

Some results on the Möbius gyrovector space

メビウスジャイロベクトル空間に関するいくつかの結果

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アブストラクト. Hilbert 空間の間の線形作用素に対応する Möbius ジャイロベクトル空間の間の写像について考察する. また, Möbius の演算に関連する Cauchy 型の不等式, Möbius の和に関する級数の収束条件, さらに Möbius の和に関連する実または複素内積空間における Cauchy-Bunyakovsky-Schwarz 型の不等式についても述べる.

1 Möbius gyrovector space

Definition.(Ungar [U]) Let $\mathbb{V} = (\mathbb{V}, \cdot)$ be a real inner product space with a positive definite inner product $\cdot$ and let $\mathbb{V}_s$ be the open ball

$$\mathbb{V}_s = \{\mathbf{a} \in \mathbb{V} : \|\mathbf{a}\| < s\}$$

for any fixed $s > 0$. Then the Möbius addition $\oplus_M$ and the Möbius scalar multiplication $\otimes_M$ are given by the equations

$$\mathbf{a} \oplus_M \mathbf{b} = \frac{\left(1 + \frac{2}{s^2} \mathbf{a} \cdot \mathbf{b} + \frac{1}{s^2} \|\mathbf{b}\|^2\right) \mathbf{a} + \left(1 - \frac{1}{s^2} \|\mathbf{a}\|^2\right) \mathbf{b}}{1 + \frac{2}{s^2} \mathbf{a} \cdot \mathbf{b} + \frac{1}{s^4} \|\mathbf{a}\|^2 \|\mathbf{b}\|^2}$$

$$r \otimes_M \mathbf{a} = s \tanh \left(r \tanh^{-1} \frac{\|\mathbf{a}\|}{s} \right) \frac{\mathbf{a}}{\|\mathbf{a}\|} \quad (\text{if } \mathbf{a} \neq \mathbf{0}), \quad r \otimes_M \mathbf{0} = \mathbf{0}$$

for $\mathbf{a}, \mathbf{b} \in \mathbb{V}_s$, $r \in \mathbb{R}$. The operations $\oplus_M$, $\otimes_M$ in the interval $(-s, s)$ are defined by

$$a \oplus_M b = \frac{a + b}{1 + \frac{1}{s^2} ab}$$

$$r \otimes_M a = s \tanh \left(r \tanh^{-1} \frac{a}{s} \right)$$

for $a, b \in (-s, s)$, $r \in \mathbb{R}$.

Then, $(\mathbb{V}_s, \oplus_M, \otimes_M)$ satisfies the axioms of gyrovector space.

We simply denote $\oplus_M, \otimes_M$ as $\oplus, \otimes$, respectively.

We use $\oplus_s, \otimes_s$ also, if we want to indicate the parameter s .

If several kinds of operations appear in a formula simultaneously, we always give priority by the following order (i) ordinary scalar multiplication (ii) gyroscalar multiplication $\otimes$ (iii) gyroaddition $\oplus$, that is,

$$r_1 \otimes w_1 \mathbf{a}_1 \oplus r_2 \otimes w_2 \mathbf{a}_2 = \{r_1 \otimes (w_1 \mathbf{a}_1)\} \oplus \{r_2 \otimes (w_2 \mathbf{a}_2)\},$$

and parentheses are omitted in such cases.

Gyro-operations are generally not commutative, associative or distributive:

$$\begin{aligned} \mathbf{a} \oplus \mathbf{b} &\neq \mathbf{b} \oplus \mathbf{a} \\ \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c}) &\neq (\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} \\ r \otimes (\mathbf{a} \oplus \mathbf{b}) &\neq r \otimes \mathbf{a} \oplus r \otimes \mathbf{b} \\ t(\mathbf{a} \oplus \mathbf{b}) &\neq t\mathbf{a} \oplus t\mathbf{b}. \end{aligned}$$

Proposition.(Ungar)

$$\begin{aligned} \mathbf{a} \oplus_s \mathbf{b} &\rightarrow \mathbf{a} + \mathbf{b} \quad (s \rightarrow \infty) \\ r \otimes_s \mathbf{a} &\rightarrow r\mathbf{a} \quad (s \rightarrow \infty). \end{aligned}$$

2 On mappings between Möbius gyrovector spaces corresponding to Hilbert space operators

T. Abe raised the following problem in an oral presentation [A].

Problem. What are mappings between gyrolinear spaces corresponding to linear mappings between linear spaces ?

The author thinks that the following two results (due to the author) will not provide any satisfactory answer to the problem above.

Theorem. Let $\mathbb{V}$ be a real Hilbert space with $\dim \mathbb{V} \geq 2$. If $f : \mathbb{V}_1 \rightarrow (-1, 1)$ satisfies that

$$\begin{aligned} f(\mathbf{x} \oplus \mathbf{y}) &= f(\mathbf{x}) \oplus f(\mathbf{y}) \\ f(r \otimes \mathbf{x}) &= r \otimes f(\mathbf{x}) \end{aligned}$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{V}_1$, $r \in \mathbb{R}$, then we have $f \equiv 0$.

Theorem. Let $\mathbb{V}$ be a real Hilbert space. If $f : \mathbb{V} \rightarrow \mathbb{R}$ is a continuous map and satisfies that

$$\begin{aligned} f(\mathbf{x} \oplus_s \mathbf{y}) - \{f(\mathbf{x}) \oplus_s f(\mathbf{y})\} &\rightarrow 0 \quad (s \rightarrow \infty) \\ f(r \otimes_s \mathbf{x}) - r \otimes_s f(\mathbf{x}) &\rightarrow 0 \quad (s \rightarrow \infty) \end{aligned}$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{V}$, $r \in \mathbb{R}$, then there exists a unique $\mathbf{c} \in \mathbb{V}$ such that $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{c}$ ($\mathbf{x} \in \mathbb{V}$). The converse is also true.

Theorem.(Molnár and Virosztek, 2015) Let $\beta : \mathbb{R}_1^3 \rightarrow \mathbb{R}_1^3$ be a continuous map. We have β is an algebraic endomorphism with respect to the operation $\oplus_E$, i.e., β satisfies

$$\beta(\mathbf{u} \oplus_E \mathbf{v}) = \beta(\mathbf{u}) \oplus_E \beta(\mathbf{v}) \quad (\mathbf{u}, \mathbf{v} \in \mathbb{R}_1^3)$$

if and only if either (i) or (ii) of the following holds:

(i) there is an orthogonal matrix $O \in M_3(\mathbb{R})$ such that

$$\beta(\mathbf{v}) = O\mathbf{v}, \quad \mathbf{v} \in \mathbb{R}_1^3$$

(ii) $\beta(\mathbf{v}) = \mathbf{0}$, $\mathbf{v} \in \mathbb{R}_1^3$.

Theorem.(Frenkel, 2016) For $n \geq 2$, continuous endomorphisms of the Einstein gyrogroup $(\mathbb{R}_1^n, \oplus_E)$ are precisely the restrictions to $\mathbb{R}_1^n$ of the orthogonal transformations of $\mathbb{R}^n$ and 0-map.

• **An attempt to formulate a class of mappings between Möbius gyrovector spaces that is corresponding to Hilbert space operators.**

Definition. We denote by $M_{n,m}(\mathbb{R})$ the set of all $n \times m$ matrices whose entries are real numbers.

The ordinary operation of matrices on vectors:

$$A\mathbf{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$

$$A : x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \mapsto (a_{11}x_1 + a_{12}x_2)\mathbf{f}_1 + (a_{21}x_1 + a_{22}x_2)\mathbf{f}_2.$$

For simplicity, let us assume that $\mathbb{U}$ and $\mathbb{V}$ are finite dimensional real inner product spaces. Suppose that $\{\mathbf{e}_i\}_{i=1}^m$ (resp. $\{\mathbf{f}_j\}_{j=1}^n$) be an orthonormal basis in $\mathbb{U}$ (resp. $\mathbb{V}$). For an arbitrary element $\mathbf{x} \in \mathbb{U}_1$, we can apply the orthogonal gyroexpansion (see [W2]) to get a unique m -tuple $(r_1, \dots, r_m)$ of real numbers such that

$$\mathbf{x} = r_1 \otimes_1 \frac{\mathbf{e}_1}{2} \oplus_1 \dots \oplus_1 r_m \otimes_1 \frac{\mathbf{e}_m}{2}.$$

Then we can define a map $f : \mathbb{U}_1 \rightarrow \mathbb{V}_1$ by the equation

$$f(\mathbf{x}) = (a_{11}r_1 + \dots + a_{1m}r_m) \otimes_1 \frac{\mathbf{f}_1}{2} \oplus_1$$

$$\dots \oplus_1 (a_{n1}r_1 + \dots + a_{nm}r_m) \otimes_1 \frac{\mathbf{f}_n}{2}.$$

We say that f is the induced map from the matrix A .

For any map $f : \mathbb{U}_1 \rightarrow \mathbb{V}_1$ and any $s > 0$, we can define a map $f_s : \mathbb{U}_s \rightarrow \mathbb{V}_s$ by the equation

$$f_s(\mathbf{x}) = sf\left(\frac{\mathbf{x}}{s}\right) \quad (\mathbf{x} \in \mathbb{U}_s). \quad (1)$$

Theorem. Let $\mathbb{U}$ and $\mathbb{V}$ be two real Hilbert spaces and $A \in M_{n,m}(\mathbb{R})$. If $f : (\mathbb{U}_1, \oplus_1, \otimes_1) \rightarrow (\mathbb{V}_1, \oplus_1, \otimes_1)$ is the induced map from the matrix A and f_s is defined by (1), then, for arbitrary $\mathbf{x}, \mathbf{y} \in \mathbb{U}$ and $r \in \mathbb{R}$, we have

$$f_s(\mathbf{x} \oplus_s \mathbf{y}) \rightarrow A(\mathbf{x} + \mathbf{y})$$

$$f_s(\mathbf{x}) \oplus_s f_s(\mathbf{y}) \rightarrow A\mathbf{x} + A\mathbf{y}$$

$$f_s(r \otimes_s \mathbf{x}) \rightarrow Ar\mathbf{x}$$

$$r \otimes_s f_s(\mathbf{x}) \rightarrow rA\mathbf{x}$$

as $s \rightarrow \infty$.

Theorem. Let $\mathbb{U}, \mathbb{V}, \mathbb{W}$ be real Hilbert spaces. Let $A = (a_{ij}) \in M_{n,m}(\mathbb{R})$, $B = (b_{ij}) \in M_{p,n}(\mathbb{R})$. Suppose that $\{\mathbf{e}_i\}_{i=1}^m$, $\{\mathbf{f}_j\}_{j=1}^n$, $\{\mathbf{g}_k\}_{k=1}^p$ be an orthonormal basis

in $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W}$, respectively. Let f (resp. g) be the induced map from matrix A (resp. B). Then the composed map $g \circ f$ is also an induced map from the matrix BA .

Theorem. Let f be the induced map from a matrix A and f^* be the induced map from the adjoint matrix A^* . Then

$$-f_s^*(\mathbf{x}) \cdot \mathbf{y} \oplus_s \mathbf{x} \cdot f_s(\mathbf{y}) \rightarrow 0 \quad (s \rightarrow \infty).$$

Definition. Let $\mathbb{U}$, $\mathbb{V}$ be two real Hilbert spaces. A mapping $f : \mathbb{U}_1 \rightarrow \mathbb{V}_1$ is said to be *quasi gyrolinear operator* if there exists an linear operator $T : \mathbb{U} \rightarrow \mathbb{V}$ such that, for any $\mathbf{x}, \mathbf{y} \in \mathbb{U}$ and $r \in \mathbb{R}$,

$$\begin{aligned} f_s(\mathbf{x} \oplus_s \mathbf{y}) &\rightarrow T(\mathbf{x} + \mathbf{y}) \\ f_s(\mathbf{x}) \oplus_s f_s(\mathbf{y}) &\rightarrow T\mathbf{x} + T\mathbf{y} \\ f_s(r \otimes_s \mathbf{x}) &\rightarrow Tr\mathbf{x} \\ r \otimes_s f_s(\mathbf{x}) &\rightarrow rT\mathbf{x} \end{aligned}$$

as $s \rightarrow \infty$. It might be necessary to impose some conditions on order of convergence, such as $o(s^\alpha), O(s^\alpha)$, in some contexts.

3 Some inequalities

Properties of operations $\oplus$ and $\otimes$ on the open interval $(-1, 1)$. For $a, b, c \in (-1, 1)$, $r, r_1, r_2 \in \mathbb{R}$,

- (1) $a \oplus b = b \oplus a$
- (2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
- (3) $0 \oplus a = a \oplus 0 = a, \quad (-a) \oplus a = a \oplus (-a) = 0$
- (4) $1 \otimes a = a$
- (5) $(r_1 r_2) \otimes a = r_1 \otimes (r_2 \otimes a)$
- (6) $r \otimes (a \oplus b) = r \otimes a \oplus r \otimes b$
- (7) $(r_1 + r_2) \otimes a = r_1 \otimes a \oplus r_2 \otimes a.$

It seems not appropriate to say “ $((-1, 1), \oplus, \otimes)$ is a one-dimensional real linear space”.

Theorem.(Cauchy, 1821) If x_j, y_j are real numbers, then

$$x_1 y_1 + \cdots + x_n y_n \leq (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}} (y_1^2 + \cdots + y_n^2)^{\frac{1}{2}}.$$

Theorem. If $r_1, \cdots, r_n \geq 0$ and $0 \leq x_1, \cdots, x_n < 1$, then we have

$$r_1 \otimes x_1 \oplus \cdots \oplus r_n \otimes x_n \leq (r_1^2 + \cdots + r_n^2)^{\frac{1}{2}} \otimes (x_1^2 \oplus \cdots \oplus x_n^2)^{\frac{1}{2}}.$$

The equality holds if and only if one of the following conditions is satisfied:

- (i) $r_j = 0$ ($j = 1, \cdots, n$)
- (ii) $x_j = 0$ ($j = 1, \cdots, n$)
- (iii) $r_j = x_j = 0$ except for precisely one j .

Theorem. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence satisfying $0 \leq x_n < 1$ ($n = 1, 2, \cdots$). Then, the series $x_1 \oplus x_2 \oplus \cdots \oplus x_n \oplus \cdots$ converges in the open interval $(-1, 1)$ if and only if the series $\sum_{n=1}^{\infty} x_n < \infty$ in the ordinary sense.

Example. $x_n = \frac{1}{2n}$. Then $x_1 \oplus \cdots \oplus x_n = 1 - \frac{1}{n+1}$.

Theorem.(Cauchy, Bunyakovsky 1859, Schwarz 1885) Let $\mathbb{V}$ be an inner product space. Then

$$|\langle u, v \rangle| \leq \|u\| \|v\| \quad (u, v \in \mathbb{V}).$$

The following theorem is a CBS type inequality related to the Möbius addition in complex inner product spaces. One can get the classical CBS inequality by putting $v = 0$.

Theorem. Let $\mathbb{V}$ be a complex inner product space and let $w \in \mathbb{V}$ be a fixed element with $\|w\| \leq 1$. For any $u, v \in \mathbb{V}$ and for any $s > \max\{\|u\|, \|v\|\}$, the following inequality holds

$$\left| \frac{\langle u, w \rangle - \langle v, w \rangle}{1 - \frac{1}{s^2} \overline{\langle u, w \rangle} \langle v, w \rangle} \right| \leq \sqrt{\frac{\|u\|^2 - 2\operatorname{Re}\langle u, v \rangle + \|v\|^2}{1 - \frac{2}{s^2} \operatorname{Re}\langle u, v \rangle + \frac{1}{s^4} \|u\|^2 \|v\|^2}}.$$

The equality holds if and only if one of the following conditions is satisfied:

- (i) $u = v$
- (ii) $\|w\| = 1$ and $u = \lambda w, v = \mu w$ for some complex numbers λ, μ .

We state the following theorem for the real inner product spaces, showing relation between the Möbius addition (subtraction) and inner product.

Theorem. Let $\mathbb{V}$ be a real inner product space and let $w \in \mathbb{V}$ be a fixed element with $\|w\| \leq 1$. For any $u, v \in \mathbb{V}$ and for any $s > \max\{\|u\|, \|v\|\}$, the following inequality holds

$$|\langle u, w \rangle \ominus_s \langle v, w \rangle| \leq \|u \ominus_s v\|.$$

That is,

$$\left| \frac{\langle u, w \rangle - \langle v, w \rangle}{1 - \frac{1}{s^2} \langle u, w \rangle \langle v, w \rangle} \right| \leq \sqrt{\frac{\|u\|^2 - 2\langle u, v \rangle + \|v\|^2}{1 - \frac{2}{s^2} \langle u, v \rangle + \frac{1}{s^4} \|u\|^2 \|v\|^2}}.$$

The equality holds if and only if one of the following conditions is satisfied:

- (i) $u = v$
- (ii) $u = \lambda w$ and $v = \mu w$ for some real numbers λ, μ .

Remark. If $\|w\| \leq 1$, then the CBS inequality trivially shows that

$$|\langle u, w \rangle - \langle v, w \rangle| = |\langle u - v, w \rangle| \leq \|u - v\|. \quad (2)$$

On the contrary, note that generally

$$\langle u, w \rangle \ominus_s \langle v, w \rangle \neq \langle u \ominus_s v, w \rangle.$$

So we cannot prove Theorem as (2).

References

- [A] Toshikazu Abe, On gyrolinear spaces (Oral presentation), the 90th Yonezawa Mathematics Seminar, July 2nd, 2016.
- [AW] Toshikazu Abe and Keiichi Watanabe, Finitely generated gyrovectors subspaces and orthogonal gyrodecomposition in the Möbius gyrovectors space, J. Math. Anal. Appl. **449** (2017), no. 1, 77–90. <http://dx.doi.org/10.1016/j.jmaa.2016.11.039>
- [F] Péter E. Frenkel, On endomorphisms of the Einstein gyrogroup in arbitrary dimension, J. Math. Phys. **57** (2016), no. 3, 032301, 3 pp.
- [MV] Lajos Molnár and Dániel Virosztek, On algebraic endomorphisms of the Einstein gyrogroup, J. Math. Phys. **56** (2015), no. 8, 082302, 5 pp.

- [U] Abraham Albert Ungar, *Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2008.
- [W1] Keiichi Watanabe, A confirmation by hand calculation that the Möbius ball is a gyrovector space. *Nihonkai Math. J.* **27**(2016), 99–115.
- [W2] Keiichi Watanabe, Orthogonal Gyroexpansion in Möbius Gyrovector Spaces, *Journal of Function Spaces*, vol. 2017, Article ID 1518254, 13 pages (2017). doi:10.1155/2017/1518254.
- [W3] Keiichi Watanabe, A Cauchy type inequality for Möbius operations, *J. Inequal. Appl.* (2018) 2018:97. 9 pages. doi:10.1186/s13660-018-1690-2.
- [W4] Keiichi Watanabe, A Cauchy-Bunyakovsky-Schwarz type inequality related to the Möbius addition, preprint.
- [W5] Keiichi Watanabe, On mappings between Möbius gyrovector spaces corresponding to Hilbert space operators, preprint.