

Weakly o-minimal structures

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1 Introduction

Let $(M, <)$ be a dense linear ordering without endpoints and A a subset of M . The set A is said to be *convex* if for all $a, b \in A$ and $c \in M$ with $a < c < b$ we have $c \in A$. A structure $(M, <, \dots)$ equipped with a dense linear ordering $<$ without endpoints is said to be *o-minimal* (*weakly o-minimal*) if every definable¹ subset of M is a finite union of intervals (convex sets) in $(M, <)$, respectively. A theory T is said to be *weakly o-minimal* if every model of T is weakly o-minimal.

It is well-known that the monotonicity theorem of [3] fails in a weakly o-minimal structure. However Arfiev [1] showed that the “weaker” version of the monotonicity theorem of [3] holds in any weakly o-minimal structure. In this paper we survey Arfiev’s results.

2 Preliminaries

Let M be a weakly o-minimal structure. For each $A, B \subseteq M$ we write $A < B$ if $a < b$ whenever $a \in A$ and $b \in B$. An ordered pair $\langle C, D \rangle$ of non-empty definable subsets in M is called a *definable cut* if $C < D$, $C \cup D = M$ and D has no lowest elements. The set of all definable cuts in M will be denoted by $\overline{M}$. Moreover we define a linear order on $\overline{M}$ by $\langle C_1, D_1 \rangle < \langle C_2, D_2 \rangle$ if and only if $C_1 \subsetneq C_2$. Then we may treat $(M, <)$

¹Throughout this paper ‘definable’ means ‘definable with parameters’.

as a substructure of $(\overline{M}, <)$ by identifying an element $a \in M$ with the definable cut $\langle (-\infty, a], (a, \infty) \rangle$.

Let A be a definable subset of M^n . Then a function $f : A \rightarrow \overline{M}$ is said to be *definable* if the set $\{(\overline{x}, y) \in A \times M : f(\overline{x}) > y\}$ is definable.

Remark 1 Let A be a definable subset of M^n . Suppose that f is a function from A into $\overline{M}$. Then the following conditions are equivalent:

1. f is definable;
2. there exists a formula $\varphi(\overline{x}, y)$ with parameters such that $f(\overline{a}) = \sup\varphi(\overline{a}, M)$ whenever $\overline{a} \in A$.

Definition 2 Let $f : A \rightarrow \overline{M}$ be a function, where A is a subset of M . Then f is said to be *tidy* if one of the following holds:

1. for each $a \in A$ there exists an open interval $I \subseteq A$ with $a \in I$ such that $f \upharpoonright I$ is strictly increasing, in which case f is said to be *locally increasing* on A ;
2. for each $a \in A$ there exists an open interval $I \subseteq A$ with $a \in I$ such that $f \upharpoonright I$ is strictly decreasing, in which case f is said to be *locally decreasing* on A ;
3. for each $a \in A$ there exists an open interval $I \subseteq A$ with $a \in I$ such that $f \upharpoonright I$ is constant, in which case f is said to be *locally constant* on A .

Definition 3 Let $f : A \rightarrow \overline{M}$ be a function, where A is a subset of M . Then f is said to be *have the local minimum throughout* A if for each $a \in A$ there exist $b_0, b_1 \in A$ with $b_0 < a < b_1$ such that for each $c \in (b_0, b_1) \setminus \{a\}$ we have $f(a) < f(c)$. Similarly, we define that f *has the local maximum throughout* A .

Definition 4 A weakly o-minimal structure M is said to be *have monotonicity* if for each definable function $f : A \subseteq M \rightarrow \overline{M}$ there exists $n \in \mathbb{N}$ and a partition of A into definable sets $X, I_0, \dots, I_n$ such that X is finite, $I_0, \dots, I_n$ are open convex sets and for each $i \leq n$ the function $f \upharpoonright I_i$ is tidy.

Arfiev showed the following.

Theorem 5 ([1]) *Every weakly o-minimal structure M has monotonicity.*

In the next section we give the proof for Theorem 5.

3 Proof of Theorem 5

Throughout this section we assume that $(M, <, \dots)$ is a weakly o-minimal structure and f is a definable function from definable subset A of M into $\overline{M}$. We now define the following formulas:

$$\begin{aligned}\varphi_0(x) &:\equiv \exists x_1 > x (\forall y \in (x, x_1) f(y) < f(x)); \\ \varphi_1(x) &:\equiv \exists x_1 > x (\forall y \in (x, x_1) f(y) = f(x)); \\ \varphi_2(x) &:\equiv \exists x_1 > x (\forall y \in (x, x_1) f(y) > f(x)); \\ \psi_0(x) &:\equiv \exists x_0 < x (\forall y \in (x_0, x) f(y) < f(x)); \\ \psi_1(x) &:\equiv \exists x_0 < x (\forall y \in (x_0, x) f(y) = f(x)); \\ \psi_2(x) &:\equiv \exists x_0 < x (\forall y \in (x_0, x) f(y) > f(x)); \\ \theta_{ij}(x) &:\equiv \psi_i(x) \wedge \varphi_j(x) \text{ for each } i, j \leq 2.\end{aligned}$$

To show Theorem 5, we first prove some lemmas needed later.

Lemma 6 ([2]) *For each $x \in \text{Int}(A)$, there exist $i, j \leq 2$ such that $\theta_{ij}(x)$ holds.*

Proof. Suppose that there exists some $x \in \text{Int}(A)$ such that $\varphi_j(x)$ does not hold for some $j \leq 2$. Then the set $\{y \in A : f(y) \leq f(x)\}$ cannot be written as a union of finitely many convex sets, contradicting that M is weakly o-minimal. Thus, for each $x \in \text{Int}(A)$ there exists some $j \leq 2$ such that $\varphi_j(x)$ holds. Similarly, for each $x \in \text{Int}(A)$ there exists some $i \leq 2$ such that $\psi_i(x)$ holds. $\square$

Lemma 7 ([2]) *There exists a partition of A into finitely many points and open convex sets such that each open convex set lies in the solution set of some formula θ_{ij} .*

Proof. By Lemma 6, there exists a finite subset X of A such that we have $X \cup \bigcup_{i,j \leq 2} \theta_{ij} = A$. For each $i, j \leq 2$, by weak o-minimality of M , the set θ_{ij} can be written as a union of finitely many points and open convex sets. This finishes the proof. $\square$

Lemma 8 ([2, Lemma 3.6]) *Let I be an open interval of M . Then it cannot happen that one of the formulas $\theta_{01}, \theta_{10}, \theta_{12}, \theta_{21}$ holds throughout I .*

Proof. Suppose that the formula θ_{01} holds throughout I . Let x be an element of I . Since $\varphi_1(x)$ holds, there exists some $x_1 > x$ with $x_1 \in I$ such that for each $y \in (x, x_1)$ we have $f(x) = f(y)$. Let z be an element of open interval (x, x_1) . Since $\psi_0(z)$ holds, there exists some w with $x < w < z$ such that for each $y \in (w, z)$ we have $f(y) < f(z)$, a contradiction.

The other cases are similar. $\square$

We show the next lemma later.

Lemma 9 *Let I be an open interval of M . Suppose that $h : I \rightarrow \overline{M}$ has the local minimum or maximum throughout I . Then h is not definable.*

Lemma 10 *Let I be an open interval of M . Then it cannot happen that one of the formulas θ_{00} and θ_{22} holds throughout I .*

Proof. This lemma follows from Lemma 9. $\square$

Lemma 11 *Let I be a non-empty open definable convex subset of M such that θ_{02} holds throughout I . Then there exists $n \in \mathbb{N}$ and a partition of I into definable sets $X, I_0, \dots, I_n$ such that X is finite, $I_0, \dots, I_n$ are open convex sets and for each $i \leq n$ the function $f \upharpoonright I_i$ is locally increasing. Similarly, if θ_{20} holds throughout I , the same conclusion holds with ‘locally increasing’ replaced by ‘locally decreasing’.*

Proof. Suppose that $\theta_{02}(x)$ holds throughout I . We define the following formulas:

$$\begin{aligned}\chi_0(x) &\equiv \forall x_1 > x [\exists y, z (x < y < z < x_1 \wedge f(z) \leq f(y))]; \\ \chi_2(x) &\equiv \forall x_0 < x [\exists y, z (x_0 < y < z < x \wedge f(z) \leq f(y))].\end{aligned}$$

Claim $\chi_0(x)$ and $\chi_2(x)$ cannot hold throughout a subinterval of I .

Proof of Claim. Suppose for a contradiction that $\chi_0(x)$ holds throughout a subinterval $I_0 \subseteq I$. The argument for $\chi_2(x)$ is similar. For each $a \in I_0$, we define the following:

$$\begin{aligned} V_a &:= \{x \in I_0 : x < a \text{ and if } y \in [x, a), \text{ then } f(y) < f(a)\} \\ &\quad \cup \{x \in I_0 : x > a \text{ and if } y \in (a, x], \text{ then } f(y) > f(a)\} \cup \{a\}; \\ g(a) &:= \inf V_a. \end{aligned}$$

Since $\theta_{02}(x)$ holds throughout I_0 , the set V_a is an infinite definable convex set and a is not a boundary point of V_a . Then, by Lemma 9, it suffices to show that g has the local minimum throughout I_0 . We define the following formulas:

$$\begin{aligned} \mu_0(x, a) &:= x < a \wedge g(x) \leq g(a); \\ \mu_1(x, a) &:= x < a \wedge g(x) > g(a); \\ \nu_0(x, a) &:= x > a \wedge g(x) \leq g(a); \\ \nu_1(x, a) &:= x > a \wedge g(x) > g(a). \end{aligned}$$

By weak o-minimality of M , for each $a \in I_0$, there exist open interval $J \subseteq I_0$ and $K \subseteq I_0$ with $J < a < K$ such that a is a boundary point of J and K , either $\mu_0(x, a)$ or $\mu_1(x, a)$ holds throughout J , and either $\nu_0(x, a)$ or $\nu_1(x, a)$ holds throughout K . Then, it suffices to show that $\mu_1(x, a)$ holds throughout J and $\nu_1(x, a)$ holds throughout K . Suppose for a contradiction that $\mu_0(x, a)$ holds throughout J . The argument for $\nu_0(x, a)$ is similar. Since a is not a boundary point of V_a , there exists $b \in V_a \cap J$. Since $\chi_0(b)$ holds, there exist $c, d \in V_a \cap J$ such that $b < c < d < a$ and $f(d) \leq f(c)$. Hence, by the definition of g , we have $b < c \leq g(d)$. Now, since b is an element of V_a , we have $g(a) \leq b < g(d)$, contradicting that $\mu_0(d, a)$ holds. $\square$

By the claim, the set $\{x \in I : \chi_0(x) \vee \chi_2(x)\}$ is finite. Hence, we finish the proof. $\square$

Proof of Theorem 5. By Lemma 6 through 11, the theorem follows. $\square$

Finally, we show Lemma 9.

Proof of Lemma 9. Suppose that $h : I \rightarrow \overline{M}$ has the local minimum throughout I . Suppose for a contradiction that h is definable.

Claim 1 We may assume that h is injective.

Proof of Claim 1. Define the following equivalence relation on I^2 :

$$E(x, y) \iff h(x) = h(y).$$

We first verify that every equivalence class on E is finite. Let A be an infinite class. Then, by weak o-minimality of M , there exists an open subinterval J of A . Since h has the local minimum throughout J , for each $x \in J$ there exists $y \in J$ such that we have $h(y) > h(x)$, a contradiction. Hence, every equivalence class on E is finite. Therefore the set $Z := \{x \in I : \forall y (E(x, y) \rightarrow x \leq y)\}$ is infinite. By weak o-minimality of M , there exists an open subinterval J' of Z . We may assume $\text{dom}(h) = J'$. $\square$

From now on, by Claim 1, we assume that the function h is injective. For each $a, b \in I$, we define the following:

$$\begin{aligned} U_a &:= \{x : x > a \wedge \forall y \in (a, x] (h(y) > h(a))\} \\ &\quad \cup \{x : x < a \wedge \forall y \in [x, a) (h(y) > h(a))\} \cup \{a\}; \\ a < b &\iff U_a \supsetneq U_b. \end{aligned}$$

Then, since h has the local minimum throughout I , for each $a \in I$ the set U_a is an infinite definable convex set. The predicate $<$ is a partial ordering.

Claim 2 Let $a, b, c \in I$. Suppose that a, b, c are pairwise distinct. Then the following hold.

1. $U_a \neq U_b$;
2. a is not a boundary point of U_a ;
3. $b \in U_a \iff a < b$;
4. If $U_a \cap U_b \neq \emptyset$, then either $a < b$ or $b < a$;
5. If $a < b < c$ and $a < c$, then $a < b$;
6. If $a < b < c$ and $a < b$, then $a < c$;
7. If $b < a$ and $c < a$, then $b < c$ or $c < b$;
8. $C_a := \{x \in I : x < a\}$ is finite.

Proof of Claim 2.

(1): h は単射より, $h(a) \neq h(b)$ となる. したがって $U_a \neq U_b$ である.

(2): h の仮定より, これはいえる.

(3): $(\Leftarrow)$ 明らか.

$(\Rightarrow)$ $b \in U_a$ とする. このとき $h(b) > h(a)$ である. ここで, 一般性を失うことなしに $a < b$ とする. $c \in U_b$ を任意にとる. このとき, $a \leq c \leq b$ ならば U_a は convex なので, $c \in U_a$ である. また, $b < c$ ならば $c \in U_b$ より, 任意の $d \in (b, c]$ に対して $h(d) > h(b) > h(a)$ である. よって $c \in U_a$ となる. 同様に, $c < a$ ならば $c \in U_a$ となる. このことから $U_a \supseteq U_b$ がいえる.

(4): 一般性を失うことなしに $a < b$ とする. 仮定より $c \in U_a \cap U_b$ かつ $a < c < b$ を満たす元が存在する. まず $h(a) < h(b)$ と思う. 任意に $d \in U_b$ をとる. $a \leq d \leq c$ ならば U_a は convex なので, $d \in U_a$ である. また, $c < d$ ならば任意の $e \in (c, d]$ に対し $h(e) \geq h(b) > h(a)$ となる. よって, $d \in U_a$ となる. 同様に, $d < a$ ならば $d \in U_a$ となる. したがって $U_b \subseteq U_a$ が成り立つ. 同様に $h(b) < h(a)$ ならば $U_a \subseteq U_b$ が成り立つ.

(5): $b < a$ になったとする. すると仮定より, $U_a \supseteq U_b \supseteq U_c$ かつ $b < a < c$ となる. よって $b, c \in U_b$ となり, U_b は convex なので $a \in U_b$ がいえる. これは矛盾する.

(6): (5) と同様に示せる.

(7): 仮定より $a \in U_b \cap U_c$ である. よって, $U_b \cap U_c \neq \emptyset$ が成り立つ. すると, (4) より結論がいえる.

(8): C_a が無限集合だったとする. このとき M は weakly o-minimal より, ある开区間 J が存在して $J \subseteq C_a$ となる. $b \in J$ を任意にとる. b は U_b の境界ではないので, $c, d \in U_b \cap J$ かつ $c < b < d$ を満たすものが存在する. すると $c, d \in C_a$ より, $c < a$ かつ $d < a$ となる. ここで (7) より, $c < d$ または $d < c$ である. $c < d$ とする ($d < c$ の場合も同様に示せる). すると, $d \in U_c$ かつ $c < b < d$ となる. U_c は convex より, $b \in U_c$ である. これは $c \in U_b$ に反する. $\square$

Claim 2 の (8) より $\prec$ は離散順序である. ここで

$$K := \{x \in I : \text{任意の } y \in I \text{ に対し, } y \neq x\};$$

$$\tilde{a} := \{x \in I : a \prec x \text{ かつ } a \prec y \prec x \text{ を満たす } y \text{ は存在しない}\}$$

と定義する.

Claim 3 The following conditions hold:

1. $I \setminus K = \bigsqcup_{a \in I} \bar{a}$;
2. the set K is finite;
3. the set $\bar{a}$ is finite.

Proof of Claim 3.

(1): 任意に $a \in I$ と $b \in K$ をとる. すると K の定義より, $a \not\leq b$ である. よって $b \notin \bar{a}$ がいえる.

次に, 任意に $c \in I \setminus K$ をとると, $c \notin K$ より, ある元 $d \in I$ が存在して $d < c$ が成り立つ. ここで $<$ は離散順序より, ある元 $d' \in I$ が存在して $c \in \bar{d'}$ がいえる.

また $e_1 \neq e_2$ を任意にとる. もし $\bar{e}_1 \cap \bar{e}_2 \neq \emptyset$ であったとすると, ある元 $x \in \bar{e}_1 \cap \bar{e}_2$ がとれる. すると $e_1 < x$ かつ $e_2 < x$ だから Claim 2 の (7) より, $e_1 < e_2$ または $e_2 < e_1$ である. $e_1 < e_2$ と思う ($e_2 < e_1$ の場合も同様). すると, $e_1 < e_2 < x$ となるが, これは $x \in \bar{e}_1$ に反する. したがって, $\bar{e}_1 \cap \bar{e}_2 = \emptyset$ である.

(2): K が無限集合だったとする. このとき M は weakly o-minimal より, ある开区間 J が存在して $J \subseteq K$ となる. $b \in J$ を任意にとる. b は U_b の境界ではないので, ある元 $c \in U_b \cap J$ が存在する. すると, $c \in U_b$ より, $b < c$ となる. これは $c \in J \subseteq K$ に反する.

(3): $\bar{a}$ が無限集合だったとする. このとき M は weakly o-minimal より, ある开区間 J が存在して $J \subseteq \bar{a}$ となる. $b \in J$ を任意にとる. b は U_b の境界ではないので, ある元 $c \in U_b \cap J$ が存在する. $b \in \bar{a}$ かつ $c \in U_b$ より, $a < b < c$ となる. これは $c \in J \subseteq \bar{a}$ に反する. $\square$

Claim 3 の (2) より, $I \setminus K$ は無限集合である. さて任意の $a, b \in I \setminus K$ に対して,

$$E'(a, b) \iff M \models \exists c \in I (a \in \bar{c} \wedge b \in \bar{c})$$

と定義すると, $E'(x, y)$ は Claim 3 の (1) より $(I \setminus K)^2$ 上の同値関係になる. また Claim 3 の (3) より, $E'(x, y)$ の各クラスは有限集合である. よって $X := \{x \in I \setminus K : M \models \forall y \in I \setminus K (E'(x, y) \rightarrow x \leq y)\}$ は無限集合になる.

さて X の definable convex な構成要素で $<$ に関して最大のものを Y とする. $a \in Y$ をとる. a は U_a の境界ではないので, $b_1, b_2 \in U_a$ かつ $b_1 < a < b_2$ となる元が存在する. すると $<$ は離散順序より, $a < b \leq b_2$ か

つ $a < b' \leq b_1$ となる元たちが存在する. このとき, $E'(b, b')$ である. また $b_1 < a < b_2$ だから, Claim 2 の (5) より $b' < a < b$ となる. よって, $b \notin X$ がいえる. ところで b は U_b の境界ではないので, $c \in U_b$ かつ $b < c$ となる元が存在する. すると, $\bar{c}$ は有限より, $d \in \bar{c}$ かつ $d \in X$ となるものがとれる. よって $b < c < d$ かつ $b < c$ なので, Claim 2 の (6) より $b < d$ となる. これは Y の性質に反する.

したがって, h は definable ではない. □

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