

Non-relativistic Limit of a Dirac particle Interacting with the Quantum Radiation Field

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Abstract

The non-relativistic (scaling) limit of a Hamiltonian of a Dirac particle interacting with the quantum radiation field yields a self-adjoint extension of the Pauli-Fierz Hamiltonian with spin $1/2$ in non-relativistic quantum electrodynamics.

Keywords: quantum electrodynamics, Dirac operator, Dirac-Maxwell operator, Pauli-Fierz Hamiltonian, non-relativistic limit, scaling limit, Fock space, strongly anticommuting self-adjoint operators

1 Introduction

A Hamiltonian H of a Dirac particle — a relativistic charged particle with spin $1/2$ — interacting with the quantum radiation field is called a *Dirac-Maxwell operator*. In this note we report a result on the non-relativistic limit of H .

The Dirac-Maxwell operator H is of the form $H = H_D + H_{\text{rad}} + H_I$, where H_D is a Dirac operator describing the Dirac particle system only, H_{rad} is the free Hamiltonian of the quantum radiation field (a quantum version of the Maxwell Hamiltonian in the Coulomb gauge) and H_I is the interaction term between the Dirac particle and the quantum radiation field. As for the Dirac operator H_D , the non-relativistic limit has already been investigated and well understood ([10, Chapter 6] and references therein). We extend the methods used in the case of the Dirac operator H_D to the case of H . This can be done in an abstract framework with further developments of the theory of scaling limits on strongly anticommuting self-adjoint operators [1]. The main result we report in this note is that the non-relativistic limit of H yields a self-adjoint extension of the Pauli-Fierz Hamiltonian with spin $1/2$ in non-relativistic quantum electrodynamics.

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2 The Dirac-Maxwell Operator and The Pauli-Fierz Hamiltonian

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$, and its adjoint by T^* (provided that T is densely defined). For two objects $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ such that products $a_j b_j$ ($j = 1, 2, 3$) and their sum can be defined, we set $\mathbf{a} \cdot \mathbf{b} := \sum_{j=1}^3 a_j b_j$.

We use the physical unit system in which c (the speed of light) = 1 and $\hbar = 1$ ($\hbar := h/(2\pi)$; h is the Planck constant).

2.1 The Dirac operator

Let D_j ($j = 1, 2, 3$) be the generalized partial differential operator in the variable x_j , the j -th component of $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}^3$, and $\nabla := (D_1, D_2, D_3)$.

We denote the mass and the charge of the Dirac particle by $m > 0$ and $q \in \mathbf{R} \setminus \{0\}$ respectively. We consider the situation where the Dirac particle is in a potential V which is a *Hermitian-matrix-valued Borel measurable function* on $\mathbf{R}^3$. Then the Hamiltonian of the Dirac particle is given by the Dirac operator

$$H_D := \boldsymbol{\alpha} \cdot (-i\nabla) + m\beta + V \quad (2.1)$$

acting in the Hilbert space

$$\mathcal{H}_D := \oplus^4 L^2(\mathbf{R}^3) \quad (2.2)$$

with domain $D(H_D) := [\oplus^4 H^1(\mathbf{R}^3)] \cap D(V)$ ($H^1(\mathbf{R}^3)$ is the Sobolev space of order 1), where α_j ($j = 1, 2, 3$) and β are 4×4 Hermitian matrices satisfying the anticommutation relations

$$\{\alpha_j, \alpha_k\} = 2\delta_{jk}, \quad j, k = 1, 2, 3, \quad (2.3)$$

$$\{\alpha_j, \beta\} = 0, \quad \beta^2 = 1, \quad j = 1, 2, 3, \quad (2.4)$$

$\{A, B\} := AB + BA$ and δ_{jk} is the Kronecker delta. We assume the following:

Hypothesis (A)

Each matrix element of V is almost everywhere (a.e.) finite with respect to the three-dimensional Lebesgue measure $d\mathbf{x}$ and the subspace $\cap_{j=1}^3 [D(D_j) \cap D(V)]$ is dense in $\mathcal{H}_D$.

Under this hypothesis, H_D is a symmetric operator. For detailed analyses of the Dirac operator, see, e.g., [10].

2.2 The quantum radiation field

The Hilbert space of one-photon states in momentum representation is given by

$$\mathcal{H}_{\text{ph}} := L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3), \quad (2.5)$$

where $\mathbf{R}^3 := \{\mathbf{k} = (k_1, k_2, k_3) | k_j \in \mathbf{R}, j = 1, 2, 3\}$ physically means the momentum space of photons. Then a Hilbert space for the quantum radiation field in the Coulomb gauge is given by

$$\mathcal{F}_{\text{rad}} := \bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathcal{H}_{\text{ph}}, \quad (2.6)$$

the Boson Fock space over $\mathcal{H}_{\text{ph}}$, where $\bigotimes_s^n \mathcal{H}_{\text{ph}}$ denotes the n -fold symmetric tensor product of $\mathcal{H}_{\text{ph}}$ and $\bigotimes_s^0 \mathcal{H}_{\text{ph}} := \mathbf{C}$. For basic facts on the theory of the Boson Fock space, we refer the reader to [8, §X.7].

We denote by $a(F)$ ($F \in \mathcal{H}_{\text{ph}}$) the annihilation operator with test vector F on $\mathcal{F}_{\text{rad}}$; its adjoint is given by

$$(a(F)^* \Psi)^{(n)} = \sqrt{n} S_n(F \otimes \Psi^{(n-1)}), \quad n \geq 0, \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in D(a(F)^*),$$

where S_n is the symmetrization operator on $\bigotimes^n \mathcal{H}_{\text{ph}}$ and $\Psi^{-1} := 0$.

For each $f \in L^2(\mathbf{R}^3)$, we define

$$a^{(1)}(f) := a(f, 0), \quad a^{(2)}(f) := a(0, f). \quad (2.7)$$

The mapping $f \rightarrow a^{(r)}(f^*)$ restricted to $\mathcal{S}(\mathbf{R}^3)$ (the Schwartz space of rapidly decreasing C^∞ -functions on $\mathbf{R}^3$) defines an operator-valued distribution (f^* denotes the complex conjugate of f). We denote its symbolical kernel by $a^{(r)}(\mathbf{k})$: $a^{(r)}(f) = \int a^{(r)}(\mathbf{k}) f(\mathbf{k})^* d\mathbf{k}$.

We take a nonnegative Borel measurable function ω on $\mathbf{R}^3$ to denote the one free photon energy. We assume that, for a.e. $\mathbf{k} \in \mathbf{R}^3$ with respect to the Lebesgue measure on $\mathbf{R}^3$, $0 < \omega(\mathbf{k}) < \infty$. Then the function ω defines uniquely a multiplication operator on $\mathcal{H}_{\text{ph}}$ which is nonnegative, self-adjoint and injective. We denote it by the same symbol ω . The free Hamiltonian of the quantum radiation field is then defined by

$$H_{\text{rad}} := d\Gamma(\omega), \quad (2.8)$$

the second quantization of ω [7, p.302, Example 2] and [8, §X.7]. The operator H_{rad} is a nonnegative self-adjoint operator. The symbolical expression of H_{rad} is $H_{\text{rad}} = \sum_{r=1}^2 \int \omega(\mathbf{k}) a^{(r)}(\mathbf{k})^* a^{(r)}(\mathbf{k}) d\mathbf{k}$.

Remark 2.1 Usually ω is taken to be of the form $\omega_{\text{phys}}(\mathbf{k}) := |\mathbf{k}|$, $\mathbf{k} \in \mathbf{R}^3$, but, in this paper, for mathematical generality, we do not restrict ourselves to this case.

There exist $\mathbf{R}^3$ -valued Borel measurable functions $\mathbf{e}^{(r)}$ ($r = 1, 2$) on $\mathbf{R}^3$ such that, for a.e. $\mathbf{k}$

$$\mathbf{e}^{(r)}(\mathbf{k}) \cdot \mathbf{e}^{(s)}(\mathbf{k}) = \delta_{rs}, \quad \mathbf{e}^{(r)}(\mathbf{k}) \cdot \mathbf{k} = 0, \quad r, s = 1, 2. \quad (2.9)$$

These vector-valued functions $\mathbf{e}^{(r)}$ are called the polarization vectors of a photon.

The time-zero quantum radiation field is given by $\mathbf{A}(\mathbf{x}) := (A_1(\mathbf{x}), A_2(\mathbf{x}), A_3(\mathbf{x}))$ with

$$A_j(\mathbf{x}) := \sum_{r=1}^2 \int d\mathbf{k} \frac{e_j^{(r)}(\mathbf{k})}{\sqrt{2(2\pi)^3\omega(\mathbf{k})}} \left\{ a^{(r)}(\mathbf{k})^* e^{-i\mathbf{k}\cdot\mathbf{x}} + a^{(r)}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \right\}, \quad j = 1, 2, 3, \quad (2.10)$$

in the sense of operator-valued distribution.

Let ρ be a real tempered distribution on $\mathbf{R}^3$ such that

$$\frac{\hat{\rho}}{\sqrt{\omega}}, \quad \frac{\hat{\rho}}{\omega} \in L^2(\mathbf{R}^3), \quad (2.11)$$

where $\hat{\rho}$ denotes the Fourier transform of ρ . The quantum radiation field

$$\mathbf{A}^\rho := (A_1^\rho, A_2^\rho, A_3^\rho) \quad (2.12)$$

with momentum cutoff $\hat{\rho}$ is defined by

$$A_j^\rho(\mathbf{x}) := \sum_{r=1}^2 \int d\mathbf{k} \frac{e_j^{(r)}(\mathbf{k})}{\sqrt{2\omega(\mathbf{k})}} \left\{ a^{(r)}(\mathbf{k})^* e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{\rho}(\mathbf{k})^* + a^{(r)}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \hat{\rho}(\mathbf{k}) \right\}. \quad (2.13)$$

Symbolically $A_j^\rho(\mathbf{x}) = \int A_j(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}) d\mathbf{y}$.

2.3 The Dirac-Maxwell operator

The Hilbert space of state vectors for the coupled system of the Dirac particle and the quantum radiation field is taken to be

$$\mathcal{F} := \mathcal{H}_D \otimes \mathcal{F}_{\text{rad}}. \quad (2.14)$$

This Hilbert space can be identified as

$$\mathcal{F} = L^2(\mathbf{R}^3; \oplus^4 \mathcal{F}_{\text{rad}}) = \int_{\mathbf{R}^3}^{\oplus} \oplus^4 \mathcal{F}_{\text{rad}} d\mathbf{x} \quad (2.15)$$

the Hilbert space of $\oplus^4 \mathcal{F}_{\text{rad}}$ -valued Lebesgue square integrable functions on $\mathbf{R}^3$ (the constant fibre direct integral with base space $(\mathbf{R}^3, d\mathbf{x})$ and fibre $\oplus^4 \mathcal{F}_{\text{rad}}$ [9, §XIII.6]). We freely use this identification. The total Hamiltonian of the coupled system — a *Dirac-Maxwell operator* — is defined by

$$H := H_D + H_{\text{rad}} - q\boldsymbol{\alpha} \cdot \mathbf{A}^\rho = \boldsymbol{\alpha} \cdot (-i\nabla - q\mathbf{A}^\rho) + m\beta + V + H_{\text{rad}}. \quad (2.16)$$

The (essential) self-adjointness of H is discussed in [2].

2.4 The Pauli-Fierz Hamiltonian with spin 1/2

A Hamiltonian which describes a quantum system of non-relativistic charged particles interacting with the quantum radiation field is called a Pauli-Fierz Hamiltonian [6]. Here

we consider a non-relativistic charged particle with mass m , charge q and spin $1/2$. Suppose that the particle is in an external electromagnetic vector potential $A^{\text{ex}} = (\mathbf{A}^{\text{ex}}, \phi)$, where $\mathbf{A}^{\text{ex}} := (A_1^{\text{ex}}, A_2^{\text{ex}}, A_3^{\text{ex}}) : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ and $\phi : \mathbf{R}^3 \rightarrow \mathbf{R}$ are Borel measurable and a.e. finite with respect to $d\mathbf{x}$. Let

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.17)$$

the Pauli spin matrices, and set

$$\boldsymbol{\sigma} := (\sigma_1, \sigma_2, \sigma_3). \quad (2.18)$$

Then the Pauli-Fierz Hamiltonian of this quantum system is defined by

$$H_{\text{PF}} := \frac{\{\boldsymbol{\sigma} \cdot (-i\nabla - q\mathbf{A}^e - q\mathbf{A}^{\text{ex}})\}^2}{2m} + \phi + H_{\text{rad}} \quad (2.19)$$

acting in the Hilbert space

$$\mathcal{F}_{\text{PF}} := L^2(\mathbf{R}^3; \mathbf{C}^2) \otimes \mathcal{F}_{\text{rad}} = L^2(\mathbf{R}^3; \oplus^2 \mathcal{F}_{\text{rad}}) = \int_{\mathbf{R}^3}^{\oplus} \oplus^2 \mathcal{F}_{\text{rad}} d\mathbf{x}. \quad (2.20)$$

3 Main Results

3.1 A Dirac operator coupled to the quantum radiation field

We use the following representation of α_j and β [10, p.3]:

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad (3.1)$$

where I_2 is the 2×2 identity matrix. Hence the eigenspaces $\mathcal{H}_D^{\pm}$ of β with eigenvalue ± 1 take the forms respectively

$$\mathcal{H}_D^+ = \left\{ \begin{pmatrix} f \\ g \\ 0 \\ 0 \end{pmatrix} \middle| f, g \in L^2(\mathbf{R}^3) \right\}, \quad \mathcal{H}_D^- = \left\{ \begin{pmatrix} 0 \\ 0 \\ f \\ g \end{pmatrix} \middle| f, g \in L^2(\mathbf{R}^3) \right\}. \quad (3.2)$$

and we have

$$\mathcal{H}_D = \mathcal{H}_D^+ \oplus \mathcal{H}_D^-. \quad (3.3)$$

Let $P_{\pm}$ be the orthogonal projections onto $\mathcal{H}_D^{\pm}$. Then we have

$$V = V_0 + V_1 \quad (3.4)$$

with

$$V_0 = P_+ V P_+ + P_- V P_-, \quad V_1 = P_+ V P_- + P_- V P_+. \quad (3.5)$$

Note that

$$[V_0, \beta] = 0, \quad \{V_1, \beta\} = 0,$$

where $[A, B] := AB - BA$. In operator-matrix form relative to the orthogonal decomposition (3.3), we have

$$V_0 = \begin{pmatrix} U_+ & 0 \\ 0 & U_- \end{pmatrix}, \quad V_1 = \begin{pmatrix} 0 & W^* \\ W & 0 \end{pmatrix}, \quad (3.6)$$

where $U_{\pm}$ are 2×2 Hermitian matrix-valued functions on $\mathbf{R}^3$ and W is a 2×2 complex matrix-valued function on $\mathbf{R}^3$.

Let

$$\mathcal{D}(V_1) := \boldsymbol{\alpha} \cdot (-i\nabla - q\mathbf{A}^e) + V_1 \quad (3.7)$$

Then, recalling that A_j^e is $H_{\text{rad}}^{1/2}$ -bounded [2], we see that $\mathcal{D}(V_1)$ is densely defined and symmetric with $D(\mathcal{D}(V_1)) \supset (\cap_{j=1}^3 [D(D_j) \cap D(V)]) \otimes_{\text{alg}} D(H_{\text{rad}}^{1/2})$, where $\otimes_{\text{alg}}$ means algebraic tensor product.

By (3.3), we have the following orthogonal decomposition of $\mathcal{F}$:

$$\mathcal{F} = \mathcal{F}_+ \oplus \mathcal{F}_-, \quad (3.8)$$

where

$$\mathcal{F}_{\pm} := \mathcal{H}_{\text{D}}^{\pm} \otimes \mathcal{F}_{\text{rad}} \cong \mathcal{F}_{\text{PF}}. \quad (3.9)$$

Relative to this orthogonal decomposition, we can write

$$\mathcal{D}(V_1) = \begin{pmatrix} 0 & D_{W^*} \\ D_W & 0 \end{pmatrix}, \quad (3.10)$$

where

$$D_W := \boldsymbol{\sigma} \cdot (-i\nabla - q\mathbf{A}^e) + W, \quad (3.11)$$

$$D_{W^*} := \boldsymbol{\sigma} \cdot (-i\nabla - q\mathbf{A}^e) + W^* \quad (3.12)$$

acting in $\mathcal{F}_{\text{PF}}$.

For a closable linear operator T on a Hilbert space, we denote its closure by $\bar{T}$ unless otherwise stated.

Note that D_W is densely defined as an operator on $\mathcal{F}_{\text{PF}}$ and $(D_W)^* \supset D_{W^*}$. Hence $(D_W)^*$ is densely defined. Thus D_W is closable. Based on this fact, we can define

$$\tilde{\mathcal{D}}(V_1) := \begin{pmatrix} 0 & (\bar{D}_W)^* \\ \bar{D}_W & 0 \end{pmatrix}. \quad (3.13)$$

Lemma 3.1 *Under Hypothesis (A), $\tilde{\mathcal{D}}(V_1)$ is a self-adjoint extension of $\mathcal{D}(V_1)$.*

3.2 A scaled Dirac-Maxwell operator

For a self-adjoint operator A , we denote the spectrum and the spectral measure of A by $\sigma(A)$ and $E_A(\cdot)$ respectively. In the case where A is bounded from below, we set

$$\mathcal{E}_0(A) := \inf \sigma(A), \quad A' := A - \mathcal{E}_0(A) \geq 0.$$

Let $\Lambda : (0, \infty) \rightarrow (0, \infty)$ be a nondecreasing function such that $\Lambda(\kappa) \rightarrow \infty$ as $\kappa \rightarrow \infty$ and A be a self-adjoint operator on a Hilbert space. Then, for each $\kappa > 0$, we define $A^{(\kappa)}$ by

$$A^{(\kappa)} := \begin{cases} E_{A'}([0, \Lambda(\kappa)])A'E_{A'}([0, \Lambda(\kappa)]) + \mathcal{E}_0(A) & \text{if } A \text{ is bounded from below} \\ & \text{and } \mathcal{E}_0(A) < 0 \\ E_{|A|}([0, \Lambda(\kappa)])AE_{|A|}([0, \Lambda(\kappa)]) & \text{if } A \text{ is nonnegative} \\ & \text{or } A \text{ is not bounded from below} \end{cases} \quad (3.14)$$

Then $A^{(\kappa)}$ is a bounded self-adjoint operator with

$$\|A^{(\kappa)}\| \leq \Lambda(\kappa). \quad (3.15)$$

Proposition 3.2 *The following hold:*

- (i) For all $\psi \in D(A)$, $s - \lim_{\kappa \rightarrow \infty} A^{(\kappa)}\psi = A\psi$, where $s - \lim$ means strong limit.
- (ii) For all $z \in \mathbb{C} \setminus \mathbb{R}$, $s - \lim_{\kappa \rightarrow \infty} (A^{(\kappa)} - z)^{-1} = (A - z)^{-1}$.

With this preliminary, we define for $\kappa > 0$ a scaled Dirac-Maxwell operator

$$H(\kappa) := \kappa \tilde{\mathcal{D}}(V_1) + \kappa^2 m \beta - \kappa^2 m + V_{0,\kappa} + H_{\text{rad}}^{(\kappa)}, \quad (3.16)$$

where

$$V_{0,\kappa} := \begin{pmatrix} U_+^{(\kappa)} & 0 \\ 0 & U_-^{(\kappa)} \end{pmatrix}. \quad (3.17)$$

Some remarks may be in order on this definition. The parameter κ in $H(\kappa)$ means the speed of light *concerning the Dirac particle only*. The speed of light related to the external potential $V = V_0 + V_1$ and the quantum radiation field $\mathbf{A}^e$ is absorbed in them respectively. The third term $-\kappa^2 m$ on the right hand side of (3.16) is a subtraction of the rest energy of the Dirac particle. Hence taking the scaling limit $\kappa \rightarrow \infty$ in $H(\kappa)$ in a suitable sense corresponds in fact to a *partial* non-relativistic limit of the quantum system under consideration.

If one considers the non-relativistic limit in a way similar to the usual Dirac operator H_D , then one may define

$$\widehat{H}(\kappa) := \kappa \tilde{\mathcal{D}}(V_1) + \kappa^2 m \beta - \kappa^2 m + V_0 + H_{\text{rad}} \quad (3.18)$$

as a scaled Dirac-Maxwell operator, where no cutoffs on V_0 and H_{rad} are made. In this form, however, we find that, besides the (essential) self-adjointness problem of $\widehat{H}(\kappa)$, the

methods used in the usual Dirac type operators ([10, Chapter 6] or those in [1]) seem not to work. This is because of the existence of the operator H_{rad} in $\widehat{H}(\kappa)$ which is singular as a perturbation of $H_0(\kappa) := \kappa \tilde{\mathcal{D}}(V_1) + \kappa^2 m \beta - \kappa^2 m + V_0$ (if one would try to apply the methods on scaling limits discussed in the cited literatures, then one would have to treat H_{rad} as a perturbation of $H_0(\kappa)$). To avoid this difficulty, we replace H_{rad} in $\widehat{H}(\kappa)$ by a bounded self-adjoint operator which is obtained by cutting off H_{rad} . This is one of the basic ideas of the present paper. We apply the same idea to V_0 which also may be singular as a perturbation of $\kappa \tilde{\mathcal{D}}(V_1) + \kappa^2 m \beta - \kappa^2 m$. In this way we arrive at Definition (3.16) of a scaled Dirac-Maxwell operator.

Lemma 3.3 *Under Hypothesis (A), $H(\kappa)$ is self-adjoint with $D(H(\kappa)) = D(\tilde{\mathcal{D}}(V_1))$.*

3.3 Self-adjoint extension of the Pauli-Fierz Hamiltonian

Essential self-adjointness of the the Pauli-Fierz Hamiltonian H_{PF} given by (2.19) and its generalizations is discussed in [4, 5]. These papers show that, under additional conditions on $\hat{\rho}, \omega, \mathbf{A}^{\text{ex}}$ and ϕ , the Pauli-Fierz Hamiltonians are essentially self-adjoint. In this note we define a self-adjoint extension of H_{PF} , which may not be known before.

We define

$$H_{\text{PF}}(\kappa; W, U_+) := \frac{(\overline{D}_W)^* \overline{D}_W}{2m} + U_+^{(\kappa)} + H_{\text{rad}}^{(\kappa)}, \quad \kappa > 0 \quad (3.19)$$

acting in $\mathcal{F}_{\text{PF}}$.

Lemma 3.4 *Under Hypotheses (A), $H_{\text{PF}}(\kappa; W, U_+)$ is self-adjoint and bounded from below.*

A generalization of the Pauli-Fierz Hamiltonian H_{PF} is defined by

$$H_{\text{PF}}(W, U_+) := \frac{D_W \bullet D_W}{2m} + U_+ + H_{\text{rad}} \quad (3.20)$$

acting in $\mathcal{F}_{\text{PF}}$.

We formulate additional conditions:

Hypothesis (B)

The function U_+ is bounded from below. In this case we set

$$u_0 := \mathcal{E}_0(U_+).$$

Remark 3.1 Under Hypothesis (A), $D(H_{\text{PF}}(W, U_+))$ is not necessarily dense in $\mathcal{F}_{\text{PF}}$, but, $D(\overline{D}_W) \cap D(U_+) \cap D(H_{\text{rad}})$ is dense in $\mathcal{F}_{\text{PF}}$. Hence $D(\overline{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{\text{rad}}^{1/2})$

also is dense in $\mathcal{F}_{\text{PF}}$. Therefore we can define a densely defined symmetric form $\mathbf{s}_{\text{PF}}$ as follows:

$$D(\mathbf{s}_{\text{PF}}) := D(\bar{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{\text{rad}}^{1/2}) \text{ (form domain)}, \quad (3.21)$$

$$\mathbf{s}_{\text{PF}}(\Psi, \Phi) := \frac{1}{2m}(\bar{D}_W \Psi, \bar{D}_W \Phi) + (\Psi, U_+ \Phi) + (H_{\text{rad}}^{1/2} \Psi, H_{\text{rad}}^{1/2} \Phi), \quad (3.22)$$

$$\Psi, \Phi \in D(\mathbf{s}_{\text{PF}}). \quad (3.23)$$

Assume Hypothesis (B) in addition to Hypothesis (A). Then it is easy to see that $\mathbf{s}_{\text{PF}}$ is closed. Let $H_{\text{PF}}^{(f)}$ be the self-adjoint operator associated with $\mathbf{s}_{\text{PF}}$. Then $H_{\text{PF}}^{(f)} \geq u_0$ and $H_{\text{PF}}^{(f)}$ is a self-adjoint extension of $H_{\text{PF}}(W, U_+)$.

Theorem 3.5 *Under Hypotheses (A) and (B), there exists a self-adjoint extension of $\widetilde{H}_{\text{PF}}(W, U_+)$ of $H_{\text{PF}}(W, U_+)$ which have the following properties:*

- (i) $\widetilde{H}_{\text{PF}}(W, U_+) \geq u_0$.
- (ii) $D(|\widetilde{H}_{\text{PF}}(W, U_+)|^{1/2}) \subset D(\bar{D}_W) \cap D(|U_+|^{1/2}) \cap D(H_{\text{rad}}^{1/2})$
- (iii) For all $z \in (\mathbf{C} \setminus \mathbf{R}) \cup \{\xi \in \mathbf{R} | \xi < u_0\}$,

$$s - \lim_{\kappa \rightarrow \infty} (H_{\text{PF}}(\kappa; W, U_+) - z)^{-1} = (\widetilde{H}_{\text{PF}}(W, U_+) - z)^{-1},$$

where $s - \lim$ means strong limit.

- (iv) For all $\xi < u_0$ and $\Psi \in D(|\widetilde{H}_{\text{PF}}(W, U_+)|^{1/2})$,

$$s - \lim_{\kappa \rightarrow \infty} (H_{\text{PF}}(\kappa; W, U_+) - \xi)^{1/2} \Psi = (\widetilde{H}_{\text{PF}}(W, U_+) - \xi)^{1/2} \Psi.$$

Remark 3.2 As for conditions for $\hat{\rho}$ and ω for Theorem 3.5 to hold, we only need condition (2.11); no additional conditions is necessary.

Remark 3.3 In the same manner as in Theorem 3.5, we can define a self-adjoint extension of the Pauli-Fierz Hamiltonian without spin.

Remark 3.4 Under Hypotheses (A), (B) and that $D(H_{\text{PF}}(W, U_+))$ is dense, $H_{\text{PF}}(W, U_+)$ is a symmetric operator bounded from below. Hence it has the Friedrichs extension $\widehat{H}_{\text{PF}}(W, U_+)$. But it is not clear that, in the case where $H_{\text{PF}}(W, U_+)$ is not essentially self-adjoint, $\widetilde{H}_{\text{PF}}(W, U_+) = \widehat{H}_{\text{PF}}(W, U_+)$ or $\widetilde{H}_{\text{PF}}(W, U_+) = H_{\text{PF}}^{(f)}$ (Remark 3.1) or both of them do not hold.

3.4 Main theorems

We now state main results on the non-relativistic limit of $H(\kappa)$.

Theorem 3.6 *Let Hypotheses (A) and (B) be satisfied. Suppose that*

$$\lim_{\kappa \rightarrow \infty} \frac{\Lambda(\kappa)^2}{\kappa} = 0. \quad (3.24)$$

Then, all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$s - \lim_{\kappa \rightarrow \infty} (H(\kappa) - z)^{-1} = \begin{pmatrix} (\widetilde{H}_{\text{PF}}(W, U_+) - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.25)$$

In the case where U_+ is not necessarily bounded from below, we have the following.

Theorem 3.7 *Let Hypothesis (A) and (3.24) be satisfied. Suppose that $H_{\text{PF}}(W, U_+)$ is essentially self-adjoint. Then, all $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$s - \lim_{\kappa \rightarrow \infty} (H(\kappa) - z)^{-1} = \begin{pmatrix} (\overline{H_{\text{PF}}(W, U_+) - z})^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.26)$$

Remark 3.5 Under additional conditions on ϱ, ω, W and U_+ , one can prove that $H_{\text{PF}}(W, U_+)$ is essentially self-adjoint for all values of the coupling constant q [4, 5].

We now apply Theorems 3.6 and 3.7 to the case where $V = V_{\text{em}} = \phi - q\boldsymbol{\alpha} \cdot \mathbf{A}^{\text{ex}}$, i.e., the case where $W = -q\boldsymbol{\sigma} \cdot \mathbf{A}^{\text{ex}}$ and $U_{\pm} = \phi I_2$. We assume the following.

Hypothesis (C)

(C.1) The subspace $\cap_{j=1}^3 [D(D_j) \cap D(A_j^{\text{ex}}) \cap D(\phi)]$ is dense in $L^2(\mathbb{R}^3)$.

(C.2) ϕ is bounded from below. In this case we set $\phi_0 := \inf \sigma(\phi)$.

Under Hypothesis (C), we have a self-adjoint operator

$$\widetilde{H}_{\text{PF}} := \widetilde{H}_{\text{PF}}(-q\boldsymbol{\sigma} \cdot \mathbf{A}^{\text{ex}}, \phi), \quad (3.27)$$

which is a self-adjoint extension of the original Pauli-Fierz Hamiltonian H_{PF} given by (2.19).

Let

$$H_{\text{DM}}(\kappa) := \kappa \not{D}(-q\boldsymbol{\alpha} \cdot \mathbf{A}^{\text{ex}}) + \kappa^2 m \beta - \kappa^2 m + \phi^{(\kappa)} + H_{\text{rad}}^{(\kappa)}, \quad (3.28)$$

Then $H_{\text{DM}}(\kappa)$ is the Dirac-Maxwell operator $H(\kappa)$ with $V_1 = -q\boldsymbol{\alpha} \cdot \mathbf{A}^{\text{ex}}$ and $V_0 = \phi$.

Theorems 3.6 and 3.7 immediately yield the following results on the non-relativistic limit of $H_{\text{DM}}(\kappa)$.

Corollary 3.8 *Let Hypothesis (C) and (3.24) be satisfied. Then, for all $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$s - \lim_{\kappa \rightarrow \infty} (H_{\text{DM}}(\kappa) - z)^{-1} = \begin{pmatrix} (\widetilde{H}_{\text{PF}} - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.29)$$

Corollary 3.9 Assume (C.1) and (3.24). Suppose that H_{PF} is essentially self-adjoint. Then, all $z \in \mathbb{C} \setminus \mathbb{R}$,

$$s - \lim_{\kappa \rightarrow \infty} (H_{\text{DM}}(\kappa) - z)^{-1} = \begin{pmatrix} (\overline{H}_{\text{PF}} - z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \quad (3.30)$$

Thus a mathematically rigorous connection of relativistic QED to non-relativistic QED is established.

Proofs of these results are given in [3]. The method used is an extension of a theory [1] of scaling limits of strongly anticommuting self-adjoint operators.

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