

Dynamical representations of substituted Sturmian sequences

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1 Introduction

We announce some theorems about Sturmian words in this report. The proofs and details will be published elsewhere. We need some notations. Let L be an alphabet, i.e., a non-empty finite set of letters. Now, we set $L = \{0, 1\}$. Let $W = \bigcup_{n=0}^{\infty} \{0, 1\}^n$, $W^* = \bigcup_{n=0}^{\infty} \{0, 1\}^n \cup \{0, 1\}^{\mathbb{N}}$. For $x, y \in [0, 1]$ we define $G(x, y), \hat{G}(x, y) \in W^*$ by

$$\begin{aligned} G(x, y) &= G_0(x, y)G_1(x, y) \dots, \\ \hat{G}(x, y) &= \hat{G}_0(x, y)\hat{G}_1(x, y) \dots, \end{aligned}$$

where $G_j(x, y) = [(j+1)x + y] - [jx + y]$, $\hat{G}_j(x, y) = [(j+1)x + y] - [jx + y]$ for each integer j and $[u]$ is an integral part of u and $[u] = -[-u]$ for each $u \in \mathbb{R}$.

Examples

$$x = \frac{1}{3}$$

$$G(x, 0) = 001001 \dots = (001)_{\infty}$$

$$x = \sqrt{2} - 1, y = \frac{1}{2}$$

$$G(x, y) = 0101001010 \dots$$

For $w \in L^{\mathbb{N}}$, we define $Sub(w)$ by

$$Sub(w) = \{u \in W \mid u \text{ is a subword of } w\}.$$

A Sturmian word is defined to be a word $w \in L^{\mathbb{N}}$ satisfying

$$||A|_1 - |B|_1| \leq 1$$

for any $A, B \in Sub(w)$ with $|A| = |B|$, where for $u \in W$ $|u|$ is a length of u and $|u|_1$ is a number of the occurrences of the letter 1 in u . In this lecture we consider only non periodic Sturmian word.

Theorem. (Morse and Hedlund [2]; Coven and Hedlund [3]) *w is Sturmian if and only if w is equal to $G(x, y)$ or $\hat{G}(x, y)$ for some $x, y \in [0, 1]$.*

A transformation f on W^* is called substitution on W^* , if f satisfies following conditions:

1. $f(0), f(1) \in W$,
2. for any $a \in W$ and $b \in W^*$, $f(ab) = f(a)f(b)$.

Example

Let f be a substitution on W^* defined by

$$f : \begin{cases} 0 \rightarrow 01 \\ 1 \rightarrow 010 \end{cases}$$

Then,

$$f(00101) = f(0)f(0)f(1)f(0)f(1) = 010101001010.$$

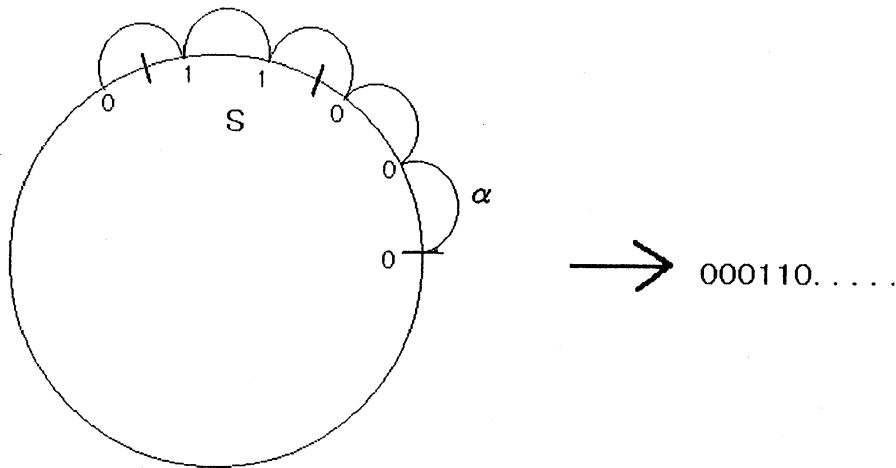
Let α be a real number. Define right infinite word $\chi(S, \alpha) \in W^*$ for a set S in interval $[0, 1]$ and α by

$$\chi(S, \alpha) = \lambda(S, \alpha; 0)\lambda(S, \alpha; 1) \cdots,$$

where

$$\lambda(S, \alpha; n) = \begin{cases} 1 & \text{if } \langle n\alpha \rangle \in S, \\ 0 & \text{if } \langle n\alpha \rangle \notin S, \end{cases}$$

where $\langle x \rangle$ is a fractional part of x .



Example of $\chi(S, \alpha)$

We define a mod 1 semiclosed interval $[x, y)^\sim$ for $0 \leq x, y \leq 1$ by

$$[x, y)^\sim = \begin{cases} [x, y) & \text{if } 0 \leq x \leq y, \\ [0, y) \cup [x, 1) & \text{if } 0 \leq y < x. \end{cases}$$

We can define a mod 1 semiclosed interval $(x, y]^\sim$ in the same manner as above.

Our main result is as follows.

Theorem 1. Let S be a Sturmian sequence. Let F be a substitution with $\text{GCD}(|F(0)|, |F(1)|) = 1$. Then, there exist $x, y \in \mathbb{E}$ and integers $m_1, \dots, m_k$ and $n_1, \dots, n_k$ such that x is irrational and $0 < x < 1$ and

$$\chi(I, x) = F(S),$$

where

$$I = \bigcup_{i=1}^k [\langle m_i x - y \rangle, \langle n_i x - y \rangle)^\sim,$$

or

$$I = \bigcup_{i=1}^k (\langle m_i x - y \rangle, \langle n_i x - y \rangle]^\sim.$$

The converse is also true.

2 An algorithm on inhomogeneous Diophantine approximation

We introduce the following algorithm on inhomogeneous Diophantine approximation to prove main Theorem. Let us define functions t_0, t_1, t_2 on $\mathbb{R}^2$ by

$$\begin{aligned} t_0(x, y) &= \left(\frac{x}{1+x}, \frac{y}{1+x} \right), \\ t_1(x, y) &= \left(\frac{1}{2-x}, \frac{y}{2-x} \right), \\ t_2(x, y) &= (1-x, 1-y). \end{aligned}$$

Let us a domain X by

$$X = \{(x, y) | 0 \leq x, y \leq 1 \text{ and } y \neq mx + n \text{ for any integers } m, n\}.$$

We define domains S_i^0 ($i = 0, \dots, 5$) by

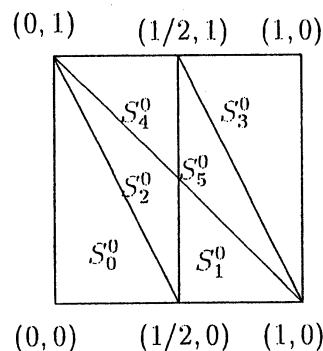


Figure of X

We define transformation T_0 on X as follows:

$$T_0(x, y) = \begin{cases} t_0^{-1}(x, y) & \text{if } (x, y) \in S_0^0, \\ t_1^{-1}(x, y) & \text{if } (x, y) \in S_1^0, \\ t_2^{-1} \circ t_0^{-1}(x, y) & \text{if } (x, y) \in S_2^0, \\ t_2^{-1} \circ t_0^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_3^0, \\ t_2^{-1} \circ t_1^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_4^0, \\ t_0^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_5^0. \end{cases}$$

We define domains S_i^1 ($i = 0, \dots, 5$) as follows:

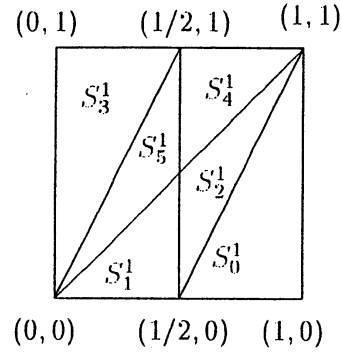
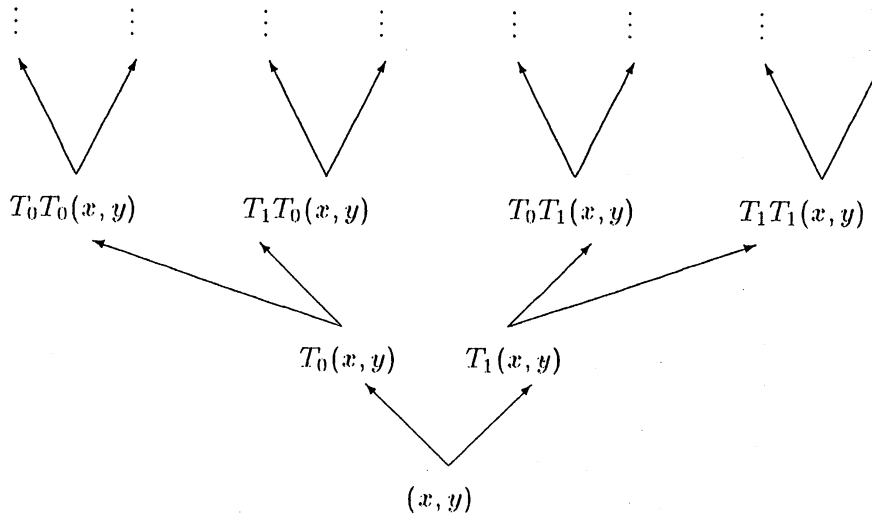


Figure of X

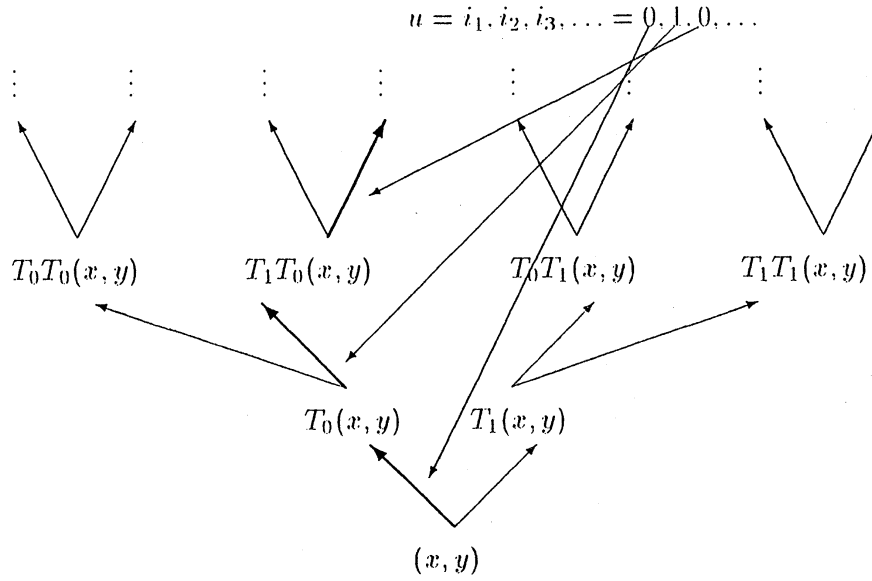
We define transformation T_1 on X as follows:

$$T_1(x, y) = \begin{cases} t_1^{-1}(x, y) & \text{if } (x, y) \in S_0^1, \\ t_0^{-1}(x, y) & \text{if } (x, y) \in S_1^1, \\ t_2^{-1} \circ t_1^{-1}(x, y) & \text{if } (x, y) \in S_2^1, \\ t_2^{-1} \circ t_1^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_3^1, \\ t_2^{-1} \circ t_0^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_4^1, \\ t_1^{-1} \circ t_2^{-1}(x, y) & \text{if } (x, y) \in S_5^1. \end{cases}$$

For $(x, y) \in X$ we consider the following binary tree:



We associate $u = \{i_1, i_2, \dots\} \in \{0, 1\}^{\mathbb{N}}$ with a path in the tree like the following example:

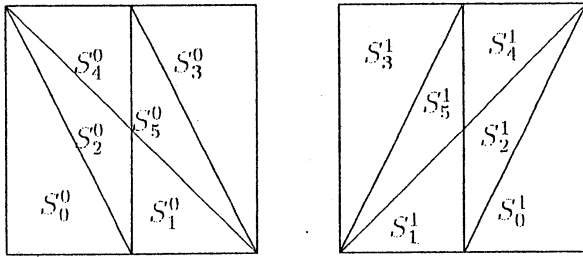


For $u = \{i_1, i_2, \dots\} \in \{0, 1\}^{\mathbb{N}}$ and a positive integer n , we define $g(u, n, (x, y)) \in X$ by

$$g(u, n, (x, y)) = T_{i_n} \cdots T_{i_1}(x, y).$$

We define a sequence $S(u, (x, y)) = \{j_n\}_{n=1}^{\infty} \in \{0, 1, 2, 3, 4, 5\}^{\mathbb{N}}$ which is called the name of (x, y) related to $u \in \{0, 1\}^{\mathbb{N}}$ as follows: for $n = 1, 2, \dots$

$$g(u, n-1, (x, y)) \in S_{j_n}^{i_n}.$$



Figures of S_j^i ($0 \leq j \leq 5$, $i \in \{0, 1\}$)

$u = \{i_1, i_2, \dots\} \in \{0, 1\}^{\mathbb{N}}$ is called good related to (x, y) if there exists infinitely many positive integers k such that $i_k = i_{k+1}$ and j_k and j_{k+1} satisfy one of following two conditons (1) and (2):

(1) $j_k = 1$ and $j_{k+1} \in \{0, 2\}$,

(2) $j_k = 4$ and $j_{k+1} \in \{3, 5\}$.

where $\{j_1, j_2, \dots\}$ is the name of (x, y) related to u .

We define substitutions s_0, s_1, e on L by

$$s_0 : \begin{cases} 0 \rightarrow 0 \\ 1 \rightarrow 01 \end{cases}, \quad s_1 : \begin{cases} 0 \rightarrow 01 \\ 1 \rightarrow 1 \end{cases}, \quad e : \begin{cases} 0 \rightarrow 1 \\ 1 \rightarrow 0 \end{cases}.$$

For $(i, k) \in \{0, 1\} \times \{0, 1, 2, 3, 4, 5\}$, we define substitutions $\phi(i, k)$ as follows:

$$\phi(i, j) = \begin{cases} s_0 & \text{if } (i, j) = (0, 0), \\ s_1 & \text{if } (i, j) = (0, 1), \\ s_0 e & \text{if } (i, j) = (0, 2), \\ e s_0 e & \text{if } (i, j) = (0, 3), \\ e s_1 e & \text{if } (i, j) = (0, 4), \\ e s_0 & \text{if } (i, j) = (0, 5), \\ s_1 & \text{if } (i, j) = (1, 0), \\ s_0 & \text{if } (i, j) = (1, 1), \\ s_1 e & \text{if } (i, j) = (1, 2), \\ e s_1 e & \text{if } (i, j) = (1, 3), \\ e s_0 e & \text{if } (i, j) = (1, 4), \\ e s_1 & \text{if } (i, j) = (1, 5). \end{cases}$$

By the theory [1] we have the following important Lemma:

Lemma

$$\phi(i_1, j_1) \cdots \phi(i_n, j_n) G(g(u, n, (x, y))) = G(x, y).$$

For substitutions f and g on W^* we say that f is equivalent to g , if for any $w \in W$ $|f(w)| = |g(w)|$.

We have the following Theorem 2.

Theorem 2 *Let $(x, y) \in X$. Let $u = \{i_1, i_2, \dots\} \in \{0, 1\}^{\mathbb{N}}$ be a good path in the previous tree related to (x, y) . Let I be a finite union of intervals $[\langle m_1 \alpha - \beta \rangle, \langle m_2 \alpha - \beta \rangle) \sim (m_1, m_2 \in \mathbb{Z})$. Then, there exist a integer $k \geq 0$ and a substitution ψ on W^* which is equivalent to $\phi(i_1, j_1) \cdots \phi(i_k, j_k)$ such that*

$$\chi(I, x) = \psi(G(g(u, k, (x, y))))),$$

where $\{j_1, j_2, \dots\}$ is the name of (x, y) related to u . The converse also holds.

From Theorem 2 and considering homogeneous cases ($y = mx + n$), we get Theorem 1.

References

- [1] S.Ito and S.Yasutomi, On continued fraction, substitution and characteristic sequences $[nx + y] - [(n - 1)x + y]$, Japan. J. Math.16(1990),287-306.
- [2] M.Morse and G.A.Hedlund, Symbolic dynamics II. Sturmian trajectories. Amer.J.Math 62(1940), 1-42
- [3] E.M.Coven and G.A.Hedlund, Sequences with minimal block growth. Math. systems theory 7(1973),138-153

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