

THE SCHWARZIAN DERIVATIVE AND STARLIKENESS

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ABSTRACT. In this paper, we obtain sufficient conditions for starlikeness by applying the Schwarzian derivative.

1. Introduction

Let $\mathcal{A}$ be the class of the functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$.

2. Preliminaries

In [1, p.304], Miller and Mocanu obtained the following result:

Theorem A. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$\operatorname{Re} \left[\frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} + z^2\{f, z\} \right) \right] > 0 \quad \text{in } \mathcal{U}$$

or

$$\operatorname{Re} \left[\frac{zf'(z)}{f(z)} (1 + z^2\{f, z\}) \right] > 0 \quad \text{in } \mathcal{U}$$

where $\{f, z\}$ is the Schwarzian derivative

$$\{f, z\} = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

Then it follows that

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad \text{in } \mathcal{U}$$

or $f(z)$ is starlike in $\mathcal{U}$.

We will improve Theorem A.

3. Main result

Theorem 1. *Let $f(z) \in \mathcal{A}$ and suppose that*

$$(1) \quad \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} + z^2\{f, z\} \right) \right] \geq -\frac{1}{2} \quad \text{in } \mathcal{U}.$$

Then $f(z)$ is starlike in $\mathcal{U}$.

Proof. Let us put

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1.$$

Then it follows that

$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)},$$

and

$$z^2\{f, z\} = \frac{zp'(z)}{p(z)} + \frac{z^2p''(z)}{p(z)} - \frac{3}{2} \left(\frac{zp'(z)}{p(z)} \right)^2 + \frac{1}{2}(1 - p(z)^2).$$

Applying the same method as the proof of [2, Lemma 1], we have

$$f'(z) \neq 0 \quad \text{in } \mathcal{U},$$

because if $f'(z)$ has zero in $\mathcal{U}$, then it contradicts (1). Therefore we have

$$p(z) \neq 0 \quad \text{in } \mathcal{U}.$$

On the other hand, if there exists a point $z_0 \in \mathcal{U}$ such that

$$\operatorname{Re} p(z) > 0 \quad \text{for } |z| < |z_0|$$

and

$$\operatorname{Re} p(z_0) = 0 \quad (p(z_0) \neq 0).$$

Then, from [3], we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where

$$k \geq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } p(z_0) = ia, \quad a > 0$$

and

$$k \leq \frac{1}{2} \left(a + \frac{1}{a} \right) \quad \text{when } p(z_0) = ia, \quad a < 0.$$

This shows that

$$p'(z_0) \neq 0.$$

Therefore we have

$$\frac{z_0^2 p''(z_0)}{p(z_0)} = \frac{z_0 p''(z_0)}{p'(z_0)} \frac{z_0 p'(z_0)}{p(z_0)} = ik \left(1 + \frac{z^2 p''(z_0)}{p'(z_0)} - 1 \right)$$

and it follows

$$\begin{aligned} (2) \quad & \operatorname{Re} \left[\frac{z_0 f'(z_0)}{f(z_0)} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} + z_0^2 \{f, z_0\} \right) \right] \\ &= \operatorname{Re} \left[ia \left(ia + ik + ik + ik \left(1 + \frac{z_0 p''(z_0)}{p'(z_0)} - 1 \right) + \frac{1}{2} (3k^2 + a^2 + 1) \right) \right] \\ &= -a^2 - ak - ak \operatorname{Re} \left(1 + \frac{z_0 p''(z_0)}{p'(z_0)} \right) \end{aligned}$$

From [1, Theorem 4, (ii)] or [4, p.3], we have

$$(3) \quad 1 + \operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} \geq 0.$$

On the other hand, we have

$$(4) \quad -ak \leq -\frac{1}{2}(1 + a^2) < -\frac{1}{2}$$

where $p(z_0) = ia$ and

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik.$$

From (2), (3) and (4), we have

$$\operatorname{Re} \left[\frac{z_0 f'(z_0)}{f(z_0)} \left(1 + \frac{z_0 f''(z_0)}{f'(z_0)} + z_0^2 \{f, z_0\} \right) \right] < -\frac{1}{2}.$$

This contradicts (1). Therefore we have

$$\operatorname{Re} p(z) = \operatorname{Re} \frac{z f'(z)}{f(z)} > 0 \quad \text{in} \quad \mathcal{U}.$$

This completes the proof and improves Theorem A. Applying the same method as the proof of Theorem 1, we have the following theorem.

Theorem 2. *Let $f(z) \in A$ and suppose that*

$$\operatorname{Re} \left[\frac{z f'(z)}{f(z)} z^2 \{f, z\} \right] \geq 0 \quad \text{in} \quad \mathcal{U}.$$

Then $f(z)$ is starlike in $\mathcal{U}$.

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