

PRIME GRAPHS

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1. Prime graphs. Let G be a finite group and $\Gamma(G)$ be the prime graph of G . This is the graph such that the vertex-set $V(\Gamma(G)) = \pi(G)$, the set of prime divisors of $|G|$ and two distinct primes p and r are joined by an edge if and only if there exists an element of order pr in G . The concept of prime graph arose from cohomological questions associated with integral representation of finite groups (See Gruenberg[4],[5], Gruenberg-Roggenkamp[6],[7]). Let $n(\Gamma(G))$ be the number of connected components of $\Gamma(G)$ and $d_G(p, r)$ the length of the shortest path between p and r . If there is no path between p and r , then $d_G(p, r)$ is defined to be infinite.

Theorem 1 ([10],[13],[14]).

$$n(\Gamma(G)) = \begin{cases} 1, \\ 2, \\ 3, \\ 4, \\ 5, \\ 6 \end{cases}$$

Theorem 2 ([11]).

$$d_G(p, r) = \begin{cases} 1, \\ 2, \\ 3, \\ 4, \\ \infty \end{cases}$$

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Remark 1. Theorems 1 and 2 hold for any finite group G . The proofs depend upon the classification of finite simple groups. Theorem 1 is the solution of Gruenberg-Kegel's conjecture. We classify not only the number of connected components but also the components themselves. The significance of Theorem 1 can be found in [5],[8],[9],[12] and [15].

Remark 2. If G is solvable or simple, then $d_G(p, r) = 1, 2, 3$ or $d_G(p, r) = \infty$. For the sporadic simple group G , $d_G(p, r) = 3$ if and only if $G = F_1$ and $p = 29, r = 47$ or $G = M_{23}$ and $p = 3, r = 7$. Unfortunately we have no application of Theorem 2. We are trying to find applications of Theorem 2.

2. Related topics. Let χ be a character(*resp.* p -Brauer character) of G and L be the set of values of χ on nonidentity elements(*resp.* nonidentity p -regular elements) of G . We say that χ is sharp(*resp.* p -Brauer sharp) if $f_L(\chi(1)) = |G|$ (*resp.* $f_L(\chi(1)) = |G|_{p'}$) where $f_L(x)$ is the monic polynomial of least degree whose set of roots is L . We note that $|G|$ (*resp.* $|G|_{p'}$) always divides $f_L(\chi(1))$ by Blichfeldt's theorem(See [1]). Recently Alvis and Nozawa[1] classified the groups with sharp character χ such that χ takes an irrational value and $(\chi, 1_G) = 1$. Therefore we can assume that L is contained in $\mathbf{Z}$. Let $L = \{l_1, l_2, \dots, l_t\}$. The (p -Brauer) sharp character χ is said to be t -connected if and only if $L \subseteq \mathbf{Z} - \{\chi(1) - 1, \chi(1) + 1\}$ and $(\chi(1) - l_i, \chi(1) - l_j) = 1$ for $i \neq j$.

Theorem 3 ([3],[8]). *The following two conditions are equivalent.*

- (1) G has a 2-connected (p -Brauer) sharp character.
- (2) $\Gamma(G) - \{p\}$ is disconnected.

Remark 3. $\Gamma(G) - \{p\}$ is a subgraph of $\Gamma(G)$ such that the vertex-set is $V(\Gamma(G)) - \{p\}$. If p does not divide $|G|$, then $\Gamma(G) - \{p\} = \Gamma(G)$ and the result is for ordinary (generalized)

characters.

Remark 4. In [1] the authors assume that χ is the character of its representation. However in [3] and [8] χ may not have its representation.

Let $\mathfrak{N}(G) = \{n \in \mathbf{Z} \mid G \text{ has a conjugacy class } C \text{ with } |C| = n\}$. Thompson made the following conjecture.

Thompson's conjecture. Let G be a finite group and M a non abelian simple group. If $\mathfrak{N}(G) = \mathfrak{N}(M)$ and $Z(G) = 1$, then G is isomorphic with M .

Theorem 4 ([2]). *Thompson's conjecture holds for a finite simple group M with $n(\Gamma(M)) > 1$.*

The proof heavily depends upon the classification of the connected components of prime graphs of finite simple groups in Theorem 1.

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