

# A note on the multiple elliptic Dedekind sum

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## 1 Introduction

An elliptic analogue of the *multiple Dedekind sum* was introduced in [E], and reported in the talk of the author at this(RIMS, 1994) conference. In this note we generalize the above results in a form suggested by professors Akiyama, Tanigawa, and Wakabayashi after the conference. The author would like to appreciate their valuable suggestion.

## 2 Notations

Let  $\tau$  be an element of the complex upper half plane. Set  $\omega_1 = \pi i$ ,  $\omega_2 = -\pi i(1+\tau)$ , and  $\omega_3 = \pi i\tau$ . We denote by  $\wp(\tau, z)$  the Wierstrass  $\wp$ -function associated to the lattice  $L = L_\tau = \mathbb{Z}2\pi i + \mathbb{Z}2\pi i\tau$ . And for  $i = 1, 2, 3$  we set  $\Theta_i = 2\omega_i$ ,  $\Omega_i = 2\omega_{i+2}$ , (should be interpreted mod. 3),  $L_i = \mathbb{Z}2\Omega_i + \mathbb{Z}\Theta_i$ . Note that  $L = \mathbb{Z}\Omega_i + \mathbb{Z}\Theta_i$  for any  $i = 1, 2, 3$ .

We define elliptic functions  $\varphi_i(\tau, z)$  associated to the lattices  $L_i$  by the following two conditions:

$$\varphi_i(\tau, z)^2 = \wp(\tau, z) - e_i(\tau), \quad (1)$$

$$\varphi_i(\tau, z) = \frac{1}{z} + \cdots, \quad (2)$$

where  $e_i(\tau) = \wp(\tau, \omega_i)$ .

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As usual we denote

$$g_2(\tau) = 60 \sum_{w \in L - \{0\}} \frac{1}{w^4},$$

$$g_3(\tau) = 140 \sum_{w \in L - \{0\}} \frac{1}{w^6}.$$

Then it is easily seen

**Proposition.** For  $i = 1, 2, 3$ ,

1.

$$\varphi_i(\tau, z + \Omega_i) = -\varphi_i(\tau, z), \quad \varphi_i(\tau, z + \Theta_i) = \varphi_i(\tau, z), \quad \varphi_i(\tau, -z) = -\varphi_i(\tau, z)$$

2. At  $z = 0$

$$\varphi_i(\tau, z) = \frac{1}{z} (1 + H_2^{(i)} z^2 + H_4^{(i)} z^4 + \dots)$$

, where  $H_{2n}^{(i)}$  is a polynomial in  $g_2, g_3$ , and  $e_i$  with rational coefficients.

We define polynomials  $M_{n,\tau}^{(i)}(t_0, \dots, t_r)$  with coefficients in  $\mathbf{Q}[e_i, g_2, g_3]$  by

$$z^{r+1} \prod_{k=0}^r \varphi_i(\tau, t_k z) = \sum_{n=0}^{\infty} M_{n,\tau}^{(i)}(t_0, \dots, t_r) z^{2n}.$$

### 3 Multiple elliptic Dedekind sum and the reciprocity

Let  $p$  be a natural number and let  $a_1, \dots, a_r$  integers coprime to  $p$  such that  $p + a_1 + \dots + a_r$  is even. We define the *multiple elliptic Dedekind sum* by

$$D_{\tau}^{(i)}(p : a_1, \dots, a_r) = \sum_{\substack{m, n=0 \\ (m, n) \neq (0, 0)}}^{p-1} (-1)^m \prod_{k=1}^r \varphi_i(\tau, \frac{a_k}{p}(m\Omega_i + n\Theta_i)).$$

Then we have

**Theorem. (Reciprocity law)** Let  $a_0, a_1, \dots, a_r$  be pairwise coprime natural numbers such that  $a_0 + a_1 + \dots + a_r$  is even. Then

$$\sum_{k=0}^r \frac{1}{a_k} D_{\tau}^{(i)}(a_0, \dots, a_{k-1}, a_{k+1}, \dots, a_r) = -M_{r,\tau}^{(i)}(a_0, \dots, a_r).$$

The proof is similar to **Theorem 1** of [E].

## 4 Remarks

In [E] and the talk we treated only  $D_r^{(1)}$  without assigning index "1". As was explained there,  $D_r^{(1)}$  has a relation to the multiple Dedekind sum in the sense of [Z] and  $M_{r,\tau}^{(1)}$  has a significance in topology [HBJ]. We do not know the corresponding results for indices 2,3.

## References

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