

INTEGRABLE PLURICANONICAL FORMS
and
KODAIRA DIMENSIONS OF COMPLEMENTS OF DIVISORS

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Let X be a complex manifold (possibly non-compact) of dimension n and ω a holomorphic m -ple n -form on X . We write ω as $\omega = \psi(w) (dw_1 \wedge \dots \wedge dw_n)^m$, using local coordinates $(w_1, \dots, w_n)$. We associate with ω the continuous (n, n) -form $(\omega \wedge \bar{\omega})^{1/m}$, given locally by $|\psi(w)|^{2/m} \prod_{i=1}^n (\sqrt{-1}/2\pi) dw_i \wedge d\bar{w}_i$. Then ω is called integrable ($L_{2/m}$ -integrable) if $\int_X (\omega \wedge \bar{\omega})^{1/m} < \infty$. Let $F_m(X)$ be the set of all integrable holomorphic m -ple n -forms on X . When X has a compactification, $F_m(X)$ becomes a vector space. Using $F_m(X)$, we shall define the Kodaira dimension $\kappa(X)$ of X , which is a generalization of the notion of Kodaira dimension of compact complex manifolds introduced by Iitaka [8] (cf. Ueno [19]). Here we want to discuss the properties of $\kappa(X)$ and some related aspects. Details will appear in [17].

1. Kodaira Dimension.

Let X be a complex manifold of dimension n and $F_m(X)$ the set of all integrable holomorphic m -ple n -forms on X as above. Set $N(X) = \{m > 0 \mid F_m(X) \neq \{0\}\}$. If $m \in N(X)$, for a finite set of elements $\omega_0, \dots, \omega_N \in F_m(X)$, we can define a meromorphic map $\Phi_{\{\omega_0, \dots, \omega_N\}}: X \ni w \longrightarrow [\omega_0(w) : \dots : \omega_N(w)]$ of X into $\mathbb{P}^N$. Next we put $\text{rk}_m = \max[\text{rank } \Phi_{\{\omega_0, \dots, \omega_N\}}]$, where the maximum is taken over all choices of finite elements in $F_m(X)$ for $N=0, 1, 2, \dots$. The rank of a meromorphic map is the maximum rank of the Jacobian matrix where it is holo-

morphic. Now we define the Kodaira dimension $\kappa(X)$ of X by

$$(1.1) \quad \kappa(X) = \begin{cases} \max_{m \in N(X)} \{rk_m\} & \text{if } N(X) \neq \emptyset, \\ -\infty & \text{if } N(X) = \emptyset. \end{cases}$$

Note that $\kappa(X)$ takes one of the values $-\infty, 0, 1, \dots, n$.

(1.2) Theorem ([17]). The Kodaira dimension $\kappa(X)$ is a bimeromorphic (in the sense of Remmert) invariant of a complex manifold X .

Proof. Let X' be a complex manifold such that there exists a bimeromorphic map $f: X' \rightarrow X$. Then f^* induces an isomorphism of $F_m(X)$ onto $F_m(X')$. To see this, take an element $\omega \in F_m(X)$, then $f^*\omega$ is a holomorphic m -ple n -form on $X' - S(f)$, where $S(f)$ is the set of points where f is not holomorphic. Since $\text{codim } S(f) \geq 2$, it extends holomorphically on X' . Clearly $\int_X (f^*\omega \wedge \overline{f^*\omega})^{1/m} = \int_{X'} (\omega \wedge \overline{\omega})^{1/m} < \infty$, which implies $f^*\omega \in F_m(X')$. Considering the inverse map, we get the surjectivity. Consequently we have, by definition $\kappa(X') = \kappa(X)$. Q.E.D.

We list some properties of $\kappa(X)$ (cf. [17]).

1. Let $\mathbb{C}$ be the complex plane and $\mathbb{C}^* = \mathbb{C} - \{0\}$. Then $\kappa(\mathbb{C}) = -\infty$, $\kappa(\mathbb{C}^*) = -\infty$. Further $\kappa(\mathbb{C} \times Y) = -\infty$, $\kappa(\mathbb{C}^* \times Y) = -\infty$, for any complex manifold Y .
2. Let X, Y be complex manifolds of the same dimension such that $X \subset Y$. Then $\kappa(X) \leq \kappa(Y)$. In particular, if $\kappa(X) = -\infty$, we get $\kappa(Y) = -\infty$.
3. Let X be a complex manifold and Z an analytic subset of X with $\text{codim } Z \geq 2$. Then $\kappa(X - Z) = \kappa(X)$.
4. Let X, Y be complex manifolds of the same dimension.

Suppose that there is a surjective proper meromorphic map $f: X \rightarrow Y$. Then $\kappa(X) \geq \kappa(Y)$.

In case X is a complex space, we define $\kappa(X)$ to be $\kappa(X^*)$, using a desingularization X^* of X .

2. Complements of Divisors.

In this section, we deal with the case in which X has a compactification $\bar{X}$. We assume that $\bar{X}$ is a smooth compactification in the sense that $\bar{X}$ is a compact complex manifold and $D = \bar{X} - X$ is a divisor of normal crossings. Let $K_{\bar{X}}$ be the canonical bundle of $\bar{X}$ and $[D]$ the line bundle determined by D . In this case, we have

$$(2.1) \text{Theorem}([17]). \quad F_m(X) \cong H^0(\bar{X}, O(mK_{\bar{X}} + (m-1)[D])).$$

The proof is based on the fact that if $f(z)(dz)^m$ is integrable on the punctured disc Δ^* , then the Laurent expansion of $f(z)$ becomes as $\sum_{j=-\infty}^{\infty} a_j z^j$ ([14], Appendix).

$$(2.2) \text{Definition.} \quad \gamma_m(X) = \dim F_m(X) = \dim H^0(\bar{X}, O(mK_{\bar{X}} + (m-1)[D])).$$

We can redefine the Kodaira dimension as follows.

(2.3) Definition. Let $\psi_0, \dots, \psi_N$ be a basis of $H^0(\bar{X}, O(mK_{\bar{X}} + (m-1)[D]))$. Let ϕ_m be the meromorphic map defined by $[\psi_0 : \dots : \psi_N]$ of $\bar{X}$ into $\mathbb{P}_N$. Put $N(X) = \{m > 0 \mid \dim H^0(\bar{X}, O(mK_{\bar{X}} + (m-1)[D])) > 0\}$. Then

$$\kappa(X) = \begin{cases} \max_{m \in N(X)} \{\dim \phi_m(\bar{X})\} & \text{if } N(X) \neq \emptyset, \\ -\infty & \text{if } N(X) = \emptyset. \end{cases}$$

(2.4) Example. Let D be a hypersurface of degree d in $\mathbb{P}_n$ which has at most normal crossings. Then $\kappa(\mathbb{P}_n - D) = n$ if $d > n+1$ and $\kappa(\mathbb{P}_n - D)$

$= -\infty$ if $d \leq n+1$. Next put $U_a = \{z_1^{a_1} + \dots + z_{n+1}^{a_{n+1}} = 1\}$ in $\mathbb{C}^{n+1}$. Then $\kappa(U_a) = n$ if $\sum_i 1/a_i < 1$. Here we represent a classification of complements of finite points on compact curves.

κ	$\gamma_1 = g$	$\gamma_m \quad (m \geq 2)$	structure
$-\infty$	0	0	$\mathbb{P}_1, \mathbb{P}_1 - \{a_1\}, \mathbb{P}_1 - \{a_1\} - \{a_2\}$
0	1	1	elliptic curve
1	0	$m(k-2) - k + 1$ (except $k=3, m=2$)	$\mathbb{P}_1 - \bigcup_{i=1}^k \{a_i\}, k \geq 3$
	1	$mk - k$	elliptic curve $-\bigcup_{i=1}^k \{a_i\}, k \geq 1$
	$g \geq 2$	$m(k+2g-2) - k + 1 - g$	curve of genus $\geq 2 - \bigcup_{i=1}^k \{a_i\}, k \geq 0$

Let X be again a complex manifold of dimension n and $\bar{X}$ a smooth compactification of X with $D = \bar{X} - X$. We remark that $F_m(X)$ has an invariant Hermitian metric. So $F_m(X)$ is a finite dimensional Hilbert space ([17]). The Kodaira dimension $\kappa(X)$ has the following relation with $\kappa(K_{\bar{X}} + D, \bar{X})$.

(2.5) Proposition. If $\kappa(X) \geq 0$, then $\kappa(X) = \kappa(K_{\bar{X}} + D, \bar{X})$. Further $\kappa(X) = n$ if and only if $\kappa(K_{\bar{X}} + D, \bar{X}) = n$.

The first part of this relation also holds without the assumption that D has normal crossings (See [17], Appendix).

(2.6) Remark. When $\bar{X}$ is a smooth compactification of X , Iitaka calls $\kappa(K_{\bar{X}} + D, \bar{X})$ the logarithmic Kodaira dimension of X and writes it by $\bar{\kappa}(X)$ ([9]). He proves that $\bar{\kappa}(X)$ is a proper birational invariant of X . From Theorem (1.2) and Proposition (2.5), it follows that if $\kappa(X) \geq 0$, then $\bar{\kappa}(X)$ is a bimeromorphic invariant. But the following examples show that in case $\kappa(X) = -\infty$, $\bar{\kappa}(X)$ need not be a bimeromorphic invariant of X . We consider several

compactifications of $\mathbb{C}^{*2}$. We have, by (1.3) $\kappa(\mathbb{C}^{*2}) = -\infty$. 1)

$\mathbb{C}^{*2} = \mathbb{P}_2 - \bigcup_{i=1}^3 H_i$, with three lines H_1, H_2, H_3 in general position.

In this case $\bar{\kappa}(\mathbb{P}_2 - \bigcup_{i=1}^3 H_i) = 0$. 2) $\mathbb{C}^{*2} = \mathbb{P}_1 \times \mathbb{P}_1 - \bigcup_{i=1}^4 L_i$, where

$L_i = a_i \times \mathbb{P}_1$, $i=1,2$ and $L_i = \mathbb{P}_1 \times b_i$, $i=3,4$. We have $\bar{\kappa}(\mathbb{P}_1 \times \mathbb{P}_1 - \bigcup_{i=1}^4 L_i) = 0$.

3) $\mathbb{C}^{*2} = S - E$, where S is a Hopf surface given by $S = \mathbb{C}^2 - \{0\} / \{g\}$ with $g: (z_1, z_2) \rightarrow (\alpha^p z_1 + \lambda z_2^p, \alpha z_2)$, $\lambda \neq 0$, $0 < |\alpha| < 1$, for a positive integer p and E is an elliptic curve given by $E = (\mathbb{C}^2 - \{0\}) \cap \{z_2 = 0\} / \{g\}$ (See [7], for details). In this case, we have $K_S = -(p+1)[E]$ and then $\bar{\kappa}(S - E) = -\infty$.

4) $\mathbb{C}^{*2} = F - D$, where F is a $\mathbb{P}_1$ -bundle over an elliptic curve constructed by Serre ([5], p232) and D is a section with $D^2 = 0$. We also have $\bar{\kappa}(F - D) = -\infty$.

In case X is given by $X = \bar{X} - D$ with a singular divisor D on a compact complex manifold $\bar{X}$, it is not so easy to calculate $\kappa(X)$. Here we give a method. According to Hironaka, there exists a desingularization $\pi: X^* \rightarrow X$ such that $\pi^{-1}(D) = D^*$ has normal crossings. Let $\pi^{-1}(\text{Sing } D) = \sum_i S_i$ be the irreducible decomposition of the exceptional set of π . Let R_π be the ramification divisor of π . Set $\mathcal{E}_D = \pi^* D - D^* - R_\pi$. We can write $\mathcal{E}_D = \sum_i t_i S_i$ with integers t_i .

(2.7) Definition (Shiffman [18]). Let A be a divisor on $\bar{X}$ passing through the non-normal crossing points of D . If we write $\pi^* A = \bar{A} + \sum_i p_i^A S_i$, where $\bar{A}$ is the strict transform of A , then $p_i^A \geq 1$. Define

$$\gamma_{A,D} = \max_i \{t_i^+ / p_i^A\}, \quad \text{where } x^+ \text{ means } \max(x, 0).$$

(2.8) Proposition. We have

$$\gamma_m(X) \geq \dim H^0(\bar{X}, \mathcal{O}(mK_{\bar{X}} + (m-1)\{[D] - \gamma_{A,D}[A]\})).$$

$$\bar{\kappa}(X) = \kappa(K_{\bar{X}} + D^*, \bar{X}^*) \geq \kappa(K_{\bar{X}} + D - \gamma_{A,D} A, \bar{X}).$$

Proof. Note that $[\xi_D] = \pi^*(K_{\bar{X}} + [D]) - (K_{\bar{X}^*} + [D^*])$. We have, by definition $\kappa(\gamma_{A,D} \pi^* A - \xi_D, \bar{X}^*) \geq 0$. The assertion follows from this.

(2.9) Corollary. Let D be a singular hypersurface of degree d in $\mathbb{P}_n$. Let A be a hypersurface of degree a in $\mathbb{P}_n$ passing through the non-normal crossing points of D . If $(d - n - 1 - \gamma_{A,D} a) > 0$, then $\kappa(\mathbb{P}_n - D) = \bar{\kappa}(\mathbb{P}_n - D) = n$.

3. Quasi-Projective Manifolds with $\kappa(X) = \dim X$.

A complex manifold is called a quasi-projective manifold if it is given as a complement of an analytic subset of a projective algebraic manifold. In [17], we prove the following facts.

(3.1) Theorem. Let X be a quasi-projective manifold of dimension n . Assume that $\kappa(X) = n$. Then X satisfies

1. Any non-degenerate holomorphic map $f: \Delta^* \times \Delta^{n-1} \rightarrow X$ can be extended to a meromorphic map from Δ^n to any compactification of X . Here Δ is the unit disc and $\Delta^* = \Delta - \{0\}$.
An equidimensional holomorphic map is called non-degenerate if the Jacobian does not vanish identically.
2. Every biholomorphic transformation of X extends as a meromorphic transformation of any compactification of X .
3. Let $\text{Aut}(X)$ be the group of biholomorphic transformations of X . Then $\text{Aut}(X)$ is a finite group.
4. X is measure-hyperbolic.
- 4'. Every holomorphic map $f: \mathbb{C} \times \Delta^{n-1} \rightarrow X$ degenerates.

These properties show that in this case X behaves like a projective algebraic manifold of general type.

4. Concluding Remarks.

A. Let $\mathcal{D}$ be a bounded symmetric domain of dimension n and Γ a totally discontinuous group operating on $\mathcal{D}$ such that $X = \mathcal{D}/\Gamma$ has a compactification. Let $\pi: \mathcal{D} \rightarrow X$ be the projection. In many cases, the space $\pi^* F_m(X)$ corresponds to the vector space of cusp forms on $\mathcal{D}$ (For instance, see [6], [10]). So it is expected that this phenomenon holds in general. Moreover we have the following question: Let X be a complex manifold of dimension n . If the universal covering manifold of X is a bounded domain in $\mathbb{C}^n$, is it true that $\kappa(X) = n$?

B. Let Y be a complex manifold of dimension n and Z an analytic subset of Y . We set $F_m^Z(Y) = \{\omega \in H^0(Y-Z, \mathcal{O}(mK)) / H^0(Y, \mathcal{O}(mK)) \mid \omega \text{ is locally integrable across } Z, \text{ i.e., for every point } x \in Z, \text{ there is a neighborhood } U \text{ of } x \text{ in } Y \text{ such that } \omega \text{ is integrable on } U-Z \cap U\}.$ If $\text{codim } Z \geq 2$, then $F_m^Z(Y) = \{0\}$. In case Z is a divisor D having normal crossings, then we obtain

$$F_m^D(Y) \cong H^0(Y, \mathcal{O}(mK + (m-1)[D])) / H^0(Y, \mathcal{O}(mK))$$

(cf. Theorem (2.1)). Take neighborhoods U, U' of Z in Y . If $U \supset U'$, then we have an inclusion $F_m^Z(U) \hookrightarrow F_m^Z(U')$. Hence we can define $\hat{F}_m^Z(Y) = \varinjlim_U F_m^Z(U)$. Put $\gamma_m^Z(Y) = \dim F_m^Z(Y)$ and $\hat{\gamma}_m^Z(Y) = \dim \hat{F}_m^Z(Y)$. Then $\gamma_m^Z(Y) \leq \hat{\gamma}_m^Z(Y) \leq O(m^n)$. Further we can define $\kappa^Z(Y)$ and $\hat{\kappa}^Z(Y)$ in a similar manner as in (1.1).

Next in case Y is a complex space, letting $\pi: Y^* \rightarrow Y$ be a desingularization of Y , we put $\gamma_m^Z(Y) = \gamma_m^{Z^*}(Y^*)$, $\hat{\gamma}_m^Z(Y) = \hat{\gamma}_m^{Z^*}(Y^*)$ with $Z^* = \pi^{-1}(Z)$

(4.1) Proposition. Let $\bar{X}$ be a compact complex manifold of dimension n and D an effective divisor on $\bar{X}$. Put $X = \bar{X} - D$. Then

$$P_m(\bar{X}) \leq \gamma_m(X) \leq P_m(\bar{X}) + \hat{\gamma}_m^D(\bar{X}).$$

Proof. This follows from the exact sequence

$$0 \longrightarrow H^0(\bar{X}^*, O(mK_{\bar{X}^*})) \longrightarrow H^0(\bar{X}^*, O(mK_{\bar{X}^*} + (m-1)[D^*])) \longrightarrow F_m^D(\bar{X}) \longrightarrow 0,$$

where $\bar{X}^*, D^*$ is a desingularization of $\bar{X}$, D such that $\bar{X}^*$ is a smooth compactification of X . Q.E.D.

We consider the special case in which $Z=y$ is an isolated singularity of an n -dimensional complex space Y . For simplicity, put $\gamma_m = \hat{\gamma}_m^Y(Y)$. Let $\pi: Y^* \longrightarrow Y$ be a desingularization of Y .

For a neighborhood U of y , put $U^* = \pi^{-1}(U)$. Define

$$r_m = \dim \varinjlim_U H^0(U-y, O(mK))/H^0(U^*, O(mK)). \text{ Put } \sigma_m = r_m - \gamma_m. \text{ Then } \sigma_m \geq 0.$$

and $\gamma_1 = 0$. It is easily seen that if y is a quotient singularity, then $\sigma_m = 0$ for all m (cf. [1], [2]). Question: When $\sigma_m = 0$? When $\gamma_m = 0$?

(4.2) Example. Suppose that $\pi^{-1}(y) = E$ is $\mathbb{P}_{n-1}$ and $E|E \sim (-e)$, where (1) means the hyperplane bundle on $\mathbb{P}_{n-1}$. In this case, we get easily that $\sigma_m = 0$ for all m and if $e \leq n$, then $\gamma_m = 0$ for all m .

In case $\dim Y = 2$, Laufer showed in [15] that $\sigma_1 = 0$ if and only if y is a rational singularity. Precisely he proved $\dim R^1 \pi_* O_{U^*} = \sigma_1$. Knöller [11] calculates r_m and $\lim_{m \rightarrow \infty} r_m/m^2$ for several singularities. In particular, the condition $r_m = 0$ for all m characterizes the rational double points (See also [12], for an application).

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