sup_{0 < \varepsilon \leq \varepsilon_0, \xi' \in \mathbb{R}^{n-1}} \int_0^T \frac{1}{\varepsilon} \cdot |Y_m(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'}| dx_1 \leq C_0,$$

$$(E2) \quad \sup_{1 \leq j \leq m', 0 < \varepsilon \leq \varepsilon_0, 0 \leq x_1 \leq T, \xi' \in \mathbb{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'}| \leq C_0.$$

Proof. First we shall show (C8) implies (C7) Put

$$v_\varepsilon(x) = u_{\varepsilon,1}(x) - u_0(x),$$

$$g_\varepsilon(x) = L_0(D)u_0(x) - L_\varepsilon(D)u_0(x) + f_\varepsilon(x) - f_0(x)$$

Denote by $\hat{u}(x_1, \xi')$ the Fourier transform of $u(x)$ with respect to x' and by $F_{\xi' \rightarrow x'}^{-1}$, the inverse Fourier transformation. Then $v_\varepsilon(x)$ is given by

$$v_\varepsilon(x) = F_{\xi' \rightarrow x'}^{-1} \left\{ \sum_{j=1}^{m'} Y_j(\varepsilon, x_1, \xi') (\hat{\phi}_{\varepsilon,j}(\xi') - \hat{\phi}_{0,j}(\xi')) \right\} \\ + F_{\xi' \rightarrow x'}^{-1} \left\{ \int_0^{x_1} \frac{1}{p_{1,0} \cdot \varepsilon} \cdot Y_m(\varepsilon, x_1 - t, \xi') \hat{g}_\varepsilon(t, \xi') dt \right\}$$

Since

$$|\hat{v}_\varepsilon(x_1, \xi')| \langle \xi' \rangle^s \\ \leq \sum_{j=1}^{m'} |Y_j(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'}| |\hat{\phi}_{\varepsilon,j}(\xi') - \hat{\phi}_{0,j}(\xi')| \langle \xi' \rangle^{s'} \\ + \int_0^{x_1} \frac{1}{|p_{1,0}| \cdot \varepsilon} \cdot |Y_m(\varepsilon, x_1 - t, \xi') \langle \xi' \rangle^{s-s'}| |\hat{g}_\varepsilon(t, \xi')| \langle \xi' \rangle^{s'} dt,$$

it implies that

$$\begin{aligned} & \|v_\epsilon(x_1, \cdot)\|_s \\ & \leq C_0 \cdot \sum_{j=1}^{m'} \|\phi_{\epsilon,j} - \phi_{0,j}\|_{s'} + \frac{C_0}{|P_{1,0}|} \int_0^{x_1} \|g_\epsilon(t, \cdot)\|_s dt. \end{aligned}$$

By (D6), we have $\sum_{j=1}^{m'} \|\phi_{\epsilon,j} - \phi_{0,j}\|_{s'} \rightarrow 0$. Since u_0 belongs to $C^m([0, T]; H^{\max\{s, s'\} + k})$, it implies that

$$\sup_{0 \leq x_1 \leq T} \|L_0(D)u_0(x_1, \cdot) - L_\epsilon(D)u_0(x_1, \cdot)\|_{s'} \rightarrow 0.$$

Hence (D5) implies that $\sup_{0 \leq x_1 \leq T} \|g_\epsilon(x_1, \cdot)\|_{s'} \rightarrow 0$. Thus we have

$$\sup_{0 \leq x_1 \leq T} \|v_\epsilon(x_1, \cdot)\|_s \rightarrow 0.$$

Next we shall show (C7) implies (C8). Assume that (E2) is not satisfied. Then, for a certain j with $1 \leq j \leq m'$, there exist sequences $\{\epsilon_n\}$ with $\epsilon_n \rightarrow 0$ and $\{t_n\}$ with $0 \leq t_n \leq T$ and a sequence of open balls $\{S_n\}$, $S_n = \{|\xi' - \xi'_n| < r_n\}$ such that

$$(2.1) \quad |Y_j(\epsilon_n, t_n, \xi')| \langle \xi' \rangle^{s-s'} > n \quad \text{for } \xi' \text{ in } S_n,$$

$$(2.2) \quad 2^{-1} < (\langle \xi' \rangle / \langle \xi'_n \rangle)^{s'} < 2 \quad \text{for } \xi' \text{ in } S_n$$

Put

$$u_n(x) = c_n \cdot F_{\xi'_n \rightarrow x}^{-1} (Y_j(\epsilon_n, x_1, \xi') \cdot \chi(\xi'; S_n)),$$

where $c_n = n^{-1} \cdot |S_n|^{-1/2} \langle \xi'_n \rangle^{-s'}$. Then $u_n(x)$ satisfies

$$L_{\epsilon_n}(D)u(x) = 0. \quad \text{Since}$$

$$\begin{aligned} & |\hat{u}_n(t_n, \xi')| \langle \xi' \rangle^s \\ & = n^{-1} \cdot |S_n|^{-1/2} \langle \xi'_n \rangle^{-s'} |Y_j(\epsilon_n, t_n, \xi')| \chi(\xi'; S_n) \langle \xi' \rangle^s \\ & = n^{-1} \cdot |S_n|^{-1/2} (\langle \xi' \rangle / \langle \xi'_n \rangle)^{s'} |Y_j(\epsilon_n, t_n, \xi')| \langle \xi' \rangle^{s-s'} \chi(\xi'; S_n), \end{aligned}$$

(2.1) and (2.2) imply that

$$\sup_{0 \leq x_1 \leq T} \|u_n(x_1, \cdot)\|_S \geq \|u_n(t_n, \cdot)\|_S \geq 1/2$$

Since

$$\begin{aligned} |D_1^{j-1} \hat{u}_n(0, \xi')|_{\langle \xi' \rangle^{s'}} &= c_n \cdot \chi(\xi'; S_n)_{\langle \xi' \rangle^{s'}} \\ &= n^{-1} \cdot |S_n|^{-1/2} \cdot \chi(\xi'; S_n) (\langle \xi' \rangle / \langle \xi'_n \rangle)^{s'}, \end{aligned}$$

(2.2) implies that $\|D_1^{j-1} u_n(0, \cdot)\|_{S'} \leq 2/n \rightarrow 0$. For $k \neq j$, we have

$$\|D_1^{k-1} u_n(0, \cdot)\|_{S'} = 0. \quad \text{Put } u_{\varepsilon_n}(x) = u_n(x) + u_0(x) \quad \text{Then we}$$

have a contradiction to (D1), (D5), (D6), and (D7)

Assume that (E1) is not satisfied. Then there exist a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \rightarrow 0$ and a sequence of open balls $\{S_n\}$, $S_n = \{\xi' \in \mathbb{R}^{n-1}; |\xi' - \xi'_n| < r_n\}$ such that

$$(2.3) \quad \int_0^T \frac{1}{|P_{1,0}| \cdot \varepsilon_n} \cdot |Y_m(\varepsilon_n, T-x_1, \xi')_{\langle \xi' \rangle^{s-s'}}| dx_1 > n,$$

for ξ' in S_n . We choose $\phi_{\varepsilon, j}(x') = D_1^{j-1} u_0(0, x')$, $j=1, \dots, m$.

Then the solutions of (CP1) for $\{\varepsilon_n\}$ are given by

$$u_n(x) = u_0(x) + F_{\xi', x}^{-1} \left(\int_0^{x_1} \frac{1}{P_{1,0} \cdot \varepsilon_n} \cdot Y_m(\varepsilon_n, x_1-t, \xi') \hat{g}_{\varepsilon_n}(t, \xi') dt \right)$$

Put

$$Y_n(x_1, \xi') = \frac{1}{P_{1,0} \cdot \varepsilon_n} \cdot Y_m(\varepsilon_n, T-x_1, \xi')$$

As we shall show later by (2.5) in the proof of Lemma 3 that

$Y_m(\varepsilon, x_1, \xi')$ is continuous in (x_1, ξ') for fixed ε , it implies that

$Y_n(x_1, \xi')$ is continuous in (x_1, ξ') for every positive integer n .

For $E = \{(x_1, \xi'); Y_n(x_1, \xi') \neq 0\}$, denote by $\chi((x_1, \xi'); E)$ the

characteristic function of the set E . Put

$$H_n(x_1, \xi') = \chi((x_1, \xi'); E) \cdot \overline{Y_n(x_1, \xi')} / |Y_n(x_1, \xi')|$$

Then $|H_n(x_1, \xi')| \leq 1$ and (2.3) implies

$$\left| \int_0^T y_n(x_1, \xi') \langle \xi' \rangle^{s-s'} H_n(x_1, \xi') dx_1 \right| > n,$$

for ξ' in S_n . Approximate $H_n(x_1, \xi')$ in the sense of $L^1([0, T])$ valued in bounded functions in ξ' by the mollifier $\rho_\delta(x_1)_*$ with respect to x_1 . Put

$$h_{\delta, n}(x_1, \xi') = \int_{\mathbb{R}} \rho_\delta(x_1 - t) H_n(t, \xi') dt.$$

Then $h_{\delta, n}(x_1, \xi')$ are continuous functions with respect to x_1 in $[0, T]$ satisfying $|h_{\delta, n}(x_1, \xi')| \leq 1$. Since

$$\begin{aligned} & \left| \left| \int_0^T y_n(x_1, \xi') \langle \xi' \rangle^{s-s'} H_n(x_1, \xi') dx_1 \right| \right. \\ & \quad \left. - \left| \int_0^T y_n(x_1, \xi') \langle \xi' \rangle^{s-s'} h_{\delta, n}(x_1, \xi') dx_1 \right| \right| \\ & \leq \sup_{0 \leq x_1 \leq T} |y_n(x_1, \xi')| \cdot \langle \xi' \rangle^{s-s'} \cdot \int_0^T |h_{\delta, n}(x_1, \xi') - H_n(x_1, \xi')| dx_1, \end{aligned}$$

it implies that for ξ' in S_n there exist positive numbers $\delta_n(\xi')$ such that

$$\left| \int_0^T y_n(x_1, \xi') \langle \xi' \rangle^{s-s'} h_{\delta_n(\xi'), n}(x_1, \xi') dx_1 \right| > n,$$

for ξ' in S_n . Put

$$h_n(x_1, \xi') = h_{\delta_n(\xi'), n}(x_1, \xi'),$$

$$g_{\varepsilon_n}(x) = F_{\xi' \rightarrow x}^{-1} \cdot (n^{-1} |S_n|^{-1/2} h_n(x_1, \xi') \langle \xi' \rangle^{-s'} \chi(\xi'; S_n)),$$

where $|S_n|$ denotes the measure of S_n and $\chi(\xi'; S_n)$ is the

characteristic function of the ball S_n . We set $f_{\varepsilon_n} = f_0 + g_{\varepsilon_n}$

Then

$$\|g_{\varepsilon_n}(x_1, \cdot)\|_{S'} \leq \frac{1}{n} \rightarrow 0.$$

Since

$$\begin{aligned} & (\hat{u}_n(T, \xi') - \hat{u}_0(T, \xi'))_{\langle \xi' \rangle^S} \\ &= \int_0^T Y_n(x_1, \xi')_{\langle \xi' \rangle^{S-S'}} \cdot \hat{g}_{\varepsilon_n}(x_1, \xi')_{\langle \xi' \rangle^{S'}} dx_1 \\ &= \int_0^T Y_n(x_1, \xi')_{\langle \xi' \rangle^{S-S'}} \cdot h_n(x_1, \xi') dx_1 \cdot n^{-1} |S_n|^{-1/2} \chi(\xi'; S_n), \end{aligned}$$

it implies that $\|u_n(T, \cdot) - u_0(T, \cdot)\|_S \geq 1$. This contradicts (D1), (D5), (D6), and (D7)

[Q.E.D.]

Put

$$B_R = \{|\xi'| \leq R\}, \quad p = p_{2,0}/p_{1,0}, \quad \theta = \arg -p, \quad \theta = \exp i\theta/m'',$$

$$\zeta = \exp 2\pi i/m'', \text{ and } \tau_j^1 = \zeta^{j-m'-1}, \quad j=m'+1, \dots, m.$$

By the same argument as in Lemma 2.2 in [3], it implies the following lemma whose proof is omitted.

Lemma 2. Let (A1) in Assumption 1 be satisfied. Then, for every positive number R , there exist a positive number ε_R with $\varepsilon_R < 1$ and continuous functions $\tau_{j,1}(\varepsilon, \xi')$, $j=1, \dots, m$ on $(0, \varepsilon_R] \times B_R$ satisfying

$$\limsup_{\varepsilon \rightarrow 0} \sup_{\xi' \in B_R} |\tau_{j,1}(\varepsilon, \xi')| = 0, \text{ for } j=1, \dots, m$$

such that for $m'+1 \leq i < j \leq m$ and for $1 \leq i \leq m'$, $m'+1 \leq j \leq m$

$$\tau_i(\varepsilon, \xi') \neq \tau_j(\varepsilon, \xi') \text{ on } (0, \varepsilon_R] \times B_R,$$

and

$$\tau_j(\varepsilon, \xi') = \sigma_j(\xi') + \tau_{j,1}(\varepsilon, \xi'), \text{ for } j=1, \dots, m';$$

$$\varepsilon^{1/m''} \cdot \tau_j(\varepsilon, \xi') = \theta \tau_j' \cdot |p|^{1/m''} + \tau_{j,1}(\varepsilon, \xi'), \text{ for } j=m'+1, \dots, m.$$

Lemma 3. Let Assumption 1 be satisfied and ε_R be the same as in Lemma 2. For every positive number R , there exists a positive number $C_{1,R}$ such that

$$(2.4) \quad \sup_{0 < \varepsilon \leq \varepsilon_R, \quad 0 \leq x_1 \leq T, \quad |\xi'| \leq R} \varepsilon^{-\max\{(j-m'), 0\}/m''} |Y_j(\varepsilon, x_1, \xi')| \leq C_{1,R}, \text{ for } j=1, \dots, m.$$

Proof. Fix an arbitrary positive number R and assume that $0 < \varepsilon \leq \varepsilon_R$. For arbitrary roots $\tau_j = \tau_j(\varepsilon, \xi')$, $j=1, \dots, m$, which do not need to be distinct,

$$(2.5) \quad Y_j(\varepsilon, x_1, \xi') = (-1)^{j-1} \cdot D(0, 1, \dots, j-2, j, \dots, m-1)(\tau_1, \dots, \tau_m, x_1), \quad j=1, \dots, m.$$

As we have already shown in Theorem in [2], (A2) in Assumption 1 implies that the imaginary parts of $\theta \tau_j'$, $j=m'+1, \dots, m$ are

non-negative. Put $\eta = \varepsilon^{1/m''}$, $\eta_R = \varepsilon_R^{1/m''}$, $z_j = \tau_j(\varepsilon, \xi')$,

$j=1, \dots, m$, and $w_j = \varepsilon^{1/m''} \cdot \tau_j(\varepsilon, \xi')$, $j=1, \dots, m$. Then Assumption

1 implies that for every positive number R , there exist positive numbers M_R , M'_R , and c_R such that (A.8) in Lemma A.3 in Appendix

is satisfied for $M=M_R$, $M'=M'_R$, $c=c_R$, and $\eta_0=\eta_R$. Hence Lemma A.3

can be applied to (2.5). Since $D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1)$,

ρ in S_2 are entire in z' and continuous in x_1 for $0 \leq x_1 \leq T$, it

implies that there exists a positive number $C_{2,R}$ such that

$$\max_{\rho \in S_2} |D(\rho(1), \rho(2), \dots, \rho(m'-1))(\tau_1, \dots, \tau_m, x_1)| \leq C_{2,R},$$

on $[0, \varepsilon_R] \times [0, T] \times B_R$. Since $E(w)$ is holomorphic for $w_1 \neq w_j$, $1 \leq i \leq m'$ and $m'+1 \leq j \leq m$, Lemma 2 implies that there exists a positive number $C_{3,R}$ such that for $j=1, \dots, m'$

$$\begin{aligned} & |D(0, 1, \dots, j-2, j, \dots, m'-1)(\tau_1, \dots, \tau_m, x_1) \\ & \times ((\varepsilon^{1/m''} \cdot \tau_{m'+1}) \cdot \dots \cdot (\varepsilon^{1/m''} \cdot \tau_m))^{m'} \cdot E(\varepsilon^{1/m''} \cdot \tau_1, \dots, \varepsilon^{1/m''} \cdot \tau_m)| \\ & \leq C_{3,R}, \end{aligned}$$

on $[0, \varepsilon_R] \times [0, T] \times B_R$. Then

$$\begin{aligned} & |D(0, 1, \dots, j-2, j, \dots, m-1)(\tau_1, \dots, \tau_m, x_1)| \\ & \leq C_{3,R} + (C_1 + C_2 \cdot C_{2,R}) \cdot \varepsilon^{1/m''}, \end{aligned}$$

for $j=1, \dots, m'$ and

$$\begin{aligned} & |D(0, 1, \dots, j-2, j, \dots, m-1)(\tau_1, \dots, \tau_m, x_1)| \\ & \leq (C_1 + C_2 \cdot C_{2,R}) \cdot \varepsilon^{(j-m')/m''}, \end{aligned}$$

for $j=m'+1, \dots, m$. Put $C_{1,R} = C_{3,R} + C_1 + C_2 \cdot C_{2,R}$, then we have

(2.4)

[Q.E.D.]

Denote by $y_j(x_1, \xi')$, $j=1, \dots, m'$ the fundamental solutions of the following ordinary differential equation with parameter ξ' :

$$L_0(D_1, \xi') y(x_1, \xi') = 0$$

with initial conditions:

$$D_1^{k-1} y(0, \xi') = \delta_{j,k}, \quad j, k=1, \dots, m',$$

where $\delta_{j,k}$ is Kronecker's delta. As we have already shown

$$\begin{aligned}
(2.6) \quad & y_j(x_1, \xi') \\
& = (-1)^{j-1} \cdot D(0, 1, \dots, j-2, j, \dots, m'-1) (\sigma_1, \dots, \sigma_{m'}, x_1), \quad j=1, \dots, m', \\
& \text{where } \sigma_j = \sigma_j(\xi'), \quad j=1, \dots, m' \text{ are roots appearing in Lemma 2.}
\end{aligned}$$

Lemma 4. Let Assumption 1 be satisfied and ε_R be the same as in Lemma 2. Then

$$(2.7) \quad Y_j(\varepsilon, x_1, \xi') \rightarrow y_j(x_1, \xi'), \quad j=1, \dots, m';$$

$$(2.8) \quad Y_j(\varepsilon, x_1, \xi') \rightarrow 0, \quad j=m'+1, \dots, m,$$

uniformly on $[0, T] \times B_R$ when $\varepsilon \downarrow 0$.

Moreover, $Y_j(\varepsilon, x_1, \xi')$, $j=1, \dots, m$ satisfy

$$(E8) \quad \sup_{1 \leq j \leq m, \quad 0 < \varepsilon \leq \varepsilon_0, \quad 0 \leq x_1 \leq T, \quad \xi' \in \mathbb{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi')| \leq C_0$$

then $y_j(x_1, \xi')$, $j=1, \dots, m'$ satisfy

$$(E9) \quad \sup_{1 \leq j \leq m', \quad 0 \leq x_1 \leq T, \quad \xi' \in \mathbb{R}^{n-1}} |y_j(x_1, \xi')| \leq C_0$$

Proof. By Lemma 3, (2.8) is obvious and it suffices to show that for $j=1, \dots, m'$

$$\begin{aligned}
& (-1)^{j-1} \cdot D(0, 1, \dots, j-2, j, \dots, m'-1) (\tau_1, \dots, \tau_{m'}, x_1) \\
& \times ((\varepsilon^{1/m''} \cdot \tau_{m'+1}) \cdot \dots \cdot (\varepsilon^{1/m''} \cdot \tau_m))^{m'} \cdot E(\varepsilon^{1/m''} \cdot \tau_1, \dots, \varepsilon^{1/m''} \cdot \tau_m) \\
& \rightarrow y_j(x_1, \xi')
\end{aligned}$$

Since $\tau_j(\varepsilon, \xi') \rightarrow \sigma_j(\xi')$, $j=1, \dots, m'$ uniformly on B_R when $\varepsilon \downarrow 0$ by Lemma 2, it implies that for $j=1, \dots, m'$

$$(-1)^{j-1} \cdot D(0, 1, \dots, j-2, j, \dots, m'-1) (\tau_1, \dots, \tau_{m'}, x_1) \rightarrow y_j(x_1, \xi')$$

On the other hand,

$$\begin{aligned}
& \left((\varepsilon^{1/m''} \cdot \tau_{m'+1}) \cdot \dots \cdot (\varepsilon^{1/m''} \cdot \tau_m) \right)^{m'} \\
& \rightarrow \left((\theta \cdot \tau_{m'+1}' \cdot |p|^{1/m''}) \cdot \dots \cdot (\theta \cdot \tau_m' \cdot |p|^{1/m''}) \right)^{m'}
\end{aligned}$$

and

$$\begin{aligned}
& E(\varepsilon^{1/m''} \cdot \tau_1, \dots, \varepsilon^{1/m''} \cdot \tau_m) \\
& \rightarrow E(0, \dots, 0, (\theta \cdot \tau_{m'+1}' \cdot |p|^{1/m''}), \dots, (\theta \cdot \tau_m' \cdot |p|^{1/m''})) \\
& = 1 / \left((\theta \cdot \tau_{m'+1}' \cdot |p|^{1/m''}) \cdot \dots \cdot (\theta \cdot \tau_m' \cdot |p|^{1/m''}) \right)^{m'}
\end{aligned}$$

Thus we have (2.7)

Since R is arbitrary, (2.7) and (E8) imply (E9)

[Q.E.D.]

Let us consider the following singularly perturbed Cauchy problem:

$$\text{(CP2)} \quad \begin{cases} L_\varepsilon(D)u(x) = 0, \text{ in } [0, T] \times \mathbb{R}_{x'}^{n-1}; \\ D_1^{j-1}u(0, x') = 0, \quad j=1, \dots, m'; \\ D_1^{j-1}u(0, x') = \phi_{\varepsilon, j}(x'), \quad j=m'+1, \dots, m, \end{cases}$$

and its reduced Cauchy problem:

$$\text{(RCP2)} \quad \begin{cases} L_0(D)u(x) = 0, \text{ in } [0, T] \times \mathbb{R}_{x'}^{n-1}; \\ D_1^{j-1}u(0, x') = 0, \quad j=1, \dots, m' \end{cases}$$

Denote by $u_{\varepsilon, 2}(x)$ the solution of (CP2) and by $u_{0, 2}(x)$ the solution of (RCP2). Then $u_{0, 2}(x) = 0$.

Lemma 5. Let Assumption 1 be satisfied and ε_R be the same as in Lemma 2. Assume that every support of the datum $\hat{\phi}_{\varepsilon, j}(\xi')$,

$j=m'+1, \dots, m$ in (CP2) is contained in the closed ball B_R . Then, for arbitrary real numbers s and s' there exists a positive number K_R which is independent of ϵ such that for $0 < \epsilon \leq \epsilon_R$,

$$(2.9) \quad \sup_{0 \leq x_1 \leq T} \|u_{\epsilon,2}(x_1, \cdot)\|_s \leq K_R \cdot \sum_{\ell=m'+1}^m \epsilon^{(\ell-m')/m''} \cdot \|\phi_{\epsilon,\ell}\|_s.$$

Remark. Here we do not use any conditions on the fundamental solutions Y_j but use (A2) in Assumption 1. Lemma 4 shows that (A2) ensures the boundedness of Y_j on $[0, T] \times B_R$ when $\epsilon > 0$.

Proof of Lemma 5. It is well known that the solution $u_{\epsilon,2}(x)$ of (CP2) satisfies

$$\hat{u}_{\epsilon,2}(x_1, \xi') = \sum_{j=m'+1}^m Y_j(\epsilon, x_1, \xi') \cdot \hat{\phi}_{\epsilon,j}(\xi').$$

Lemma 3 implies

$$|\hat{u}_{\epsilon,2}(x_1, \xi')| \leq C_{1,R} \cdot \sum_{\ell=m'+1}^m \epsilon^{(\ell-m')/m''} \cdot |\hat{\phi}_{\epsilon,\ell}|,$$

on $[0, T] \times B_R$. Thus

$$\begin{aligned} (2\pi)^{-n+1} \int_{|\xi'| \leq R} |\hat{u}_{\epsilon,2}(x_1, \xi') \langle \xi' \rangle^s|^2 d\xi' \\ \leq C_{1,R}^2 \cdot m'' \end{aligned}$$

$$\times \sum_{\ell=m'+1}^m (2\pi)^{-n+1} \int_{|\xi'| \leq R} |\epsilon^{(\ell-m')/m''} \cdot \hat{\phi}_{\epsilon,\ell}(\xi') \langle \xi' \rangle^s|^2 d\xi'$$

Put $K_R = C_{1,R} \cdot m''^{1/2} \cdot \sup_{|\xi'| \leq R} \langle \xi' \rangle^{s-s'}$. Then we have (2.9)

[Q.E.D.]

The following corollary shows us that the stability is very strong when the Cauchy problem is admissible.

Corollary 2. Let Assumption 1 be satisfied and ε_R be the same as in Lemma 2. Then, for every positive number ε with $\varepsilon \leq \varepsilon_R$, there exist Cauchy data $\phi_{\varepsilon,j}$, $j=m'+1, \dots, m$ belonging to H^∞ such that for arbitrary real numbers s and s' ,

$$\|\phi_{\varepsilon,j}\|_{s'} \rightarrow \infty, \quad j=m'+1, \dots, m;$$

$$\sup_{0 \leq x_1 \leq T} \|u_{\varepsilon,2}(x_1, \cdot)\|_s \rightarrow 0,$$

where $u_{\varepsilon,2}$ are the solutions of (CP2) for these data $\phi_{\varepsilon,j}$, $j=m'+1, \dots, m$.

Proof. Choose non-trivial $C_0^\infty(B_R)$ -functions $\psi_j(\xi')$, $j=m'+1, \dots, m$ and a positive number α with $\alpha < 1/m''$. Put

$$\phi_{\varepsilon,j}(x') = \varepsilon^{-\alpha} \cdot F_{\xi' \rightarrow x'}^{-1}(\psi_j(\xi')), \quad j=m'+1, \dots, m,$$

which are rapidly decreasing functions. If $s' < 0$, then

$$\|\phi_{\varepsilon,j}\|_{s'} \geq \varepsilon^{-\alpha \cdot \langle R \rangle^{s'}} \|F^{-1}(\psi_j)\|_0 \rightarrow \infty.$$

when $\varepsilon \rightarrow 0$. If $s' > 0$, then

$$\|\phi_{\varepsilon,j}\|_{s'} \geq \|\phi_{\varepsilon,j}\|_{-s'} \geq \varepsilon^{-\alpha \cdot \langle R \rangle^{-s'}} \|F^{-1}(\psi_j)\|_0 \rightarrow \infty,$$

when $\varepsilon \rightarrow 0$. By (2.9),

$$\sup_{0 \leq x_1 \leq T} \|u_{\varepsilon,2}(x_1, \cdot)\|_s \leq \varepsilon^{1/m'' - \alpha} \cdot K_R \cdot \sum_{j=m'+1}^m \|F^{-1}(\psi_j)\|_{s'} \rightarrow 0,$$

when $\varepsilon \rightarrow 0$.

{Q.E.D.}

Lemma 6. Let the same assumption as in Theorem 1 be satisfied. Consider the singularly perturbed Cauchy problem (CP2) and the reduced Cauchy problem (RCP2) for (CP2). Assume that for the Cauchy data $\phi_{\varepsilon,j}$, $j=1, \dots, m$ there exist positive numbers δ and M such that $\sup_{1 \leq j \leq m} \|\phi_{\varepsilon,j}\|_{s'+\delta} \leq M$. Then

the following two conditions are equivalent:

(C9) The Cauchy problem (CP2) is $(s, s'+0)$ -stable in $[0, T]$ for $\varepsilon > 0$ with respect to a particular solution $u_{0,2} = 0$ of (RCP2)

(C10) For every positive number δ there exist positive numbers ε_δ and C_δ such that

$$(E3) \quad \sup_{m'+1 \leq j \leq m, 0 < \varepsilon \leq \varepsilon_\delta, 0 \leq x_1 \leq T, \xi' \in \mathbb{R}^{n-1}} |Y_j(\varepsilon, x_1, \xi') \langle \xi' \rangle^{s-s'-\delta}| \leq C_\delta$$

Proof. First we shall show (C10) implies (C9). We have only to show that if $\sup_{1 \leq j \leq m} \|\phi_{\varepsilon,j}\|_{s'+\delta} \leq M$ then

$\sup_{0 \leq x_1 \leq T} \|u_{\varepsilon,2}(x_1, \cdot)\|_s \rightarrow 0$. As we have already shown in the proof

of Lemma 1, the solution $u_{\varepsilon,2}(x)$ of (CP2) satisfies

$$\hat{u}_{\varepsilon,2}(x_1, \xi') = \sum_{j=m'+1}^m Y_j(\varepsilon, x_1, \xi') \cdot \hat{\phi}_{\varepsilon,j}(\xi')$$

Denote by $\chi(\xi'; B_R)$ the characteristic function of the ball B_R .

Put

$$\hat{v}_{\varepsilon,2}(x_1, \xi') = \hat{u}_{\varepsilon,2}(x_1, \xi') \cdot \chi(\xi'; B_R),$$

$$\hat{w}_{\varepsilon,2}(x_1, \xi') = \hat{u}_{\varepsilon,2}(x_1, \xi') \cdot (1 - \chi(\xi'; B_R))$$

Then $v_{\varepsilon,2}(x) = F_{\xi' \rightarrow x}^{-1}(\hat{v}_{\varepsilon,2}(x_1, \xi'))$ is the solution of (CP2) with

the initial conditions:

$$D_1^{j-1} u(0, x') = 0, \quad j=1, \dots, m';$$

$$D_1^{j-1} u(0, x') = F_{\xi' \rightarrow x'}^{-1} (\hat{\phi}_{\varepsilon, j}(\xi') \cdot \chi(\xi'; B_R)), \quad j=m'+1, \dots, m.$$

Since the supports of the Fourier transforms of these Cauchy data are contained in the ball B_R , we can apply Lemma 5 to $v_{\varepsilon, 2}(x)$

Obviously

$$\|F_{\xi' \rightarrow x'}^{-1} (\hat{\phi}_{\varepsilon, \ell}(\xi') \cdot \chi(\xi'; B_R))\|_{S'} \leq \|\phi_{\varepsilon, \ell}\|_{S'},$$

(2.9) and $0 < \varepsilon \leq \varepsilon_R < 1$ imply that

$$\begin{aligned} (2.10) \quad & \sup_{0 \leq x_1 \leq T} \|v_{\varepsilon, 2}(x_1, \cdot)\|_S \\ & \leq K_R \cdot \varepsilon^{1/m''} \cdot \sum_{\ell=m'+1}^m \|F_{\xi' \rightarrow x'}^{-1} (\hat{\phi}_{\varepsilon, \ell}(\xi') \cdot \chi(\xi'; B_R))\|_{S'} \\ & \leq K_R \cdot \varepsilon^{1/m''} \cdot \sum_{\ell=m'+1}^m \|\phi_{\varepsilon, \ell}\|_{S'} \end{aligned}$$

Choose a positive number δ' satisfying $\delta' < \delta$ and put $\delta'' = \delta - \delta'$

Since

$$\begin{aligned} & |\hat{w}_{\varepsilon, 2}(x_1, \xi') \cdot \langle \xi' \rangle^S| \\ & \leq \sum_{j=m'+1}^m |Y_j(\varepsilon, x_1, \xi') \cdot \langle \xi' \rangle^{S-S'-\delta'}| \\ & \quad \cdot |\hat{\phi}_{\varepsilon, j}(\xi') \cdot \langle \xi' \rangle^{S'+\delta}| \cdot |1 - \chi(\xi'; B_R)| \cdot \langle \xi' \rangle^{-\delta''}, \end{aligned}$$

the estimate (E3) for $\delta = \delta'$ implies that

$$\begin{aligned} & |\hat{w}_{\varepsilon, 2}(x_1, \xi') \cdot \langle \xi' \rangle^S| \\ & \leq \sum_{j=m'+1}^m C_{\delta'} \cdot |\hat{\phi}_{\varepsilon, j}(\xi') \cdot \langle \xi' \rangle^{S'+\delta}| \cdot |1 - \chi(\xi'; B_R)| \cdot R^{-\delta''} \end{aligned}$$

Hence

$$(2.11) \quad \sup_{0 \leq x_1 \leq T} \|w_{\varepsilon, 2}(x_1, \cdot)\|_S \leq C_{\delta'} \cdot R^{-\delta''} \cdot \sum_{j=m'+1}^m \|\phi_{\varepsilon, j}\|_{S'+\delta}$$

Thus

$$(2.12) \quad \sup_{0 \leq x_1 \leq T} \|u_{\epsilon,2}(x_1, \cdot)\|_s \leq (K_R \cdot \epsilon^{1/m''} + C_\delta \cdot R^{-\delta''}) \cdot M \cdot m''$$

First take the upper limit of ϵ in (2.12) and next let $R \rightarrow \infty$, then

$$\lim_{\epsilon \rightarrow 0} \sup_{0 \leq x_1 \leq T} \|u_{\epsilon,2}(x_1, \cdot)\|_s = 0.$$

Next we must show (C9) implies (C10). Assume that (C10) is not satisfied. Then there exists a positive number δ such that (E3) is not satisfied. Replacing s' by $s'+\delta$ in (2.2) and (2.3) in the proof of Lemma 1, we have a sequence of solutions $u_n(x)$ of (CP2) such that

$$\sup_{0 \leq x_1 \leq T} \|u_n(x_1, \cdot)\|_s \geq 1/2,$$

$$\|D_1^{j-1} u_n(0, \cdot)\|_{s'+\delta} \rightarrow 0, \quad j=1, \dots, m.$$

This contradicts (D1), (D5), (D6), and (D7)

[Q.E.D.]

Proof of Theorem 1. First we shall show the equivalence between (C1) and (C3). Denote by $u_{\epsilon,1}(x)$ the solution of (CP1) and by $u_{\epsilon,2}(x)$ the solution of (CP2) with the initial conditions:

$$D_1^{j-1} u(0, x') = 0, \quad j=1, \dots, m';$$

$$D_1^{j-1} u(0, x') = \phi_{\epsilon,j}(x') - D_1^{j-1} u_0(0, \xi'), \quad j=m'+1, \dots, m.$$

Then the solution $u_\epsilon(x)$ of (CP) is given by $u_{\epsilon,1}(x) + u_{\epsilon,2}(x)$

Apply Lemma 1 and Lemma 6. The condition (C3) is equivalent to the (s, s') -stability of (CP1) with respect to a particular solution u_0 of (RCP) and the $(s, s'+0)$ -stability of (CP2) with respect to a particular solution $u_{0,2} = 0$ of (RCP2). By the definition, the (s, s') -stability implies the $(s, s'+0)$ -stability

Hence we can easily show that (C3) is equivalent to the $(s, s'+0)$ -stability of (CP) with respect to a particular solution u_0 of (RCP)

Since (C3) is independent of the choice of a particular solution u_0 of (RCP), it implies that (C2) is equivalent to (C1)

Finally we shall show the equivalence between (C3) and (C4). We have only to show that (C4) implies (C3). Apply Lemma 3 for $R=R_0$. Then we have (E1) and (E2) for $\epsilon_0 = \min\{\epsilon'_0, \epsilon_{R_0}\}$ and

$$C_0 = \max\{1, T\} \cdot \max\{C'_0, C_{1, R_0} \cdot (1+R_0^2) \max\{(s-s'), 0\}/2\}$$

Apply Lemma 3 for $R=R_\delta$. Then we have (E6) for $\epsilon_\delta = \min\{\epsilon'_\delta, \epsilon_{R_\delta}\}$ and

$$C_\delta = \max\{C'_\delta, C_{1, R_\delta} \cdot (1+R_\delta^2) \max\{(s-s'-\delta), 0\}/2\}$$

[Q.E.D.]

By the same argument as Theorem 1 we have the following theorem whose proof is omitted.

Theorem 2. Let Assumption 1 and 2 be satisfied for $s'=s$. Then the following three conditions are equivalent:

(C5) The Cauchy problem (CP) is H^S -stable in $[0, T]$ for $\epsilon+0$ with respect to a particular solution $u_0(x)$ of (RCP) belonging to $C^m([0, T]; H^{S+l})$

(C11) The Cauchy problem (CP) is H^S -stable in $[0, T]$ for $\epsilon+0$ with respect to every solution $u_0(x)$ of (RCP) belonging to

$$C^m([0, T]; H^{s+l})$$

(C12) There exist positive numbers ε'_0 , R_0 , and C'_0 such that

$$(E10) \quad \sup_{0 < \varepsilon \leq \varepsilon_0, R_0 \leq |\xi'|} \int_0^T \frac{1}{\varepsilon} \cdot |Y_m(\varepsilon, x_1, \xi')| \, dx_1 \leq C'_0,$$

$$(E11) \quad \sup_{1 \leq j \leq m, 0 < \varepsilon \leq \varepsilon_0, 0 \leq x_1 \leq T, R_0 \leq |\xi'|} |Y_j(\varepsilon, x_1, \xi')| \leq C'_0.$$

3. An example for Nagumo's H^S -stability

Let $P_1(\xi)$ and $P_2(\xi)$ satisfy Assumption 1 and

$\text{ord } p_{1,j}(\xi') \leq j, j=0, \dots, m; \text{ord } p_{2,j}(\xi') \leq j, j=0, \dots, m$.

Then $P_1(D)$ and $P_2(D)$ are Kowalewskian operators. Put

$$L(\xi, \lambda) = P_1(\xi) + \lambda^{m''} \cdot P_2(\xi),$$

$N' = (1, 0)$ in $R_{\xi_1} \times R_{\xi'}^{n-1}$, and $N = (N', 0)$ in $R_{\xi}^n \times R_{\lambda}$. Denote by $\overset{\circ}{L}(\xi, \lambda)$

the principal symbol of $L(\xi, \lambda)$ with respect to (ξ, λ) and by

$\overset{\circ}{P}_i(\xi)$, $i=1, 2$ those of $P_i(\xi)$, $i=1, 2$, respectively. Then

$$\overset{\circ}{L}(\xi, \lambda) = \overset{\circ}{P}_1(\xi) + \lambda^{m''} \cdot \overset{\circ}{P}_2(\xi)$$

It must be remarked that $\overset{\circ}{L}(N) = p_{1,0} \neq 0$ and $\overset{\circ}{P}_2(N') = p_{2,0} \neq 0$.

Kevorkian and Cole's suggestive example in §4.1.2 in [7] is as follows.

Example 1 (Kevorkian and Cole).

Let $P_1(\xi_1, \xi_2) = \xi_1^2 - \xi_2^2$, which is the simple wave operator, and $P_2(\xi_1, \xi_2) = \sqrt{-1} \cdot (a \cdot \xi_1 + b \cdot \xi_2)$, where a and b are real numbers. Let us consider the solutions $u_\varepsilon(x_1, x_2)$ through a fixed point $P(x_1^0, x_2^0)$ of the following equation:

$$\varepsilon \cdot (P_1(D_1, D_2) + P_2(D_1, D_2))u(x_1, x_2) = 0.$$

If there exists a convergent sequence of $u_\varepsilon(x_1, x_2)$, then the limit $u_0(x_1, x_2)$ must satisfy the reduced equation

$$P_2(D_1, D_2)u(x_1, x_2) = 0.$$

Since the general solution of the reduced equation has the form:

$u_0(x_1, x_2) = f(b \cdot x_1 - a \cdot x_2)$ and the subcharacteristic of the reduced equation has the form: $b \cdot x_1 - a \cdot x_2 = \text{constant}$, if $|a/b| > 1$ then the subcharacteristic to P lies outside the usual domain of dependence of P for the simple wave operator. Hence $u_0(x_1, x_2)$ can not be approximated by $u_\varepsilon(x_1, x_2)$ when $|a/b| > 1$.

Thus even when $\overset{\circ}{P}_1$ and $\overset{\circ}{P}_2$ are strictly hyperbolic, we need some additional assumption on the propagation speeds. Therefore we require the following assumption.

Assumption 3.

(A3): The polynomial $\overset{\circ}{L}(\xi_1 + \tau, \xi', \lambda)$ has only simple real zero for every (ξ, λ) in $R^n \times R - \{(0, 0)\}$. That is, $L(\xi, \lambda)$ is a strictly hyperbolic polynomial in (ξ, λ) with respect to N .

(A4): There exists a positive number T_1 such that if $\text{Im } \tau < -T_1$ then $P_2(\xi_1 + \tau, \xi') \neq 0$ for all ξ in R^n . That is, $P_2(\xi)$ is a hyperbolic polynomial in ξ with respect to N' in the sense of Garding.

Remark. Since

$$\begin{aligned}\overset{\circ}{L}(0 + \tau, 0, \lambda) &= p_{1,0} \cdot \tau^m + \lambda^{m''} \cdot p_{2,0} \cdot \tau^{m'} \\ &= \tau^{m'} (p_{1,0} \cdot \tau^{m''} + \lambda^{m''} \cdot p_{2,0}),\end{aligned}$$

(A.3) implies that $m' \leq 1$

Theorem 3. Let Assumption 1 and 3 be satisfied and s be an

arbitrary real number. Then the Cauchy problem (CP) is H^S -stable (and therefore $(s, s+0)$ -stable) in $0 \leq x_1 \leq T$ for $\varepsilon \neq 0$ with respect to every solution u_0 of (RCP) belonging to $C^m([0, T]; H^{S+m})$.

Proof. By Theorem 2, it suffices to show that Assumption 2, which is the assumption on the unique solvability, and (C12) are satisfied. First we shall show (C12). Denote by $t_j(\xi', \lambda)$, $j=1, \dots, m$ the roots of $L(\xi, \lambda) = 0$ with respect to ξ_1 . When $\varepsilon^{-1} = \lambda^{m''}$, we may write

$$(3.1) \quad t_j(\xi', \lambda) = \tau_j(\varepsilon, \xi'), \quad j=1, \dots, m$$

for $\varepsilon \neq 0$ by choosing the suffixes $\{j\}$ of $t_j(\xi', \lambda)$ properly. The strict hyperbolicity of $L(\xi, \lambda)$ implies that there exist positive numbers R_1 , c_1 , and M_1 such that

$$(3.2) \quad \inf_{\substack{j \neq k, \quad 1 \leq j, k \leq m, \quad |(\xi', \lambda)| \geq R_1}} |t_j(\xi', \lambda) - t_k(\xi', \lambda)| / |(\xi', \lambda)| \geq c_1;$$

$$(3.3) \quad \sup_{1 \leq j \leq m, \quad |(\xi', \lambda)| \geq R_1} |t_j(\xi', \lambda)| / |(\xi', \lambda)| \leq M_1$$

(For example, if we look carefully into the proof of Theorem 4.10 in [8], we can find this fact easily.) Hence the roots $\tau_j(\varepsilon, \xi')$, $j=1, \dots, m$ of $L_\varepsilon(\xi) = 0$ with respect to ξ_1 are distinct for $\varepsilon \neq 0$ and $R_1 \leq |\xi'|$. The hyperbolicity of $L(\xi, \lambda)$ implies that there exists a positive number C_3 such that

$$(3.4) \quad \sup_{1 \leq j \leq m, \quad (\xi', \lambda) \in R^{n-1}_{\lambda R}} |\operatorname{Im} t_j(\xi', \lambda)| \leq C_3$$

Put $\rho = |(\xi', \lambda)|$. Then (A.4) in Appendix implies that for $\varepsilon \neq 0$,

$R_1 \leq |\xi'|$, $0 \leq x_1 \leq T$, and $j=1, \dots, m$,

$$\begin{aligned}
 & |Y_j(\varepsilon, x_1, \xi')| \\
 &= |(-1)^{j-1} \cdot D(0, 1, \dots, j-2, j, \dots, m-1)(t_1, \dots, t_m, x_1)| \\
 &\leq M(0, 1, \dots, j-2, j, \dots, m-1) \cdot |(t_1, \dots, t_m)|^{m-j} \\
 &\quad \times \sum_{\ell=1}^m \exp(-\operatorname{Im} t_\ell x_1) / \prod_{k \neq \ell, 1 \leq k \leq m} |t_\ell - t_k| \\
 &\leq \rho^{1-j} \cdot M(0, 1, \dots, j-2, j, \dots, m-1) \cdot |(t_1/\rho, \dots, t_m/\rho)|^{m-j} \\
 &\quad \times \sum_{\ell=1}^m \exp(-\operatorname{Im} t_\ell x_1) / \prod_{k \neq \ell, 1 \leq k \leq m} |t_\ell/\rho - t_k/\rho| \\
 &\leq \rho^{1-j} \cdot C_4,
 \end{aligned}$$

where

$$C_4 = M(0, 1, \dots, j-2, j, \dots, m-1) \cdot m^{(m-j)/2} \cdot M_1^{m-j} \cdot m \cdot (\exp C_3 T) \cdot c_1^{1-m}$$

Since $R_1 \leq |\xi'| \leq \rho$ and $\lambda \leq \rho$, it implies that $\rho^{1-j} \leq R_1^{1-j}$,

$j=1, \dots, m$ and $\varepsilon^{-1} \cdot \rho^{1-m} = \lambda^{m''} \cdot \rho^{1-m} \leq \lambda^{m''+1-m}$ Hence

$$\sup_{0 \leq \varepsilon \leq \varepsilon_{R_1}, 0 \leq x_1 \leq T, R_1 \leq |\xi'|} |Y_j(\varepsilon, x_1, \xi')| \leq C_4, \quad j=1, \dots, m;$$

$$\sup_{0 \leq \varepsilon \leq \varepsilon_{R_1}, 0 \leq x_1 \leq T, R_1 \leq |\xi'|} \frac{1}{\varepsilon} \cdot |Y_j(\varepsilon, x_1, \xi')| \leq C_4$$

Next we shall show that the unique solvability Since (C12) and Lemma 3 imply (C6), Lemma 4 can be applied. It is well known that (E8) and (E9) imply the unique solvability.

[Q.E.D.]

Remark. If $\phi_{\varepsilon, j}$, $j=1, \dots, m$ and $\phi_{0, j}$, $j=1, \dots, m'$ belong to $H^\infty(R^{n-1})$ and f_ε and f_0 belong to $H^\infty(R^n)$ then u_ε belong to $C^m([0, T]; H^s)$ and u_0 belongs to $C^m([0, T]; H^{s+m})$

Appendix

Let $z = (z_1, z_2, \dots, z_n)$ be complex variables. For a non-negative integer ℓ , denote

$$a(\ell)(z) = \{(z_j)^\ell; j=1, \dots, n\}$$

and for non-negative integers $\ell_1, \ell_2, \dots, \ell_n$ satisfying

$0 \leq \ell_1 < \ell_2 < \dots < \ell_n$, denote

$$A(\ell_1, \ell_2, \dots, \ell_n)(z) = \det(a(\ell_i)(z); i=1, \dots, n).$$

In particular, $A(0, 1, \dots, n-1)(z)$ is the Vandermonde determinant and represented as the difference product $\prod_{1 \leq i < j \leq n} (z_j - z_i)$

Let $i = \sqrt{-1}$ and x_1 be a real parameter. Denote

$$e(z, x_1) = \{\exp iz_j x_1; j=1, \dots, n\}$$

and for non-negative integers $\ell_1, \ell_2, \dots, \ell_{n-1}$ satisfying

$0 \leq \ell_1 < \ell_2 < \dots < \ell_{n-1}$, denote

$$\begin{aligned} B(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1) \\ = \det {}^t({}^t e(z, x_1), {}^t a(\ell_1)(z), \dots, {}^t a(\ell_{n-1})(z)) \end{aligned}$$

Expand the determinant $B(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1)$ with respect to the first row. Then

$$\begin{aligned} (A.1) \quad B(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1) \\ = \sum_{j=1}^n (-1)^{1+j} \cdot A(\ell_1, \ell_2, \dots, \ell_{n-1})(z(j)) \cdot \exp iz_j x_1, \end{aligned}$$

where $z(j) = (z_1, z_2, \dots, z_{j-1}, z_{j+1}, \dots, z_n)$ Denote

$$\begin{aligned} C(\ell_1, \ell_2, \dots, \ell_n)(z) \\ = A(\ell_1, \ell_2, \dots, \ell_n)(z) / A(0, 1, \dots, n-1)(z) \end{aligned}$$

and

$$\begin{aligned} D(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1) \\ = B(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1) / A(0, 1, \dots, n-1)(z) \end{aligned}$$

Then $C(\ell_1, \ell_2, \dots, \ell_n)(z)$ is a homogeneous symmetric polynomial in $z[z]$ of order $\ell_1 + \ell_2 + \dots + \ell_n - (n-1)n/2$, which is called a Schur function. Since $B(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1)$ is an entire function of z and vanishes on the zeros of irreducible polynomials $z_j - z_i$, $1 \leq i < j \leq n$, Nullstellensatz implies that

$B(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1)$ is divided by $A(0, 1, \dots, n-1)(z)$ in the ring of entire functions. Hence $D(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1)$ is an entire function. If $z_i \neq z_j$, $1 \leq i < j \leq n$, then (A.1) implies that

$$\begin{aligned} (A.2) \quad D(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1) \\ = \sum_{j=1}^n (-1)^{1+j} \cdot C(\ell_1, \ell_2, \dots, \ell_{n-1})(z(j)) \cdot \exp iz_j x_1 \cdot E_j(z), \end{aligned}$$

where $E_j(z) = 1 / \{ (-1)^{n-j} \cdot \prod_{k \neq j, 1 \leq k \leq n} (z_j - z_k) \}$

Put

$$M(\ell_1, \ell_2, \dots, \ell_n) = \max_{|z|=1} |C(\ell_1, \ell_2, \dots, \ell_n)(z)|$$

Then

$$(A.3) \quad |C(\ell_1, \ell_2, \dots, \ell_n)(z)| \leq M(\ell_1, \ell_2, \dots, \ell_n) \cdot |z|^L,$$

where $L = \ell_1 + \ell_2 + \dots + \ell_n - (n-1)n/2$ and

$$\begin{aligned} (A.4) \quad |D(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1)| \\ \leq M(\ell_1, \ell_2, \dots, \ell_{n-1}) |z|^{L'} \cdot \sum_{j=1}^n \exp(-\text{Im } z_j x_1) / \prod_{k \neq j, 1 \leq k \leq n} |z_j - z_k|, \end{aligned}$$

where $L' = \ell_1 + \ell_2 + \dots + \ell_{n-1} - (n-2)(n-1)/2$.

Let $m, m',$ and m'' be positive integers such that $m = m' + m''$. Denote $z' = (z_1, z_2, \dots, z_{m'})$, $z'' = (z_{m'+1}, z_{m'+2}, \dots, z_m)$, and $z = (z', z'')$. Let $\ell_1, \ell_2, \dots, \ell_{m-1}$ be non-negative integers satisfying $0 \leq \ell_1 < \ell_2 < \dots < \ell_{m-1}$. Let S_1 be the set of all bijections ρ from $\{1, 2, \dots, m-1\}$ onto $\{\ell_1, \ell_2, \dots, \ell_{m-1}\}$ satisfying

$$\rho(1) < \rho(2) < \dots < \rho(m');$$

$$\rho(m'+1) < \rho(m'+2) < \dots < \rho(m-1)$$

and S_2 be the set of all bijections ρ from $\{1, 2, \dots, m-1\}$ onto $\{\ell_1, \ell_2, \dots, \ell_{m-1}\}$ satisfying

$$\rho(1) < \rho(2) < \dots < \rho(m'-1);$$

$$\rho(m') < \rho(m'+1) < \dots < \rho(m-1).$$

There are one-to-one correspondence between the bijections in S_1 and the selections of $m-1$ objects taken m' at a time and between the bijections in S_2 and the selections of $m-1$ objects taken $m'-1$ at a time, respectively. Define the bijection π from $\{\ell_1, \ell_2, \dots, \ell_{m-1}\}$ onto $\{2, 3, \dots, m\}$ as

$$\pi(\ell_j) = j+1, \quad j=1, \dots, m-1.$$

Denote

$$I(\rho) = \sum_{j=1}^{m'} \pi(\rho(j)) + m'(m'+1)/2$$

and

$$J(\rho) = 1 + \sum_{j=1}^{m'-1} \pi(\rho(j)) + m'(m'+1)/2.$$

For $z_i \neq z_j$, $1 \leq i \leq m'$, $m'+1 \leq j \leq m$, denote

$$E(z) = 1/\prod_{1 \leq i \leq m', m'+1 \leq j \leq m} (z_j - z_i)$$

Lemma A.1. For $z_i \neq z_j$, $1 \leq i \leq m'$, $m'+1 \leq j \leq m$,

$$\begin{aligned}
 (A.5) \quad & D(\ell_1, \ell_2, \dots, \ell_{m-1})(z, x_1) \\
 &= \sum_{\rho \in S_1} (-1)^{I(\rho)} \cdot C(\rho(1), \rho(2), \dots, \rho(m'))(z') \\
 &\quad \times D(\rho(m'+1), \rho(m'+2), \dots, \rho(m-1))(z'', x_1) \cdot E(z) \\
 &+ \sum_{\rho \in S_2} (-1)^{J(\rho)} \cdot D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1) \\
 &\quad \times C(\rho(m'), \rho(m'+1), \dots, \rho(m-1))(z'') \cdot E(z)
 \end{aligned}$$

Proof. Apply the Laplace expansion theorem to

$B(\ell_1, \ell_2, \dots, \ell_{m-1})(z, x_1)$ The minors of order m' of the original matrix ${}^t({}^t e(z, x_1), {}^t a(\ell_1)(z), \dots, {}^t a(\ell_{m-1})(z))$ of order m are

$$A(\rho(1), \rho(2), \dots, \rho(m'))(z'), \text{ for } \rho \text{ in } S_1,$$

$$B(\rho(1), \rho(2), \dots, \rho(m'-1))(z'', x_1), \text{ for } \rho \text{ in } S_2,$$

and those cofactors of order m'' are

$$(-1)^{I(\rho)} \cdot B(\rho(m'+1), \rho(m'+2), \dots, \rho(m-1))(z', x_1), \text{ for } \rho \text{ in } S_1,$$

$$(-1)^{J(\rho)} \cdot A(\rho(m'), \rho(m'+1), \dots, \rho(m-1))(z''), \text{ for } \rho \text{ in } S_2,$$

respectively Hence

$$\begin{aligned}
 (A.6) \quad & B(\ell_1, \ell_2, \dots, \ell_{m-1})(z, x_1) \\
 &= \sum_{\rho \in S_1} (-1)^{I(\rho)} \cdot A(\rho(1), \rho(2), \dots, \rho(m'))(z') \\
 &\quad \times B(\rho(m'+1), \rho(m'+2), \dots, \rho(m-1))(z'', x_1) \\
 &+ \sum_{\rho \in S_2} (-1)^{J(\rho)} \cdot B(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1) \\
 &\quad \times A(\rho(m'), \rho(m'+1), \dots, \rho(m-1))(z'')
 \end{aligned}$$

Divide (A.6) by

$$(A.7) \quad A(0, 1, \dots, m-1)(z) \\ = A(0, 1, \dots, m'-1)(z') \cdot A(0, 1, \dots, m''-1)(z'')/E(z),$$

we have (A.5).

[Q.E.D.]

Denote

$$L'(\rho) = \begin{cases} \rho(1) + \rho(2) + \dots + \rho(m') - (m'-1)m'/2, & \text{for } \rho \text{ in } S_1 \\ \rho(1) + \rho(2) + \dots + \rho(m'-1) - (m'-1)m'/2, & \text{for } \rho \text{ in } S_2, \end{cases}$$

and

$$L''(\rho) = \begin{cases} \rho(m'+1) + \rho(m'+2) + \dots + \rho(m-1) - (m''-1)m''/2, & \text{for } \rho \text{ in } S_1 \\ \rho(m') + \rho(m'+1) + \dots + \rho(m-1) - (m''-1)m''/2, & \text{for } \rho \text{ in } S_2 \end{cases}$$

Put

$$\begin{aligned} & \tilde{M}(\ell_1, \ell_2, \dots, \ell_{m-1}) \\ &= \max \left\{ \max_{\rho \in S_1} M(\rho(1), \rho(2), \dots, \rho(m')), \right. \\ & \quad \max_{\rho \in S_1} M(\rho(m'+1), \rho(m'+2), \dots, \rho(m-1)), \\ & \quad \left. \max_{\rho \in S_2} M(\rho(m'), \rho(m'+1), \dots, \rho(m-1)) \right\} \end{aligned}$$

For a positive parameter η , put $w_j = \eta \cdot z_j$, $j=1, \dots, m$.

Lemma A.2. Assume that $z_i \neq z_j$, for $1 \leq i \leq m'$, $m'+1 \leq j \leq m$ and for $m'+1 \leq i < j \leq m$. Then

$$\begin{aligned} & |C(\rho(1), \rho(2), \dots, \rho(m'))(z') \\ & \quad \cdot D(\rho(m'+1), \rho(m'+2), \dots, \rho(m-1))(z'', x_1) \cdot E(z)| \\ & \leq \tilde{M}(\ell_1, \ell_2, \dots, \ell_{m-1})^2 \cdot |z'|^{L'(\rho)} \cdot |w''|^{L''(\rho) + (m''-1)} \cdot |E(w)| \end{aligned}$$

$$x \eta^{m'm''-L''(\rho)} \cdot \left(\sum_{j=m'+1}^m \exp(-\operatorname{Im} w_j x_1 / \eta) / \prod_{k \neq j, m'+1 \leq k \leq m} |w_j - w_k| \right),$$

for ρ in S_1 and

$$\begin{aligned} & |D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1) \\ & x C(\rho(m'), \rho(m'+1), \dots, \rho(m-1))(z'') \cdot E(z)| \\ & \leq |D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1)| \\ & \tilde{x} M(\ell_1, \ell_2, \dots, \ell_{m-1}) \cdot |w''|^{L''(\rho)} \cdot |E(w)| \cdot \eta^{m'm''-L''(\rho)}, \end{aligned}$$

for ρ in S_2

Proof. Since

$$C(\ell_1, \ell_2, \dots, \ell_n)(z) = \eta^{-L} \cdot C(\ell_1, \ell_2, \dots, \ell_n)(\eta \cdot z),$$

where $L = \ell_1 + \ell_2 + \dots + \ell_n - (n-1)n/2$,

$$D(\ell_1, \ell_2, \dots, \ell_{n-1})(z, x_1) = \eta^{-L''} \cdot D(\ell_1, \ell_2, \dots, \ell_{n-1})(\eta \cdot z, x_1 / \eta),$$

where $L'' = \ell_1 + \ell_2 + \dots + \ell_{n-1} - (n-1)n/2$, and $E(z) = \eta^{m'm''} \cdot E(w)$, it

implies that

$$\begin{aligned} & C(\rho(1), \rho(2), \dots, \rho(m'))(z') \\ & x D(\rho(m'+1), \rho(m'+2), \dots, \rho(m-1))(z'', x_1) \cdot E(z) \\ & = C(\rho(1), \rho(2), \dots, \rho(m'))(z') \\ & x D(\rho(m'+1), \rho(m'+2), \dots, \rho(m-1))(w'', x_1 / \eta) \cdot E(w) \cdot \eta^{m'm''-L''(\rho)}, \end{aligned}$$

for ρ in S_1 and

$$\begin{aligned} & D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1) \\ & x C(\rho(m'), \rho(m'+1), \dots, \rho(m-1))(z'') \cdot E(z) \\ & = D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1) \end{aligned}$$

$${}_xC(\rho(m'), \rho(m'+1), \dots, \rho(m-1))(w'') \cdot E(w) \cdot \eta^{m'm''-L''(\rho)},$$

for ρ in S_2 . By using (A.3) and (A.4), we come to the conclusion.

[Q.E.D.]

Lemma A.3. Assume that $z_i \neq z_j$, for $1 \leq i \leq m'$, $m'+1 \leq j \leq m$ and for $m'+1 \leq i < j \leq m$. Let

$$\{\ell_1, \ell_2, \dots, \ell_{m-1}\} = \{0, 1, \dots, k-1, k+1, \dots, m-1\}.$$

Assume that there exist positive numbers M , M' , c , and η_0 with $\eta_0 \leq 1$ such that for every η satisfying $0 < \eta \leq \eta_0$, the following estimates are satisfied:

$$(A.8) \quad |z'| \leq M; \quad |w''| \leq M;$$

$$\left| \sum_{j=m'+1}^m \exp(-i \operatorname{Im} w_j x_1 / \eta) \right| \leq M';$$

$$\inf_{m'+1 \leq i < j \leq m} |w_i - w_j| \geq c; \quad \inf_{1 \leq i \leq m', m'+1 \leq j \leq m} |w_i - w_j| \geq c.$$

Denote

$$\tilde{M} = \max_{0 \leq k \leq m-1} \tilde{M}(0, 1, \dots, k-1, k+1, \dots, m-1),$$

$$C_1 = \frac{(m-1)!}{m'!(m''-1)!} \cdot \tilde{M}^2 \cdot M^{m'm''-k+m''-1} \cdot c^{-m'm''-m''+1} \cdot M',$$

and

$$C_2 = \frac{(m-1)!}{(m'-1)!m''!} \cdot \tilde{M} \cdot M^{m'm''} \cdot c^{-m'm''}$$

Then

$$(A.9) \quad \begin{aligned} & |D(0, 1, \dots, k-1, k+1, \dots, m-1)(z, x_1) \\ & - D(0, 1, \dots, k-1, k+1, \dots, m'-1)(z', x_1) \end{aligned}$$

$$\times (w_{m'+1} \cdot w_{m'+2} \cdot \dots \cdot w_m)^{m'} \cdot E(w) |$$

$$\leq (C_1 + C_2 \cdot \max_{\rho \in S_2} |D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1)|) \cdot \eta,$$

for $k=0, \dots, m'-1$ and

$$(A.10) \quad |D(0, 1, \dots, k-1, k+1, \dots, m-1)(z, x_1)|$$

$$\leq (C_1 + C_2 \cdot \max_{\rho \in S_2} |D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1)|) \cdot \eta^{k-m'+1},$$

for $k=m', \dots, m-1$. Here ρ in S_2 are bijections from

$\{1, 2, \dots, m-1\}$ onto $\{0, 1, \dots, k-1, k+1, \dots, m-1\}$ satisfying

$$\rho(1) < \rho(2) < \dots < \rho(m'-1);$$

$$\rho(m') < \rho(m'+1) < \dots < \rho(m-1).$$

Proof. First it must be remarked that

$$m'm'' - L''(\rho) \geq m'm'' - m' - (m'+1) - \dots - (m-1) + (m''-1)m''/2 = 0,$$

where the equality holds if and only if

$$(A.11) \quad k=0, 1, \dots, m'-1,$$

$$\rho \in S_2,$$

$$\rho(j) = \begin{cases} j-1, & j=1, \dots, k; \\ j, & j=k+1, \dots, m-1 \end{cases}$$

Since

$$C(m', m'+1, \dots, m-1)(z'') = (z_{m'+1} \cdot z_{m'+2} \cdot \dots \cdot z_m)^{m'},$$

it implies that for ρ satisfying (A.11),

$$(-1)^{J(\rho)} \cdot D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1)$$

$$\times C(\rho(m'), \rho(m'+1), \dots, \rho(m-1))(z'') \cdot E(z)$$

$$= (-1)^{m'(m'+1)} \cdot D(0, 1, \dots, k-1, k+1, \dots, m'-1)(z', x_1)$$

$$\times (w_{m'+1} \cdot w_{m'+2} \cdot \dots \cdot w_m)^{m'} \cdot E(w)$$

For ρ not satisfying (A.11), Lemma A.2 implies that

$$\begin{aligned} & |C(\rho(1), \rho(2), \dots, \rho(m'))(z')| \\ & \times D(\rho(m'+1), \rho(m'+2), \dots, \rho(m-1))(z'', x_1) \cdot E(z)| \\ & \leq \tilde{M}^2 \cdot M^{L'(\rho)+L''(\rho)+m''-1} \cdot |E(w)| \cdot \eta^{m'm''-L''(\rho)} \cdot M' \cdot c^{-m''+1}, \end{aligned}$$

for ρ in S_1 and

$$\begin{aligned} & |D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1)| \\ & \times C(\rho(m'), \rho(m'+1), \dots, \rho(m-1))(z'') \cdot E(z)| \\ & \leq |D(\rho(1), \rho(2), \dots, \rho(m'-1))(z', x_1)| \\ & \times \tilde{M} \cdot M^{L''(\rho)} \cdot |E(w)| \cdot \eta^{m'm''-L''(\rho)}, \end{aligned}$$

for ρ in S_2 . If (A.11) is not satisfied, then $m'm''-L(\rho) \geq 1$.

If $k = m', \dots, m-1$, then

$$m'm''-L''(\rho) \geq m'm''-(m'-1)-m'-\dots-(m-1)+k+(m''-1)m''/2 = k-m'+1.$$

Since $|E(w)| \leq c^{-m'm''}$ and $L'(\rho)+L''(\rho) = m'm''-k$, for ρ in S_1 ,

Lemma A.1 implies the conclusion.

[Q.E.D.]

References

- [1] R. Ashino: The reducibility of the boundary conditions in the one-parameter family of elliptic linear boundary value problems I, Osaka J. Math., 25 (1988), 737-757.
- [2] R. Ashino: On the admissibility of singular perturbations in Cauchy problems, Osaka J. Math., 26 (1989), 387-398.
- [3] R. Ashino: The reducibility of the boundary conditions in the one-parameter family of elliptic linear boundary value problems II, Osaka J. Math., 26 (1989), 535-556.
- [4] R. Ashino: On the weak admissibility of singular perturbations in Cauchy problems, Publ. Res. Inst. Math. Sci., 25 (1989),
- [5] M. Nagumo: On singular perturbation of linear partial differential equations with constant coefficients I, Proc. Japan Acad. 35 (1959), 449-454
- [6] H. Kumano-go: On singular perturbation of linear partial differential equations with constant coefficients II, Proc. Japan Acad. 35 (1959), 541-546.
- [7] J. Kevorkian and J. D. Cole: Perturbation Methods in Applied Mathematics, Springer-Verlag (1981)
- [8] S. Mizohata: The Theory of Partial Differential Equations, Cambridge (1973)
- [9] S. Mizohata: On the Cauchy Problem, Vol. 3 in Notes and Reports in Mathematics in Science and Engineering, Academic Press (1985)