

AN EXTENSION OF GAUSS' FORMULA FOR HYPERGEOMETRIC SERIES.

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1. Introduction. A system of linear ordinary differential equations with rational coefficients:

$$dx/dt=A(t)x$$

is in Fuchsian class if all the poles of $A(t)$ are regular singular points of the system. A fundamental problem for such a system is the computation of the group of the system. But the computation has scarcely been carried out in closed form except for the case of hypergeometric system

$$(1) \quad (t-B)dx/dt=Ax$$

where B is the diagonal matrix

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and A has the form

$$A = \begin{pmatrix} 1-c & 1 \\ (c-1-a)(c-1-t) & c-1-a-b \end{pmatrix}$$

A crucial role was played in the computation of the group of the system in Riemann's paper "Beitrage zur Theorie der durch die Gauss'sche Reihe $F(a,b,c,x)$ darstellbaren Functionen", by Gauss' Formula

$$(2) \quad F(a,b,c,1)=\Gamma(c)\Gamma(c-a-b)/\Gamma(c-a)\Gamma(c-b)$$

A generalization of the formula for higher dimensional systems (1) has implicitly been used for the computations of Stokes multipliers of the system

$$tdx/dt=(A+tB)x$$

in a paper by M.Kohno ([/]). The object of the paper is to present this generalization by modifying slightly the result of him.

2. Statement of the Theorem.

Let B be a diagonal matrix with constant diagonal elements $\lambda_1, \lambda_2, \dots, \lambda_n$ and let A be the matrix with (j,k) -th element $a_{j,k}$ ($j,k=1,2,\dots,n$). We consider a system of differential equations

$$(1) (t-B)dx/dt = Ax$$

The system (1) has $n+1$ regular singular points: $\lambda_1, \dots, \lambda_n, \infty$. The characteristic exponents at $t=\lambda_k$ are

$$0, 0, \dots, a_{kk}, \dots, 0$$

So, if we denote by e_k the constant n -vector with all the elements zero except the k -th which is 1, there is a solution $x_k(t)$ which behaves like $(t-\lambda_k)^{a_{kk}} e_k$.

Theorem. If on none of the numbers $a_{11}, \dots, a_{nn}; p_1, \dots, p_n$ ($\det(A-pI)=0$) is an integer, then we have

$$(3) \quad \det(x_1(t), x_2(t), \dots, x_n(t)) = (t-\lambda_1)^{a_{11}} \dots (t-\lambda_n)^{a_{nn}} \frac{\prod_{k=1}^n \Gamma(a_{kk} + 1)}{\prod_{j=1}^n \Gamma(p_j + 1)}$$

where the branches of multi-valued factors $(t-\lambda_k)^{a_{kk}}$ are suitably chosen in a simply connected domain D whose boundary is a closed Jordan curve J carrying all the singularities $\lambda_1, \dots, \lambda_n$.

Our proof is a direct continuation of the argument of Gauss' original proof of the formula (2). but it will be carried out only for the special case

$$(4) \quad |\lambda_j - \lambda_k| > |\lambda_k| > 0 \quad (j \neq k)$$

The method of analytic continuation of functions beyond the circle of convergence across an arc of finite non-zero width, used in [2], may be used to avoid the unnecessary condition (4).

3. Proof of the Theorem.

If we expand the solution $x_k(t)$ into power series in $(t-\lambda_k)$;

$$(5) \quad x_k(t) = \sum_{s=0}^{\infty} G_k(s) (t-\lambda_k)^{a_{kk}+s}$$

the coefficients $G_k(s)$ satisfy

$$(6) \quad (a_{kk}+s-A)G_k(s) = (B-\lambda_k)(a_{kk}+s+1)G_k(s+1)$$

$$(7) \quad G_k(0) = e_k$$

A simple change of (6) by the transformation

$$(8) \quad G_k(s) = H_k(s) \frac{\Gamma(a_{kk}+1)}{\Gamma(a_{kk}+s+1)}$$

takes (6) and (7) into

$$(6)^* \quad (a_{kk}+s-A)H^k(s) = (B-\lambda_k)H^k(s+1)$$

$$(7)^* \quad H^k(0) = e_k$$

Now we introduce a new parameter m which takes non-negative integral values into the system and define new vector valued functions

$$(9) \quad x_k(t, m) = \sum_{s=0}^{\infty} H^k(s) \cdot \frac{\Gamma(a_{kk}+1)}{\Gamma(a_{kk}+m+s+1)} (t-\lambda_k)^{a_{kk}+s+m}$$

and the matrix $X(t, m)$ whose k -th vertical vector is $x_k(t, m)$.

By an easy computation using the property of the Gamma-function

$$x \Gamma(x) = \Gamma(x+1)$$

we have

$$(10) \quad (t-\lambda_k) dx_k(t, m)/dt = (t-\lambda_k) x_k(t, m-1)$$

Similarly, by using (6)*, we have

$$(11) \quad (t-\lambda_k) dx_k(t, m)/dt = (A+m) x_k(t, m) + (B-\lambda_k) x_k(t, m-1)$$

Combining (10) and (11) together, we have systems of difference equations

$$(12) \quad (t-B)x_k(t, m-1) = (A+m)x_k(t, m)$$

The most important feature of the system (12) is that their coefficients are independent of suffix k . Consequently, the matrix $X(t, m)$ satisfies:

$$(13) \quad (t-B)X(t, m-1) = (A+m)X(t, m)$$

Since we are interested only in computing the determinant of $X(t, m)$, we write $\det X(t, m) = c(t, m)$ and deduce

$$(14) \quad c(t, m) = c(t, 0) (t-\lambda_1)^m \cdots (t-\lambda_n)^m \frac{\Gamma(\rho_1+1) \cdots \Gamma(\rho_n+1)}{\Gamma(m+\rho_1+1) \cdots \Gamma(m+\rho_n+1)}$$

On the other hand, for a fixed t , $c(t, m)$ is given asymptotically by

$$(15) \quad c(t, m) = (t-\lambda_1)^{m+a_{11}} \cdots (t-\lambda_n)^{m+a_{nn}} \frac{\Gamma(a_{11}+1) \cdots \Gamma(a_{nn}+1)}{\Gamma(m+a_{11}+1) \cdots \Gamma(m+a_{nn}+1)} \left\{ 1 + O\left(\frac{1}{m}\right) \right\}$$

since the expansion (9) is a convergent inverse factorial series, for t which is in the intersection of the circles of convergence of the solutions $x_k(t)$ $k=1, 2, \dots, n$. The condition (4) shows $t=0$ is such a point. And since our domain D is a simply connected domain, we have

$$(16) \quad X(t^*) = c(t^*) = \det X(t^*) = c(t) \exp \left[\int_t^{t^*} \sum_{k=1}^n a_{kk}(t-\lambda_k) dt \right]$$

that is, (3) is valid uniformly in D . We have

$$\begin{aligned} c(t, 0) &= \lim_{m \rightarrow \infty} (t-\lambda_1)^{a_{11}} \cdots (t-\lambda_n)^{a_{nn}} \frac{\prod_{k=1}^n \Gamma(a_{kk}+1)}{\prod_{j=1}^n \Gamma(\rho_j+1)} \\ &\quad \times \frac{\Gamma(m+\rho_1+1) \cdots \Gamma(m+\rho_n+1)}{\Gamma(m+a_{11}+1) \cdots \Gamma(m+a_{nn}+1)} \left\{ 1 + O\left(\frac{1}{m}\right) \right\} \\ &= \lim_{m \rightarrow \infty} (t-\lambda_1)^{a_{11}} \cdots (t-\lambda_n)^{a_{nn}} \frac{\prod_{k=1}^n \Gamma(a_{kk}+1)}{\prod_{j=1}^n \Gamma(\rho_j+1)} \cdot m^{\sum \rho_j - \sum a_{kk}} \left\{ 1 + O\left(\frac{1}{m}\right) \right\} \end{aligned}$$

which complete the proof of (3) since

$$\sum \rho_j - \sum a_{kk} = 0$$

by invariance of the trace of matrix A (Fuchs's relation).

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4. An Application of the Theorem.

Consider the case $n=2$. Let $x_1(t)^*$, $x_2(t)^*$ be solutions

$$x_1^*(t) = \sum_{s=0}^{\infty} G_1^*(s)(t-\lambda_1)^s \quad G_1^*(0)=e_2$$

$$x_2^*(t) = \sum_{s=0}^{\infty} G_2^*(s)(t-\lambda_2)^s \quad G_2^*(0)=e_1$$

We have

$$\det(x_1, x_1^*) = (t-\lambda_1)^{a_{11}} [1 + O(t-\lambda_1)]$$

$$\det(x_2, x_2^*) = (t-\lambda_2)^{a_{22}} [1 + O(t-\lambda_2)]$$

If we assume

$$x_1(t) = p x_2 + p^* x_2^*$$

The connection constants p^* can easily be computed with the help of the theorem by taking the limit $t \rightarrow \lambda_2$.

Reference

1. M. Kohno: The convergence condition of a series appearing in connection problems and the determination of Stokes multipliers, Publ. R.I.M.S. Kyoto univ., 3(1968), 337-350.
2. K. Okubo: A global representation of a fundamental set of solutions and a Stokes phenomenon from a system of linear ordinary differential equation, J. Math. Soc. Japan, 15(1963) 268-288.