

Lurie's Topological Quantum Field Theory

Transformation Group, RIMS, 2009/08/20

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1. INTRODUCTION

[CTFT] Jacob Lurie, *On the Classification of Topological Field Theories (Draft)*, May 1, 2009, 111pages,

[GMTW] math/0605249 The homotopy type of the cobordism category. Soren Galatius, Ib Madsen, Ulrike Tillmann, Michael Weiss

In [CTFT], a part of the proof [GMTW] of the Mumford Conjecture (= the Madsen-Weiss Theorem) for a closed oriented manifold Σ_g of dimension $n = 2$:

— Mumford Conjecture = Madsen-Weiss Theorem —

The map induced by the Miller-Morita-Mumford classes

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots] \rightarrow H^*(BDiff(\Sigma_g); \mathbb{Q})$$

is an isomorphism in degrees $\leq n(g)$ with $\lim_{g \rightarrow \infty} n(g) = \infty$.

valid for general n , has been higher categorically reorganized and made transparent.

— Relavent argument valid for general n —

- (1) For a general closed oriented manifold M of dimension n , construct

$$BDiff(M) \rightarrow \Omega^n |\mathbf{Bord}_n^{SO(n)}|$$

- (2) (**Cobordism hypothesis, Group-Completed Version**) For a topological group G with a continuous homomorphism $\chi : G \rightarrow O(n)$,

$$|\mathbf{Bord}_n^G| \simeq (QS^0)_{hG}.$$

In particular,

$$|\mathbf{Bord}_n^{SO(n)}| \simeq \Omega^\infty \left(\Sigma^n BSO(n)^{-\zeta_n} \right),$$

where ζ_n is the universal rank n vector bundle over $BSO(n)$.

— An outline of the proof of the Mumford conjecture —

- (1) Specializing to the case $M = \Sigma_g$, factorize as

$$BDiff(\Sigma_g) \rightarrow Y_g$$

$$\xrightarrow{\text{a connected component inclusion}} \Omega^2 |\mathbf{Bord}_2^{SO(2)}|$$

- (2) (**The Harer stability**) $BDiff(\Sigma_g) \rightarrow Y_g$ is an $n(g)$ -equivalence with $\lim_{g \rightarrow \infty} n(g) = \infty$.

- (3) $\Omega^2 |\mathbf{Bord}_2^{SO(2)}| \simeq \Omega^\infty (BSO(2)^{-\zeta_2}) \simeq \Omega^\infty (\mathbb{C}P_1^\infty)$ is easy to understand homotopy theoretically (Galatius).

In this paper, I shall present a short introduction to some higher categorical aspect of the cobordism hypothesis presented in [CTFT], in an OHP presentation style. I claim no originality here, but I intended to convey the readers with at least a rough outline of [CTFT]. Of course, I am entirely responsible for any possible mistakes and confusions here. Also, I hope to come back with a sequel with more details.

Fortunately, Lurie's own lecture series on this subject is available as video files on the web:

<http://lab54.ma.utexas.edu:8080/video/lurie.html>

So, just google "*Jacob Lurie video*" to locate this web site!

2. WHAT IS THE COBORDISM HYPOTHESIS?

A very general form of the cobordism hypothesis is the following:

Cobordism Hypothesis for (X, ζ) -manifolds (Theorem 2.4.18)

- $\mathcal{C}$: a symmetric monoidal (∞, n) -category with duals;
 $\mathcal{C}^\sim$: its underlying ∞ -groupoid ($= (\infty, 0)$ -category), obtained by discarding all of the noninvertible morphisms [CTFT, 2.4.4];
 $T^\sim$: a topological space s.t. $\mathcal{C}^\sim \cong \pi_{\leq \infty} T^\sim$, as an ∞ -groupoid;
- (X, ζ) : a CW complex X and its n dimensional vector bundle with an inner product;
 $\tilde{X} \rightarrow X$: its associated principal $O(n)$ -bundle of orthonormal frames in ζ ;

$\Rightarrow \exists$ an equivalence of $(\infty, 0)$ -categories:

$$\mathrm{Fun}^\otimes \left(\mathbf{Bord}_n^{(X, \zeta)}, \mathcal{C} \right) \simeq \mathrm{Hom}_{O(n)} \left(\tilde{X}, T^\sim \right)$$

Natural questions concerning the cobordism hypothesis

- (a) What is an (∞, n) -category?
- (b) What is a (symmetric monoidal) functor between (symmetric monoidal) (∞, n) -categories?
- (c) What is the (∞, n) -category $\mathbf{Bord}_n^{(X, \zeta)}$?
- (d) What does it mean for a symmetric monoidal (∞, n) -category to have duals?
- (e) How can we deduce $|\mathbf{Bord}_n^G| \simeq (QS^0)_{hG}$ from the cobordism hypothesis?

3. WHAT IS AN (∞, n) -CATEGORY?

A rough definition of (∞, n) -category

An (∞, n) -category is a higher category, where all the "strictness" are dropped and only "up to coherent isomorphisms", in which all k -morphisms are invertible for $k > n$.

— A fundamental n -groupoid of a topological space X —

For each $0 \leq n \leq \infty$, one can define an n -category $\pi_{\leq n}X$, called the **fundamental n -groupoid of X** :

- The objects of $\pi_{\leq n}X$ are the points of X .
- Given a pair of objects $x, y \in X$, a 1-morphism in $\pi_{\leq n}X$ from x to y is a path in X from x to y .
- Given a pair of objects $x, y \in X$ and a pair of 1-morphisms $f, g : x \rightarrow y$, a 2-morphism from f to g in $\pi_{\leq n}X$ is a homotopy of paths in X (which is required to be fixed at the common endpoints x and y).
- ...
- An n -morphism in $\pi_{\leq n}X$ is given by a homotopy between homotopies between ... between paths between points of X . **Two such homotopies determined the same n -morphism in $\pi_{\leq n}X$ if they are homotopic to one another (via a homotopy which is fixed on the common boundaries).**

— A rough inductive definition of (∞, n) -category —

- $(\infty, 0)$ -category = ∞ -groupoid
"=" topological space
- (∞, n) -category consists of the following data:
 - a collection of objects $X, Y, Z, \dots$
 - for pairs of objects $X, Y \in \mathcal{C}$, an $(\infty, n-1)$ -category $\text{Hom}_{\mathcal{C}}(X, Y)$
 - composition law
 - Associativity (with units) (up to coherent isomorphism)

— Examples of (∞, n) -categories —

- For a topological space X , its fundamental ∞ -groupoid $\pi_{\leq \infty}X$ is an $(\infty, 0)$ -category;
- An $(\infty, n-1)$ -category is an (∞, n) -category;
- An n -category is an (∞, n) -category by considering only identity k -morphisms for $k > n$;

There is an adjunction:

$$\begin{array}{ccc}
 \text{Fun}_{n\text{-category}}(h_n \mathcal{D}, \mathcal{C}) & \cong & \text{Fun}_{(\infty, n)\text{-category}}(\mathcal{D}, i\mathcal{C}) \\
 h_n : \{(\infty, n)\text{-category}\} & \xrightleftharpoons{\quad} & \{n\text{-category}\} : i \\
 \mathcal{D} & \xrightarrow{h_n} & h_n \mathcal{D} \\
 i\mathcal{C} \text{ (only identity } k\text{-morphisms for } k > n) & \xleftarrow{i} & \mathcal{C},
 \end{array}$$

where, for an (∞, n) -category $\mathcal{D}$, $h_n \mathcal{D}$ is the **homotopy n -category** of $\mathcal{D}$, given by

- For $k < n$, the k -morphisms of $h_n \mathcal{D}$ are the k -morphisms of $\mathcal{D}$.
- The n -morphisms of $h_n \mathcal{D}$ are given by isomorphism classes of n -morphisms in $\mathcal{D}$.

— What is an $(\infty, 0)$ -category supposed to be? —

$(\infty, 0)$ -category is a topological space, or a Kan complex $\implies$ the object of the classical homotopy theory.

— What is an $(\infty, 1)$ -category supposed to be? —

Must satisfy the following conditions:

- Both $(\infty, 0)$ -category (i.e. a topological space or a Kan complex) and a strict 1-category (i.e. an ordinary category) are $(\infty, 1)$ -categories
- For every objects X, Y , $\text{Hom}(X, Y)$ is an $(\infty, 0)$ -category (i.e. a topological space or a Kan complex).

Actually, there are many different approaches to define an $(\infty, 1)$ -category, such as a topological category, a simplicial category, a quasi-category, a Segal category, a complete Segal space.

However, these are all essentially the same concepts, and there is a diagram of right Quillen equivalences

$$\begin{array}{ccc}
 \text{Set}_\Delta & \xleftarrow{N} & \text{Cat}_\Delta \\
 \uparrow G_0 & & \downarrow \\
 \text{Fun}(\Delta^{op}, \text{Set}_\Delta) & \longrightarrow & \text{SegSet}_\Delta
 \end{array}$$

where

- The category of simplicial sets Set_Δ has the Joyal model structure, whose fibrant objects are nothing but **quasi categories**.
- Cat_Δ is the category of **simplicial categories**.
- $\text{Fun}(\Delta^{op}, \text{Set}_\Delta)$ is the category of all bisimplicial sets with the **complete Segal** model structure
- SegSet_Δ is the category of **preSegal categories**: i.e. bisimplicial sets $X_{\bullet, \bullet}$ with the property that the 0th column $X_{\bullet, 0}$ is a constant simplicial set.

— The original idea of Rezk and Barwick —

Understand inductively w.r.t. n , by expressing an (∞, n) -category by $(\infty, n-1)$ -categories:

They compared the following two objects:

- (∞, n) -category $\mathcal{C}$
- simplicial $(\infty, n-1)$ -category $\mathcal{C}_\bullet$ with the Segal condition:
For each $k \geq 0$, the canonical map

$$\mathcal{C}_k \rightarrow \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \cdots \times_{\mathcal{C}_0} \mathcal{C}_1$$

is an equivalence of $(\infty, n-1)$ -categories.

— $(\infty, n) \mathcal{C} \implies \text{simplicial } (\infty, n-1) \mathcal{C}_\bullet \text{ with Segal}$ —

Given an (∞, n) -category $\mathcal{C}$, construct a simplicial $(\infty, n-1) \mathcal{C}_\bullet$ with the Segal condition by

- $\mathcal{C}_0 := (\infty, 0)$ -category (and so an $(\infty, n-1)$ -category) extracted from $\mathcal{C}$, by discarding all of the noninvertible morphisms in $\mathcal{C}$ at all levels.
- $\mathcal{C}_1 := (\infty, n-1)$ -category whose objects are given by triples $(X \in \mathcal{C}_0, Y \in \mathcal{C}_0, f \in \text{Map}_{\mathcal{C}}(X, Y))$, where, for each pair of objects $X, Y \in \mathcal{C}$, $\text{Map}_{\mathcal{C}}(X, Y)$ is an $(\infty, n-1)$ -category, depending functorially on the pair $X, Y \in \mathcal{C}_0$.
- $\mathcal{C}_k := (\infty, n-1)$ -category whose objects are given by $(2k+1)$ -tuples

$$(X_0 \in \mathcal{C}_0, X_1 \in \mathcal{C}_0, \dots, X_k \in \mathcal{C}_k,$$

$$f_1 \in \text{Map}_{\mathcal{C}}(X_0, X_1), \dots, f_k \in \text{Map}_{\mathcal{C}}(X_{k-1}, X_k)),$$

- The collection of $(\infty, n-1)$ -categories $\{\mathcal{C}_k\}_{k \geq 0}$ forms a simplicial $(\infty, n-1)$ -category $\mathcal{C}_\bullet$ satisfying the Segal condition.

— simplicial $(\infty, n-1) \mathcal{C}_\bullet$ with Segal $\implies (\infty, n) \mathcal{C}$ —

Given a simplicial $(\infty, n-1)$ category $\mathcal{C}_\bullet$ satisfying the Segal condition, construct an (∞, n) -category $\mathcal{C}$ by

- The objects of $\mathcal{C}$ are the objects of $\mathcal{C}_0$:

$$\text{Ob}(\mathcal{C}) = \text{Ob}(\mathcal{C}_0)$$

- Given a pair of objects $X, Y \in \mathcal{C}_0$, the $(\infty, n-1)$ -category of maps $\text{Map}_{\mathcal{C}}(X, Y)$ is given by the fiber product $\{X\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{Y\}$:

$$\text{Map}_{\mathcal{C}}(X, Y) = \{X\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{Y\}$$

- Given a sequence of objects $X_0, \dots, X_k \in \mathcal{C}_0$, the composition law

$$\text{Map}_{\mathcal{C}}(X_0, X_1) \times \dots \times \text{Map}_{\mathcal{C}}(X_{k-1}, X_k) \rightarrow \text{Map}_{\mathcal{C}}(X_0, X_k)$$

is given by the composite map

$$\begin{aligned} & \text{Map}_{\mathcal{C}}(X_0, X_1) \times \dots \times \text{Map}_{\mathcal{C}}(X_{k-1}, X_k) \\ &= (\{X_0\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{X_1\}) \times \dots \times (\{X_{k-1}\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{X_k\}) \\ &\xleftarrow[\text{Segal condition}]{\cong} \mathcal{C}_k \times_{\mathcal{C}_0 \times \dots \times \mathcal{C}_0} (\{X_0\} \times \dots \times \{X_k\}) \\ &\rightarrow \{X_0\} \times_{\mathcal{C}_0} \mathcal{C}_1 \times_{\mathcal{C}_0} \{X_k\} = \text{Map}_{\mathcal{C}}(X_0, X_k) \end{aligned}$$

— A bad news: a motivation of “*completeness*” —

Although the composite

$$\begin{array}{ccc} (\infty, n) \rightarrow & \text{simplicial } (\infty, n-1) \rightarrow (\infty, n) \\ \mathcal{C} \mapsto & \mathcal{C}_\bullet \mapsto \mathcal{C}' \end{array}$$

is an equivalence: $\mathcal{C} \simeq \mathcal{C}'$, the composite

$$\begin{array}{ccc} \text{simplicial } (\infty, n-1) \rightarrow & (\infty, n) \rightarrow \text{simplicial } (\infty, n-1) \\ \mathcal{C}_\bullet \mapsto & \mathcal{C} \mapsto \mathcal{C}' \end{array}$$

is NOT an equivalence $\mathcal{C}_\bullet \not\simeq \mathcal{C}'_\bullet$ in general, for $\mathcal{C}$ may have more invertible morphisms than $\mathcal{C}_0$ and $\mathcal{C}_0 \not\simeq \mathcal{C}'_0$.

— Technical preparations. —

- An *n-fold simplicial space* is an n -fold simplicial object in the category of topological spaces and continuous maps:

$$\Delta^{op} \times \cdots \Delta^{op} \rightarrow Top$$

- We will say that a map $X \rightarrow Y$ of n -fold simplicial spaces is a *weak homotopy equivalence* if the induced map $X_{k_1, \dots, k_n} \rightarrow Y_{k_1, \dots, k_n}$ is a weak homotopy equivalence of topological spaces, for every sequence of nonnegative integers $k_1; \dots; k_n \geq 0$.
- A commutative diagram of topological spaces

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is said to be a *homotopy pullback square* (or a *homotopy Cartesian diagram*) if

$$W \rightarrow X \times_Z Y \rightarrow X \times_Z^R Y := X \times_Z Z^{[0,1]} \times_Z Y$$

is a weak homotopy equivalence.

- A diagram of n -fold simplicial spaces:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \end{array}$$

is a *homotopy pullback square* if, for every sequence of nonnegative integers $k_1; \dots; k_n \geq 0$, the induced square

$$\begin{array}{ccc} X_{k_1, \dots, k_n} & \longrightarrow & Y_{k_1, \dots, k_n} \\ \downarrow & & \downarrow \\ X'_{k_1, \dots, k_n} & \longrightarrow & Y'_{k_1, \dots, k_n} \end{array}$$

is a homotopy pullback square.

- We will say that an n -fold simplicial space X is *essentially constant* if there exists a weak homotopy equivalence of n -fold simplicial spaces $X' \rightarrow X$, where X' is a constant functor.

— The case $n = 1$: the Complete Segal Space —

- Let $X_\bullet$ be a simplicial space. We say that $X_\bullet$ is a Segal space if the following condition is satisfied:
 - For every pair of integers $m, n \geq 0$, the diagram

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array}$$

is a homotopy pullback square.

- Let $X_\bullet$ be a Segal space, and let

$$\delta : X_0 \rightarrow X_1$$

be the “degeneracy map” induced by the unique nondecreasing functor $\{0, 1\} \rightarrow \{0\}$. For every point x in X_0 , the morphism $[\delta(x)]$ in the homotopy category $hX_\bullet$ coincides with the identity map $\text{id}_x : x \rightarrow x$. In particular,

$\delta(x)$ is invertible for each $x \in X_0$.

- Let $X_\bullet$ be a Segal space, and let $Z \subseteq X_1$ denote the subset consisting of the invertible elements (this is a union of path components in X_1 ; we will consider Z as endowed with the subspace topology). We will say that $X_\bullet$ is complete if the map

$$\delta : X_0 \rightarrow Z$$

is a weak homotopy equivalence.

- $(\infty, 1)$ -category is a complete Segal space.

— An (∞, n) -category is defined by generalizing “complete Segal space” —

An $(\infty; n)$ -category is an n -fold complete Segal space.

— Definition of a n -fold (complete) Segal space —

Let $n > 0$, and let X be an n -fold simplicial space. Regard X as a simplicial object $X_\bullet$ in the category of $(n-1)$ -fold simplicial spaces. X is said to be an n -fold Segal space if the following conditions are satisfied:

(A1) For every $0 \leq k \leq m$, the diagram

$$\begin{array}{ccc} X_m & \longrightarrow & X_k \\ \downarrow & & \downarrow \\ X_{m-k} & \longrightarrow & X_0 \end{array}$$

is a **homotopy pullback square** (of $(n-1)$ -fold simplicial spaces).

(A2) The $(n-1)$ -fold simplicial space X_0 is **essentially constant**.

(A3) Each of the $(n-1)$ -uple simplicial spaces X_k is an $(n-1)$ -fold Segal space.

We will say that an n -fold Segal space X is complete if it satisfies the following additional conditions:

(A4) Each of the $(n-1)$ -dimensional Segal spaces X_n is complete (we regard this condition as vacuous when $n = 1$).

(A5) Let $Y_\bullet$ be the simplicial space described by the formula $Y_k = X_{k;0;\dots;0}$; note that condition (A3) guarantees that $Y_\bullet$ is a Segal space. Then $Y_\bullet$ is complete.

— Completion —

- If X is an n -fold Segal space, then there is a **universal example of a map**

$$X \rightarrow X'$$

in the homotopy category of n -fold simplicial spaces, such that X' is an n -fold complete Segal space, i.e. an (∞, n) -category.

We will refer to X' as the completion of X .

- Intuitively, the completion is a refinement of the composition:

$$\begin{array}{ccc} \text{simplicial } (\infty, n-1) \rightarrow & & (\infty, n) \rightarrow \text{simplicial } (\infty, n-1) \\ \mathcal{C}_\bullet \mapsto & & \mathcal{C} \mapsto \mathcal{C}'_\bullet \end{array}$$

- The completion may be interpreted as a
“localization”
 in an appropriate sense.

4. WHAT IS A (SYMMETRIC MONOIDAL) FUNCTOR BETWEEN (SYMMETRIC MONOIDAL) (∞, n) -CATEGORIES?

— Just a summary! —

- Let $\mathcal{C}$ and $\mathcal{D}$ be (∞, n) -categories. There exists another (∞, n) -category $\mathbf{Fun}(\mathcal{C}; \mathcal{D})$

of functors from $\mathcal{C}$ to $\mathcal{D}$. The (∞, n) -category $\mathbf{Fun}(\mathcal{C}; \mathcal{D})$ is characterized up to equivalence by the following universal property: for every (∞, n) -category $\mathcal{C}'$, there is a bijection between the set of isomorphism classes of functors

$$\mathcal{C}' \rightarrow \mathbf{Fun}(\mathcal{C}; \mathcal{D})$$

and

the set of isomorphism classes of functors

$$\mathcal{C}' \times \mathcal{C} \rightarrow \mathcal{D}.$$

- The collection of all (small) (∞, n) -categories can be organized into a **(large) $(\infty, n+1)$ -category $\mathbf{Cat}_{(\infty, n)}$** , with mapping objects given by

$$\mathbf{Map}_{\mathbf{Cat}_{(\infty, n)}}(\mathcal{C}, \mathcal{D}) = \mathbf{Fun}(\mathcal{C}; \mathcal{D})$$

- Suppose that $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal (∞, n) -categories. Then we can also define an (∞, n) -category

$$\mathbf{Fun}^{\otimes}(\mathcal{C}; \mathcal{D})$$

of symmetric monoidal functors from $\mathcal{C}$ to $\mathcal{D}$.

5. WHAT IS THE (∞, n) -CATEGORY $\mathbf{Bord}_n^{(X, \zeta)}$?

We should first explain Atiyah's topological field theory...

— The symmetric monoidal category $\mathbf{Cob}(n)$ —

For $n \in \mathbb{N}$, define the symmetric monoidal category $\mathbf{Cob}(n)$ by:

- (1) An object of $\mathbf{Cob}(n)$ is a closed oriented $(n-1)$ -manifold M .
- (2) Given a pair of objects $M, N \in \mathbf{Cob}(n)$, a morphism from M to N in $\mathbf{Cob}(n)$ is a bordism from M to N : that is, an oriented n -dimensional manifold B equipped with an orientation-preserving diffeomorphism

$$\partial B \simeq \overline{M} \amalg N.$$

Here $\overline{M}$ denotes the manifold M equipped with the opposite orientation.

We regard two bordisms B and B' as defining the same morphism in $\mathbf{Cob}(n)$ if there is an orientation-preserving diffeomorphism $B \simeq B'$ which extends the evident diffeomorphism $\partial B \simeq \overline{M} \amalg N \simeq \partial B'$ between their boundaries.

- (3) For any object $M \in \mathbf{Cob}(n)$, the identity map id_M is represented by the product bordism

$$B = M \times [0; 1].$$

- (4) Composition of morphisms in $\mathbf{Cob}(n)$ is given by gluing bordisms together. More precisely, suppose we are given a triple of objects $M, M', M'' \in \mathbf{Cob}(n)$, and a pair of bordisms

$$B : M \rightarrow M', \quad B' : M' \rightarrow M'',$$

the composition $B' \circ B$ is defined to be the morphism represented by the manifold

$$B \amalg_{M'} B'.$$

- (5) The tensor product operation for the symmetric monoidal structure

$$\otimes : \mathbf{Cob}(n) \times \mathbf{Cob}(n) \rightarrow \mathbf{Cob}(n)$$

is given by the disjoint union of manifolds.

- (6) The unit object for the symmetric monoidal structure of $\mathbf{Cob}(n)$ is

$$\emptyset,$$

the empty set (regarded as a manifold of dimension $(n-1)$).

— Atiyah's topological field theory —

Let k be a field. A topological field theory of dimension n is a symmetric monoidal functor

$$Z : \mathbf{Cob}(n) \rightarrow \mathbf{Vect}(k),$$

where $\mathbf{Vect}(k)$ is the usual symmetric monoidal category of vector spaces over k .

— A couple of problems of the topological field theory —

- Not interesting enough for those interested in $B\text{Diff}(M)$, especially those interested in the Mumford conjecture!
- Not easy to understand for large n , because the structure of a n -dimensional manifold tends to become complicated as n becomes larger!

— For the first problem: encode the group of diffeos! —

For $n \in \mathbb{N}$, define the topological symmetric monoidal category $\mathbf{Cob}_t(n)$ by topologically enriching $\mathbf{Cob}(n)$:

- The objects of $\mathbf{Cob}_t(n)$ are closed oriented manifolds of dimension $(n-1)$.
- Given a pair of objects $M, N \in \mathbf{Cob}_t(n)$, we let $\text{Hom}_{\mathbf{Cob}_t(n)}(M, N)$ denote the classifying space $\mathcal{B}(M, N)$ of bordisms from M to N :

$$\text{Hom}_{\mathbf{Cob}_t(n)}(M, N) := \mathcal{B}(M, N) = BC,$$

where C is a topological category s.t.

- objects are oriented bordisms B from M to N ,
- for every pair of bordisms B and B' , the collection of (orientation-preserving) diffeomorphisms

$$\text{Hom}_C(B; B')$$

has a topology (the topology of uniform convergence of all derivatives) such that the composition maps are continuous.

In particular,

$$\pi_0 \text{Hom}_{\mathbf{Cob}_t(n)}(M, N) = \text{Hom}_{\mathbf{Cob}(n)}(M, N).$$

⋮

Now, for any closed oriented manifold M of dimension n ,

$$\begin{aligned} B\text{Diff}(M) &\xrightarrow{\text{connected component inclusion}} \mathcal{B}(\emptyset, \emptyset) \\ &= \text{Hom}_{\mathbf{Cob}_t(n)}(\emptyset, \emptyset) \end{aligned}$$

does show up in $\mathbf{Cob}_t(n)$, and we may instead consider a topological symmetric monoidal functor

$$\overline{Z} : \mathbf{Cob}_t(n) \rightarrow \mathcal{C},$$

where $\mathcal{C}$ is some topological symmetric monoidal category...

— For the second problem: take into account lower dim.! —

Like a CW decomposition of a CW complex or a pants decomposition of a surface, **any n -dimensional manifold is composed of very simple $n - k$ -dimensional manifolds** ($0 \leq k \leq n$)

$\Rightarrow$ Take into account lower dimensional manifolds:

Suppose given a pair of nonnegative integers $k \leq n$,

a **k -category $\mathbf{Cob}_k(n)$** is given by

- The **objects** of $\mathbf{Cob}_k(n)$ are closed oriented $(n - k)$ -manifolds
- Given a pair of objects $M, N \in \mathbf{Cob}_k(n)$,
a **1-morphism** from M to N is a bordism from M to N : that is, a $(n - k + 1)$ -manifold B equipped with a diffeomorphism $\partial B \simeq \overline{M} \amalg N$.
- Given a pair of objects $M, N \in \mathbf{Cob}_k(n)$ and a pair of bordisms $B, B' : M \rightarrow N$, a **2-morphism** from B to B' is a bordism from B to B' , which is required to be trivial along the boundary: in other words, a manifold with boundary

$$\overline{B} \amalg_{M \amalg \overline{N}} \left((\overline{M} \amalg N) \times [0, 1] \right) \amalg_{\overline{M} \amalg N} B'$$

- ...
- A **k -morphism** in $\mathbf{Cob}_k(n)$ is an n -manifold X with corners, where the structure of ∂X is determined by the source and target of the morphism.

Two n -manifolds with (specified) corners X and Y **determine the same n -morphism** in $\mathbf{Cob}_k(n)$ if they differ by an orientation-preserving diffeomorphism, relative to their boundaries.

- **Composition of morphisms (at all levels)** in $\mathbf{Cob}_k(n)$ is given by gluing of bordisms.

Then,

- $\mathbf{Cob}_0(n)$ may be identified with the set of diffeomorphism classes of closed, oriented n -manifolds.
- $\mathbf{Cob}_1(n) = \mathbf{Cob}(n)$
- Objects and morphisms of $\mathbf{Cob}(n)$ can be regarded as $(n - 1)$ -morphisms and n -morphisms of $\mathbf{Cob}_n(n)$. We may therefore regard $\mathbf{Cob}_n(n)$ as an elaboration of $\mathbf{Cob}(n)$ obtained by considering also “lower” morphisms corresponding to manifolds of dimension $< n - 1$.

So, we may instead consider a symmetric monoidal functor of n -categories

$$\overline{Z} : \mathbf{Cob}_n(n) \rightarrow \mathcal{C},$$

where $\mathcal{C}$ is some symmetric monoidal n -category...

— A good news! $\mathbf{Bord}_n$ —

- Using an (∞, n) -category $\mathbf{Bord}_n$, we may simultaneously elaborate both $\mathbf{Cob}_t(n)$ and $\mathbf{Cob}_n(n)$.
- Thus, $\mathbf{Bord}_n$ may solve the two problems:
 - (1) Not interesting; (2) Not easy;
 of $\mathbf{Cob}(n)$.
- Symmetric monoidal $(\infty; n)$ -category $\mathbf{Bord}_n$ is described *informally* as follows:
 - The objects of $\mathbf{Bord}_n$ are 0-manifolds.
 - The 1-morphisms of $\mathbf{Bord}_n$ are bordisms between 0-manifolds.
 - The 2-morphisms of $\mathbf{Bord}_n$ are bordisms between bordisms between 0-manifolds.
 - ...
 - The n -morphisms of $\mathbf{Bord}_n$ are bordisms between bordisms between ... between bordisms between 0-manifolds (in other words, n -manifolds with corners).
 - The $(n+1)$ -morphisms of $\mathbf{Bord}_n$ are **diffeomorphisms** (which reduce to the identity on the boundaries of the relevant manifolds).
 - The $(n+2)$ -morphisms of $\mathbf{Bord}_n$ are **isotopies of diffeomorphisms**.
 - ...
 - The (∞, n) -category $\mathbf{Bord}_n$ is endowed with a **symmetric monoidal structure**, given by disjoint unions of manifolds.
- $h_n(\mathbf{Bord}_n) = \mathbf{Cob}_n(n)$.
- The precise definition of $\mathbf{Bord}_n$ is given as the **completion** of a n -fold Segal space $P\mathbf{Bord}_n$, which is defined using more sophisticated geometric argument...

— (X, ζ) -structure —

- Let X be a topological space and let ζ be a real vector bundle on X of rank n .
- Let M be a manifold of dimension $m \leq n$.

An (X, ζ) -structure on M consists of the following:

- (1) A continuous map $f : M \rightarrow X$.
- (2) An isomorphism of vector bundles

$$T_M \oplus \mathbb{R}^{n-m} \simeq f^* \zeta.$$

— $\mathbf{Bord}_n^{(X,\zeta)}$ —

- A n -fold Segal space $P\mathbf{Bord}_n^{(X,\zeta)}$ is constructed just like $P\mathbf{Bord}_n$, by using manifolds with (X, ζ) -structures..

$\mathbf{Bord}_n^{(X,\zeta)} \stackrel{\text{Definition}}{=} \text{the completion of}$

the n -fold Segal space $P\mathbf{Bord}_n^{(X,\zeta)}$

- Thus, $\mathbf{Bord}_n^{(X,\zeta)}$ is an (∞, n) -category, and we may consider a symmetric monoidal functor between (∞, n) -categories

$$\tilde{Z} : \mathbf{Bord}_n^{(X,\zeta)} \rightarrow \mathcal{C},$$

or even better, the (∞, n) -category

$$\mathrm{Fun}^{\otimes} \left(\mathbf{Bord}_n^{(X,\zeta)}, \mathcal{C} \right),$$

where $\mathcal{C}$ is some symmetric monoidal (∞, n) -category.

- When $\mathcal{C}$ is a symmetric monoidal (∞, n) -category *with duals*, the cobordism hypothesis claims an equivalence of $(\infty, 0)$ -categories:

$$\mathrm{Fun}^{\otimes} \left(\mathbf{Bord}_n^{(X,\zeta)}, \mathcal{C} \right) \simeq \mathrm{Hom}_{O(n)} \left(\tilde{X}, T^{\sim} \right)$$

6. WHAT DOES IT MEAN FOR A SYMMETRIC MONOIDAL (∞, n) -CATEGORY TO HAVE DUALS?

— From Atiyah's TFT to (∞, n) -category —

- For any symmetric monoidal functor
 $Z : \mathbf{Cob}(2) \rightarrow \mathbf{Vect}(k)$,

$$Z(S^1) \in \text{Ob}(\mathbf{Vect}(k))$$

is always a **finite dimensional** k -vector space.

- This is a formal consequence of the fact that

$$X := Z(S^1), \quad X^\vee := Z(\overline{S^1})$$

are **dual** vector spaces of each other. To show this fact, use the obvious bordisms to produce the k -linear maps

$$ev_X : X \otimes X^\vee \rightarrow k$$

$$coev_X : k \rightarrow X^\vee \otimes X,$$

s.t.

$$\text{id}_X : X \xrightarrow{\text{id}_X \otimes coev_X} X \otimes X^\vee \otimes X \xrightarrow{ev_X \otimes \text{id}_X} X$$

$$\text{id}_{X^\vee} : X^\vee \xrightarrow{coev_X \otimes \text{id}_{X^\vee}} X^\vee \otimes X \otimes X^\vee \xrightarrow{\text{id}_{X^\vee} \otimes ev_X} X^\vee$$

This characterization allows us to define the concept of **have a dual** for an object of a monoidal category.

- We would like to generalize the concept of “**finite dimensionality**” of $\text{Ob}(\mathbf{Vect}(k))$ to an appropriate concept for an object of an (∞, n) -category, by defining an analogue of “**dual**” for an (∞, n) -category.
- For a monoidal category $\mathcal{C}$, define a category BC by
 - $\text{Ob}(BC) = \{*\}$.
 - $\text{Hom}_{BC}(*, *) = \text{Ob } \mathcal{C}$
 - The composition

$$\text{Hom}_{BC}(*, *) \times \text{Hom}_{BC}(*, *) \rightarrow \text{Hom}_{BC}(*, *)$$

is induced by the monoidal structure of $\mathcal{C}$.

Then,

an object $x \in \text{Ob } \mathcal{C}$ has a dual

$\iff$ **a morphism $x \in \text{Hom}_{BC}(*, *)$ has an adjoint**

- For an (∞, n) -category $\mathcal{C}$, we can also define an $(\infty, n+1)$ -category BC just as above. Then, we define an concept of “have adjoint” for an $(\infty, n+1)$ -category s.t.

$$\mathcal{C} \text{ has duals} \stackrel{\text{def}}{\iff} BC \text{ has adjoints}$$

- $\mathbf{Bord}_n^{(x, \zeta)}$ has duals given by the manifold with opposite orientation.

7. HOW CAN WE DEDUCE $|\mathbf{Bord}_n^G| \simeq (QS^0)_{hG}$?

Formal reduction of $|\mathbf{Bord}_n^G| \simeq (QS^0)_{hG}$ from CH

Specialize to the case:

$\mathcal{C}$: a symmetric monoidal $(\infty, 0)$ -category with duals

$\Rightarrow \exists$ a topological space T s.t.

($\because \mathcal{C}$ is $(\infty, 0)$ -category): $\mathcal{C} \cong \pi_{\leq \infty} T$;

($\because \mathcal{C}$ is symmetric monoidal): T is a E_∞ -space;

($\because \mathcal{C}$ with duals): $\pi_0(T)$ is an abelian group.

$\Rightarrow T$ is an infinite loop space, and as infinite loop space with $O(n)$ -actions:

$$T \simeq \mathrm{Map}_{\mathrm{infinite\ loop}}(QS^0, T),$$

where the $O(n)$ -action is through Ω^n in $T \simeq \Omega^n T(n)$ on the left, and is ONLY through QS^0 on the right.

$\Rightarrow$ For $\mathbf{Bord}_n^G := \mathbf{Bord}_n^{(BG, EG \times_G \mathbb{R}^n \rightarrow BG)}$

with $\widetilde{BG} = EG \times_G O(n)$,

the $(\infty, 0)$ -category $|\mathbf{Bord}_n^G|$,

obtained from the symmetric monoidal (∞, n) -category with duals $\mathbf{Bord}_n^G$ by inverting all higher morphisms, also yields an infinite loop space, which we also denote by $|\mathbf{Bord}_n^G|$.

$\Rightarrow$ For any infinite loop space T with $\mathcal{C} = \pi_{\leq \infty} T$,

$$\begin{aligned} & \mathrm{Map}_{\mathrm{infinite\ loop}}(|\mathbf{Bord}_n^G|, T) \\ &= \mathrm{Fun}^\otimes(|\mathbf{Bord}_n^G|, \mathcal{C}) \quad \because \mathcal{C} = (\infty, 0)\text{-category} \quad \mathrm{Fun}^\otimes(\mathbf{Bord}_n^G, \mathcal{C}) \\ &\stackrel{\mathrm{CH}}{\simeq} \mathrm{Hom}_{O(n)}(EG \times_G O(n), T) = \mathrm{Hom}_G(EG, T) \\ &= \mathrm{Hom}_G(EG, \mathrm{Map}_{\mathrm{infinite\ loop}}(QS^0, T)) \\ &= \mathrm{Map}_{\mathrm{infinite\ loop}, G}(EG \times QS^0, T) \\ &= \mathrm{Map}_{\mathrm{infinite\ loop}}(EG \times_G QS^0, T) \\ &= \mathrm{Map}_{\mathrm{infinite\ loop}}((QS^0)_{hG}, T) \end{aligned}$$

$\Rightarrow |\mathbf{Bord}_n^G| \simeq (QS^0)_{hG}$