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Partial Epstein zeta functions on binary linear codes and their functional equations

By

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Abstract

In this paper, partial Epstein zeta functions on binary linear codes, which are related with Hamming weight enumerators of binary linear codes, are newly defined. Then functional equations for those zeta functions on codes are presented. In particular, it is clarified that simple functional equations hold for partial Epstein zeta functions on binary linear self-dual codes.

§ 1. Introduction

The minimum distance of a code determines the error correction / detection capability of the code. The distance distribution of a linear code is equivalent to the weight distribution of the code, because the difference between any two codewords is equal to another codeword. The MacWilliams identity for Hamming weight enumerators provides the relationship between the Hamming weight distribution of a code and that of the dual code [10, 11]. By using the MacWilliams identity, the weight enumerator of the dual code of a code is derived from that of the code, and vice versa. Some generalizations of Hamming weight enumerators and those of the MacWilliams identity have been known [11, 13].

Broué and Enguehard provided a method of construction of elliptic modular forms using the weight enumerators of self-dual codes [3]. The relationship between several types of modular forms such as Hilbert, Jacobi, and Siegel modular forms and those

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of weight enumerators, e.g. Hamming weight enumerators and Lee weight enumerators, of codes over finite fields, finite rings have been extensively studied by Bannai, Choie, Ebeling, Nebe, Ozeki, Runge, Solé, et al. [1, 2, 4, 5, 6, 7, 12].

Modular forms closely relate to Dirichlet series and zeta functions. One classical reason why modular forms were studied is their use in investigating the number of ways of representing an integer by a quadratic form. For example, the number of ways an integer can be represented as a sum of squares is equal to the coefficient in the q-expansion of the power of a modular form. P. Epstein introduced a zeta function associated with positive definite quadratic forms [8]. In [14], partial Epstein zeta functions, which are summands of Epstein zeta functions associated with quadratic forms, have been introduced and their functional equations have been proved by using the Mellin transform of theta series which are related with modular forms on binary linear codes. In this paper, partial Epstein zeta functions for binary linear codes, which are related with Hamming weight enumerators of binary linear codes, are newly defined. Then functional equations of those zeta functions for codes are presented. In particular, it is clarified that simple functional equations hold for zeta functions for binary linear self-dual codes.

The organization of this paper is as follows: Section 2 presents some definitions and some basic facts concerning binary linear codes and explains the MacWilliams identity. Section 3 describes theta series and their transformation formulae. Section 4 presents partial Epstein zeta functions for binary linear codes and their functional equations that are the main theme of this paper.

§ 2. Preliminaries

This section presents the definitions and the basic properties of binary linear codes and explains the MacWilliams identity for Hamming weight enumerators of binary linear codes.

§ 2.1. Binary linear codes

Let $\mathbb{F}_2$ be the binary field. A k-dimensional subspace of n-dimensional vector space $\mathbb{F}_2^n$ is called a binary $[n, k]$ linear code or a binary $[n, k]$ code, where n and k are called the code length and the dimension of the code, respectively. An element of a code is called a codeword of the code. A class of binary linear codes is defined by matrices over $\mathbb{F}_2$. A binary $[n, k]$ code may be specified by a basis of k linearly independent codewords. A matrix whose rows are a basis of a code is called a generator matrix of the code. The same code, in the sense of a set of codewords or a vector space, may be described by different generator matrices or bases of the vector space. Let G be a $k \times n$

generator matrix of a binary $[n, k]$ code C and let us denote G as

$$G = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n} \\ \vdots & \vdots & & \vdots \\ g_{k,1} & g_{k,2} & \cdots & g_{k,n} \end{bmatrix},$$

where $g_{j,l}$ are elements of $\mathbb{F}_2$ and the rows of G are a basis of C. All rows of G and all linear combinations of them are codewords in C. Therefore, C contains 2^k codewords. The null space of C is spanned by the rows of the following matrix H that satisfies the relation $G\,{}^tH = O_{k\times(n-k)}$:

$$H = \begin{bmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,n} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,n} \\ \vdots & \vdots & & \vdots \\ h_{n-k,1} & h_{n-k,2} & \cdots & h_{n-k,n} \end{bmatrix},$$

where $h_{j,l}$ are elements of $\mathbb{F}_2$, ${}^tH$ denotes the transpose of H, and $O_{k\times(n-k)}$ denotes the $k \times (n-k)$ zero matrix. The matrix H is called a parity-check matrix of C and generates the dual code of C. The dual code $C^\perp$ of C is defined by

$$C^\perp := \{\mathbf{v} \in \mathbb{F}_2^n \mid \langle \mathbf{c}, \mathbf{v}\rangle = 0 \text{ for all } \mathbf{c} \in C\},$$

where $\langle \mathbf{c}, \mathbf{v}\rangle = \sum_{j=1}^n c_j v_j$ for $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, which is the inner product of $\mathbf{c}$ and $\mathbf{v}$. For a linear code C, its dual code $C^\perp$ consists of all rows of H and all linear combinations of them. In other words, H is a generator matrix of $C^\perp$. If $C = C^\perp$, then C is called a self-dual code.

Example 2.1. Let C_7 be the binary $[7, 4]$ Hamming code. A generator matrix G_7 and a parity-check matrix H_7 of C_7 are

$$G_7 = \begin{bmatrix} 1\,1\,0\,1\,0\,0\,0 \\ 1\,0\,1\,0\,1\,0\,0 \\ 0\,1\,1\,0\,0\,1\,0 \\ 1\,1\,1\,0\,0\,0\,1 \end{bmatrix} \text{ and } H_7 = \begin{bmatrix} 1\,0\,0\,1\,1\,0\,1 \\ 0\,1\,0\,1\,0\,1\,1 \\ 0\,0\,1\,0\,1\,1\,1 \end{bmatrix},$$

respectively.

Example 2.2. The binary $[8, 4]$ extended Hamming code is a self-dual code [11]. The following matrix G_8 is a parity-check matrix of C_8 as well as a generator matrix of C_8:

$$G_8 = \begin{bmatrix} 1\,0\,0\,1\,1\,0\,1\,0 \\ 0\,1\,0\,1\,0\,1\,1\,0 \\ 0\,0\,1\,0\,1\,1\,1\,0 \\ 1\,1\,1\,1\,1\,1\,1\,1 \end{bmatrix}.$$

§ 2.2. MacWilliams identity for Hamming weight enumerators of binary linear codes

The distance distribution of a code closely relates to the error correction / detection capability of the code. Any two codewords in a code have to be definitely far from each other for ensuring the specific error correction / detection capability. The distance distribution of a linear code is equivalent to the weight distribution of the code, because the distance between two codewords is equal to the weight of another codeword.

Definition 2.3 (Hamming weight [11, p.8]). Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ be an element of $\mathbb{F}_2^n$, where u_i denotes the ith component of $\mathbf{u}$. The Hamming weight of u_i, denoted by $w_{\mathrm{H}}(u_i)$, is defined by

$$w_{\mathrm{H}}(u_i) := \begin{cases} 0 & \text{if } u_i = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Then the Hamming weight of $\mathbf{u}$, denoted by $w_{\mathrm{H}}(\mathbf{u})$, is defined by

$$w_{\mathrm{H}}(\mathbf{u}) := \sum_{i=1}^{n} w_{\mathrm{H}}(u_i).$$

The Hamming weight of a vector denotes the number of nonzero components of the vector.

Definition 2.4 (Hamming weight enumerator [11, p.126]). Let x_0 and x_1 be indeterminates. For a binary $[n, k]$ code C, the Hamming weight enumerator $W_C(x_0, x_1)$ is defined by

$$W_C\,(x_0,\, x_1) := \sum_{\mathbf{c} \in C} x_0^{n - w_{\mathrm{H}}(\mathbf{c})} x_1^{w_{\mathrm{H}}(\mathbf{c})} = \sum_{i=0}^{n} W_i\, x_0^{n-i} x_1^i,$$

where W_i denotes the number of codewords of Hamming weight i in C. In the same way, for the dual code $C^\perp$ of C, the Hamming weight enumerator $W_{C^\perp}(x_0, x_1)$ is defined by

$$W_{C^\perp}\,(x_0,\, x_1) := \sum_{\mathbf{c}' \in C^\perp} x_0^{n - w_{\mathrm{H}}(\mathbf{c}')} x_1^{w_{\mathrm{H}}(\mathbf{c}')} = \sum_{j=0}^{n} W_j^\perp\, x_0^{n-j} x_1^j,$$

where $W_j^\perp$ denotes the number of codewords of Hamming weight j in $C^\perp$. Both the weight enumerators $W_C\,(x_0,\, x_1)$ and $W_{C^\perp}\,(x_0,\, x_1)$ are homogeneous polynomials of degree n in two indeterminates x_0 and x_1.

The following Theorem 2.5 holds for Hamming weight enumerators of binary linear codes.

Theorem 2.5 (MacWilliams identity [10], [11, p.127]). *Let C be a binary $[n,k]$ code with dual code $C^\perp$. Then the following relation holds between the weight enumerator of C and that of $C^\perp$:*

$$W_{C^\perp}(x_0,x_1)=\frac{1}{|C|}W_C\left(x_0+x_1,x_0-x_1\right), \tag{2.1}$$

where $|C|=2^k$ denotes the number of codewords in C.

Equation (2.1) is the **MacWilliams identity for Hamming weight enumerators of binary linear codes** and shows that the weight enumerator of $C^\perp$ is derived from that of C. Equation (2.1) is symmetric with respect to the roles of C and $C^\perp$, that is,

$$W_C(x_0,x_1)=\frac{1}{|C^\perp|}W_{C^\perp}\left(x_0+x_1,x_0-x_1\right), \tag{2.2}$$

where $|C^\perp|=2^{n-k}$ denotes the number of codewords in $C^\perp$. Equation (2.2) is obtained by putting $y_0=x_0+x_1$ and $y_1=x_0-x_1$ in Eq. (2.1). The weight enumerator of C is derived from that of $C^\perp$ by using Eq. (2.2).

If C is self-dual, then $n=2k$ and $|C|=2^k=2^{n/2}$. Therefore, Eq. (2.1) results in the following form:

$$W_C(x_0,x_1)=W_C\left(\frac{x_0+x_1}{\sqrt{2}},\frac{x_0-x_1}{\sqrt{2}}\right). \tag{2.3}$$

Equation (2.3) shows that the Hamming weight enumerators of binary linear self-dual codes are invariant under the transform

$$\sigma:\begin{pmatrix}x_0\\x_1\end{pmatrix}\longmapsto\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}x_0\\x_1\end{pmatrix}.$$

Example 2.6. The code C_7, which was given in Example 2.1, contains 16 codewords shown in Table 1. Then the weight enumerator of C_7 is

$$W_{C_7}(x_0,x_1)=x_0^7+7x_0^4x_1^3+7x_0^3x_1^4+x_1^7.$$

On the other hand, the dual code $C_7^\perp$ contains 8 codewords shown in Table 2. The weight enumerator of $C_7^\perp$ is

$$W_{C_7^\perp}(x_0,x_1)=x_0^7+7x_0^3x_1^4.$$

Substitute x_0+x_1 and x_0-x_1 into x_0 and x_1 of $W_{C_7}(x_0,x_1)$, respectively, then

$$\begin{aligned}&W_{C_7}(x_0+x_1,x_0-x_1)\\&=(x_0+x_1)^7+7(x_0+x_1)^4(x_0-x_1)^3+7(x_0+x_1)^3(x_0-x_1)^4+(x_0-x_1)^7\\&=16(x_0^7+7x_0^3x_1^4)=2^4W_{C_7^\perp}(x_0,x_1).\end{aligned}$$

Table 1. Codewords in C_7 and their Hamming weights

Codeword	Weight	Codeword	Weight	Codeword	Weight
0000000	0	1011010	4	1001101	4
1101000	3	0011001	3	0101011	4
1010100	3	1100110	4	0010111	4
0110010	3	0100101	3	0001110	3
1110001	4	1000011	3	0111100	4
1111111	7	—	—	—	—

Table 2. Codewords in $C_7^\perp$ and their Hamming weights

Codeword	Weight	Codeword	Weight	Codeword	Weight
0000000	0	0010111	4	0111100	4
1001101	4	1100110	4	1110001	4
0101011	4	1011010	4	—	—

Conversely, $W_{C_7}(x_0, x_1)$ is derived from $W_{C_7^\perp}(x_0 + x_1, x_0 - x_1)$. In fact,

$$\begin{aligned} W_{C_7^\perp}(x_0 + x_1, x_0 - x_1) &= (x_0 + x_1)^7 + 7(x_0 + x_1)^3(x_0 - x_1)^4 \\ &= 8(x_0^7 + 7x_0^4x_1^3 + 7x_0^3x_1^4 + x_1^7) = 2^3 W_{C_7}(x_0, x_1). \end{aligned}$$

Example 2.7. The code C_8, which was given in Example 2.2, contains 16 codewords shown in Table 3. Since the generator matrix of $C_8^\perp$ is identical with that of C_8, both the weight enumerators of C_8 and $C_8^\perp$ are

$$W_{C_8}(x_0, x_1) = W_{C_8^\perp}(x_0, x_1) = x_0^8 + 14x_0^4x_1^4 + x_1^8.$$

In fact, $W_{C_8}(x_0, x_1)$ is invariant under the transform σ:

$$\begin{aligned} W_{C_8}\left(\frac{x_0 + x_1}{\sqrt{2}}, \frac{x_0 - x_1}{\sqrt{2}}\right) &= \left(\frac{x_0 + x_1}{\sqrt{2}}\right)^8 + 14\left(\frac{x_0 + x_1}{\sqrt{2}}\right)^4\left(\frac{x_0 - x_1}{\sqrt{2}}\right)^4 + \left(\frac{x_0 - x_1}{\sqrt{2}}\right)^8 \\ &= x_0^8 + 14x_0^4x_1^4 + x_1^8 = W_{C_8}(x_0, x_1). \end{aligned}$$

Equation (2.1) provides the relationship between the coefficients of $W_C(x_0, x_1)$ and those of $W_{C^\perp}(x_0, x_1)$. To indicate the relationship explicitly, we introduce the

Table 3. Codewords in C_8 and their Hamming weights

Codeword	Weight	Codeword	Weight	Codeword	Weight
00000000	0	10011010	4	01010110	4
00101110	4	11001100	4	10110100	4
01100101	4	01111000	4	10101001	4
11010001	4	11100010	4	00110011	4
01001011	4	10000111	4	00011101	4
11111111	8	—	—	—	—

$(n+1)\times(n+1)$ matrix M_n as follows: we denote the expansion of the polynomial $(x_0+x_1)^{n-j}(x_0-x_1)^j$ as

$$\mu_{j,0}x_0^n+\mu_{j,1}x_0^{n-1}x_1+\cdots+\mu_{j,n}x_1^n$$

for $j=0,1,\ldots,n$. The coefficients $\mu_{j,l}=\sum_{p=0}^{l}(-1)^p\binom{j}{p}\binom{n-j}{l-p}$ are known as special cases of Krawtchouk polynomials [11, p.129, Equation (12)]. We define M_n by

$$M_n:=\begin{pmatrix}\mu_{0,0}&\mu_{0,1}&\cdots&\mu_{0,n}\\ \mu_{1,0}&\mu_{1,1}&\cdots&\mu_{1,n}\\ \vdots&\vdots&&\vdots\\ \mu_{n,0}&\mu_{n,1}&\cdots&\mu_{n,n}\end{pmatrix}.$$

Proposition 2.8. *The coefficients $\mu_{j,l}$ have the following properties:*

(i) $\sum_{m=0}^{n}\mu_{j,m}\,\mu_{m,l}=\begin{cases}2^n \text{ if } j=l,\\ 0 \quad \text{otherwise.}\end{cases}$

(ii) $\sum_{l=0}^{n}\mu_{j,l}=\begin{cases}2^n \text{ if } j=0,\\ 0 \quad \text{otherwise.}\end{cases}$

Property (i) implies that $M_n^2=2^nI_{n+1}$, where I_{n+1} denotes the $(n+1)\times(n+1)$ identity matrix.

Proof. Property (i) is given in [11, p.152, Chapter 5, Corollary 18]. Property (ii) is obtained by substituting 1 into x_0 and x_1 of the equations $(x_0+x_1)^{n-j}(x_0-x_1)^j=\mu_{j,0}x_0^n+\mu_{j,1}x_0^{n-1}x_1+\cdots+\mu_{j,n}x_1^n$ for $j=0,1,\ldots,n$. □

By using the components of the matrix M_n, the right-hand side of Eq. (2.1) is

expressed as follows:

$$\begin{aligned}
&\frac{1}{2^k} W_C\left(x_0 + x_1,\, x_0 - x_1\right) \\
&= \frac{1}{2^k} \sum_{i=0}^{n} W_i \left(x_0 + x_1\right)^{n-i} \left(x_0 - x_1\right)^i \\
&= \frac{1}{2^k} \sum_{i=0}^{n} W_i \left(\mu_{i,0} x_0^n + \mu_{i,1} x_0^{n-1} x_1 + \cdots + \mu_{i,n} x_1^n\right) \\
&= \frac{1}{2^k} \left\{ \left(\sum_{i=0}^{n} W_i \mu_{i,0} \right) x_0^n + \left(\sum_{i=0}^{n} W_i \mu_{i,1} \right) x_0^{n-1} x_1 + \cdots + \left(\sum_{i=0}^{n} W_i \mu_{i,n} \right) x_1^n \right\}. \qquad (2.4)
\end{aligned}$$

Comparing the coefficients of Eq. (2.4) with those of the left-hand side of Eq. (2.1), we see that the coefficients of $W_C(x_0, x_1)$ and those of $W_{C^\perp}(x_0, x_1)$ satisfy the following equation:

$$\left(W_0^\perp,\, W_1^\perp,\, W_2^\perp,\, \ldots,\, W_n^\perp\right) = \frac{1}{2^k} \left(W_0,\, W_1,\, W_2,\, \ldots,\, W_n\right) M_n. \qquad (2.5)$$

Multiplying both sides of Eq. (2.5) by $\frac{1}{2^n} M_n$ yields

$$\left(W_0,\, W_1,\, W_2,\, \ldots,\, W_n\right) = \frac{1}{2^{n-k}} \left(W_0^\perp,\, W_1^\perp,\, W_2^\perp,\, \ldots,\, W_n^\perp\right) M_n. \qquad (2.6)$$

Equation (2.6) is also obtained directly from Eq. (2.2). Equations (2.5) and (2.6) play important roles in Section 4.

§ 3. Theta series

In this section, we deal with one-variable theta series that are essential to derive the functional equations for partial Epstein zeta functions on binary linear codes in Section 4.

Definition 3.1. Let τ be a variable in the upper half-plane $\mathcal{H}$ of $\mathbb{C}$. Let $\mathbf{u}_{n,j} = (1, \ldots, 1, 0, \ldots, 0)$ be an element of $\mathbb{F}_2^n$, where the leftmost j components of $\mathbf{u}_{n,j}$ are 1's and the other components are 0's for $j = 0, 1, \ldots, n$. For $\tau \in \mathcal{H}$, the theta series $\theta_{n,j}(\tau)$ for $j = 0, 1, \ldots, n$ are defined by

$$\theta_{n,j}(\tau) := \sum_{\substack{\mathbf{m} \in \mathbb{Z}^n \\ \mathbf{m} \equiv \mathbf{u}_{n,j} \pmod 2}} e^{2\pi i \tau \frac{\langle \mathbf{m},\, \mathbf{m} \rangle}{4}} = \sum_{\mathbf{m} \in \mathbb{Z}^n} e^{2\pi i \tau \frac{\langle 2\mathbf{m} + \mathbf{u}_{n,j},\, 2\mathbf{m} + \mathbf{u}_{n,j} \rangle}{4}},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ is the standard inner product of $\mathbf{u}$ and $\mathbf{v}$.

Remark. For all positive integers n, the theta series $\theta_{n,j}(\tau)$ consists of the powers of $\theta_{1,0}(\tau)$ and $\theta_{1,1}(\tau)$:

$$\theta_{n,j}(\tau) = \theta_{1,0}(\tau)^{n-j}\theta_{1,1}(\tau)^{j}. \tag{3.1}$$

Proposition 3.2. *The theta series $\theta_{n,j}(\tau)$ for $j = 0, 1, \ldots, n$ satisfy the following two transformation formulae:*

(i)
$$\begin{pmatrix} \theta_{n,0}(\tau+1) \\ \theta_{n,1}(\tau+1) \\ \theta_{n,2}(\tau+1) \\ \vdots \\ \theta_{n,n}(\tau+1) \end{pmatrix} = \operatorname{diag}(1, i, i^2, \ldots, i^n) \begin{pmatrix} \theta_{n,0}(\tau) \\ \theta_{n,1}(\tau) \\ \theta_{n,2}(\tau) \\ \vdots \\ \theta_{n,n}(\tau) \end{pmatrix},$$

(ii)
$$\begin{pmatrix} \theta_{n,0}(\frac{-1}{\tau}) \\ \theta_{n,1}(\frac{-1}{\tau}) \\ \theta_{n,2}(\frac{-1}{\tau}) \\ \vdots \\ \theta_{n,n}(\frac{-1}{\tau}) \end{pmatrix} = \left(\sqrt{\frac{\tau}{i}}\right)^{n} \frac{1}{\sqrt{2^n}} M_n \begin{pmatrix} \theta_{n,0}(\tau) \\ \theta_{n,1}(\tau) \\ \theta_{n,2}(\tau) \\ \vdots \\ \theta_{n,n}(\tau) \end{pmatrix},$$

where $i = \sqrt{-1}$ and $-\pi/4 < \arg\sqrt{\tau/i} < \pi/4$. The symbol $\operatorname{diag}(1, i, i^2, \ldots, i^n)$ *denotes a diagonal matrix.*

Proof. The first formula follows directly from the definition of $\theta_{n,j}(\tau)$.

The second formula is obtained by using the Poisson summation formula. Two-variable theta series for $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$, which are extensions of $\theta_{n,j}(\tau)$, were defined and their transformation formulae were given in [14]. The proof of the second formula for $\theta_{n,j}(\tau)$ is analogous to that of the second formula for the two-variable theta series. □

§ 4. Partial Epstein zeta functions on binary linear codes

Definition 4.1 (Epstein zeta functions [8])**.** Let s be a complex variable with $\Re s > n/2$, let Y be an $n \times n$ matrix of a positive definite quadratic form, and let $\mathbf{g}$ and $\mathbf{h}$ be n-dimensional real vectors. Then the Epstein zeta function associated with $(Y, \mathbf{g}, \mathbf{h})$ is defined by

$$Z_n(Y, \mathbf{g}, \mathbf{h}, s) := \sum_{\substack{\mathbf{a}\in\mathbb{Z}^n \\ \mathbf{a}+\mathbf{g}\neq\mathbf{0}_n}} \frac{e^{2\pi i\langle \mathbf{h},\, \mathbf{a}\rangle}}{({}^t(\mathbf{a}+\mathbf{g})\, Y\, (\mathbf{a}+\mathbf{g}))^{s}},$$

where $\mathbf{a}$ runs over all elements in $\mathbb{Z}^n$ except for any vectors such that $\mathbf{a} + \mathbf{g} = \mathbf{0}_n$, and $\mathbf{0}_n$ is the n-dimensional zero vector.

Let us substitute the $n \times n$ identity matrix I_n into Y and substitute $\mathbf{0}_n$ into $\mathbf{g}$ and $\mathbf{h}$, respectively. Then we have

$$Z_n(I_n, \mathbf{0}_n, \mathbf{0}_n, s) = \sum_{\substack{\mathbf{a} \in \mathbb{Z}^n \\ \mathbf{a} \neq \mathbf{0}_n}} \frac{1}{\langle \mathbf{a}, \mathbf{a} \rangle^s},$$

where $\langle \mathbf{a}, \mathbf{a} \rangle$ denotes the standard inner product of $\mathbf{a}$ and itself. We denote the function $Z_n(I_n, \mathbf{0}_n, \mathbf{0}_n, s)$ by $Z_n(s)$ and define partial Epstein zeta functions of $Z_n(s)$ as follows.

Definition 4.2 (Partial Epstein zeta functions [14])**.** Let s be a complex variable with $\Re s > n/2$. Partial Epstein zeta functions $Z_{n,j}(s)$ for $j = 0, 1, \ldots, n$ are defined by

$$Z_{n,j}(s) := \sum_{\substack{\mathbf{v} \in \mathbb{Z}^n \\ 2\mathbf{v} + \mathbf{u}_{n,j} \neq \mathbf{0}_n}} \frac{1}{\langle 2\mathbf{v} + \mathbf{u}_{n,j}, 2\mathbf{v} + \mathbf{u}_{n,j} \rangle^s},$$

where the vectors $\mathbf{u}_{n,j}$ are binary vectors given in Definition 3.1.

Theorem 4.3 ([14])**.** *Partial Epstein zeta functions $Z_{n,j}(s)$ for $j = 0, 1, \ldots, n$, which are defined for $\Re s > n/2$, extend analytically to entire functions on the whole complex s-plane except for a simple pole at $s = n/2$ with residue $(\pi/4)^{n/2}/\Gamma(n/2)$, where $\Gamma(s)$ is the gamma function. Let*

$$\Lambda_{n,j}(s) := \left(\frac{\pi}{2}\right)^{-s} \Gamma(s)\, Z_{n,j}(s).$$

Then the following relation holds for $j = 0, 1, \ldots, n$:

$$\Lambda_{n,j}(s) = \frac{1}{\sqrt{2^n}} \sum_{l=0}^{n} \mu_{j,l} \Lambda_{n,l}\left(\frac{n}{2} - s\right), \tag{4.1}$$

where $\mu_{j,l}$ is the (j,l)th component of M_n. These relations for $j = 0, 1, \ldots, n$ are rewritten in the following single equation:

$$\left(\frac{\pi}{2}\right)^{-s} \Gamma(s) \begin{pmatrix} Z_{n,0}(s) \\ Z_{n,1}(s) \\ \vdots \\ Z_{n,n}(s) \end{pmatrix} = \left(\frac{\pi}{2}\right)^{-(n/2-s)} \Gamma\left(\frac{n}{2} - s\right) \frac{1}{\sqrt{2^n}} M_n \begin{pmatrix} Z_{n,0}\left(\frac{n}{2} - s\right) \\ Z_{n,1}\left(\frac{n}{2} - s\right) \\ \vdots \\ Z_{n,n}\left(\frac{n}{2} - s\right) \end{pmatrix}. \tag{4.2}$$

The above type of partial Epstein zeta functions are components of partial Epstein zeta functions on binary linear codes which are defined as follows.

Definition 4.4 (Partial Epstein zeta functions on binary linear codes)**.** For a binary $[n, k]$ code C, let $\Lambda(C)$ be the following set:

$$\Lambda(C) := \{\mathbf{c} + 2\mathbf{v} \mid \mathbf{c} \in C,\ \mathbf{v} \in \mathbb{Z}^n\}.$$

For a complex variable $s \in \mathbb{C}$ with $\Re s > n/2$, the partial Epstein zeta function on C is defined by

$$Z_{\Lambda(C)}(s) := \sum_{\substack{\mathbf{r}\in\Lambda(C)\\ \mathbf{r}\neq \mathbf{0}_n}} \frac{1}{\langle \mathbf{r},\, \mathbf{r}\rangle^s} = \sum_{\mathbf{c}\in C} \sum_{\substack{\mathbf{v}\in\mathbb{Z}^n\\ 2\mathbf{v}+\mathbf{c}\neq \mathbf{0}_n}} \frac{1}{\langle 2\mathbf{v}+\mathbf{c},\, 2\mathbf{v}+\mathbf{c}\rangle^s}.$$

In the same way, for the dual code $C^\perp$ of C, the set $\Lambda(C^\perp)$ is defined by

$$\Lambda(C^\perp) := \left\{ \mathbf{c}' + 2\mathbf{v} \mid \mathbf{c}' \in C^\perp,\ \mathbf{v} \in \mathbb{Z}^n \right\}.$$

For a complex variable $s \in \mathbb{C}$ with $\Re s > n/2$, the partial Epstein zeta function on $C^\perp$ is defined by

$$Z_{\Lambda(C^\perp)}(s) := \sum_{\substack{\mathbf{r}'\in\Lambda(C^\perp)\\ \mathbf{r}'\neq \mathbf{0}_n}} \frac{1}{\langle \mathbf{r}',\, \mathbf{r}'\rangle^s} = \sum_{\mathbf{c}'\in C} \sum_{\substack{\mathbf{v}\in\mathbb{Z}^n\\ 2\mathbf{v}+\mathbf{c}'\neq \mathbf{0}_n}} \frac{1}{\langle 2\mathbf{v}+\mathbf{c}',\, 2\mathbf{v}+\mathbf{c}'\rangle^s}.$$

Replacing $\mathbf{u}_{n,j}$ with a codeword $\mathbf{c}$ of Hamming weight j does not change the summation $Z_{n,j}(s)$. Therefore, if the Hamming weight enumerator of C is $W_C(x_0, x_1) = \sum_{i=0}^n W_i x_0^{n-i} x_1^i$, then $Z_{\Lambda(C)}(s)$ is expressed as follows by using the coefficients $W_0, W_1, \ldots, W_n$ of $W_C(x_0, x_1)$:

$$Z_{\Lambda(C)}(s) = \sum_{i=0}^n W_i \sum_{\substack{\mathbf{v}\in\mathbb{Z}^n\\ 2\mathbf{v}+\mathbf{u}_{n,i}\neq \mathbf{0}_n}} \frac{1}{\langle 2\mathbf{v}+\mathbf{u}_{n,i},\ 2\mathbf{v}+\mathbf{u}_{n,i}\rangle^s} = \sum_{i=0}^n W_i Z_{n,i}(s). \tag{4.3}$$

In the same way, for the Hamming weight enumerator $W_{C^\perp}(x_0, x_1) = \sum_{j=0}^n W_j^\perp x_0^{n-j} x_1^j$ of $C^\perp$, the partial Epstein zeta function $Z_{\Lambda(C^\perp)}(s)$ is expressed as

$$Z_{\Lambda(C^\perp)}(s) = \sum_{j=0}^n W_j^\perp Z_{n,j}(s). \tag{4.4}$$

The next Theorem 4.5 is the main result in this paper.

Theorem 4.5. *The partial Epstein zeta function $Z_{\Lambda(C)}(s)$ on a binary $[n,k]$ code C with dual code $C^\perp$ extends analytically to an entire function on the whole complex plane except for a simple pole at $s = n/2$ with residue $2^k (\pi/4)^{n/2} / \Gamma(n/2)$. Then $Z_{\Lambda(C)}(s)$ satisfies the following functional equation:*

$$\left(\frac{\pi}{2}\right)^{-s} \Gamma(s) \frac{1}{\sqrt{2^k}} Z_{\Lambda(C)}(s) = \left(\frac{\pi}{2}\right)^{-(n/2-s)} \Gamma\left(\frac{n}{2}-s\right) \frac{1}{\sqrt{2^{n-k}}} Z_{\Lambda(C^\perp)}\left(\frac{n}{2}-s\right).$$

In particular, if C is self-dual, we have

$$\left(\frac{\pi}{2}\right)^{-s} \Gamma(s)\, Z_{\Lambda(C)}(s) = \left(\frac{\pi}{2}\right)^{-(n/2-s)} \Gamma\left(\frac{n}{2}-s\right) Z_{\Lambda(C)}\left(\frac{n}{2}-s\right)$$

and the residue of $Z_{\Lambda(C)}(s)$ at the pole $s = n/2$ is $(\pi/2)^{n/2} / \Gamma(n/2)$.

Proof. Let us substitute Eq. (2.6) into $Z_{\Lambda(C)}(s)$. We then have

$$Z_{\Lambda(C)}(s) = \sum_{i=0}^{n} W_i Z_{n,i}(s) = \begin{pmatrix} W_0 & W_1 & \cdots & W_n \end{pmatrix} \begin{pmatrix} Z_{n,0}(s) \\ Z_{n,1}(s) \\ \vdots \\ Z_{n,n}(s) \end{pmatrix}$$

$$= \frac{1}{2^{n-k}} \begin{pmatrix} W_0^{\perp} & W_1^{\perp} & \cdots & W_n^{\perp} \end{pmatrix} M_n \begin{pmatrix} Z_{n,0}(s) \\ Z_{n,1}(s) \\ \vdots \\ Z_{n,n}(s) \end{pmatrix}. \tag{4.5}$$

Multiplying both sides of Eq. (4.5) by $(\pi/2)^{-s} \Gamma(s) / \sqrt{2^k}$ yields

(4.6)

$$\left(\frac{\pi}{2}\right)^{-s} \Gamma(s) \frac{1}{\sqrt{2^k}} Z_{\Lambda(C)}(s) = \left(\frac{\pi}{2}\right)^{-s} \Gamma(s) \frac{1}{\sqrt{2^k}} \frac{1}{2^{n-k}} \begin{pmatrix} W_0^{\perp} & W_1^{\perp} & \cdots & W_n^{\perp} \end{pmatrix} M_n \begin{pmatrix} Z_{n,0}(s) \\ Z_{n,1}(s) \\ \vdots \\ Z_{n,n}(s) \end{pmatrix}.$$

Using Eq. (4.2), we can rewrite the right-hand side of Eq. (4.6) as

$$\left(\frac{\pi}{2}\right)^{-(n/2-s)} \Gamma\left(\frac{n}{2}-s\right) \frac{1}{\sqrt{2^{n-k}}} \begin{pmatrix} W_0^{\perp} & W_1^{\perp} & \cdots & W_n^{\perp} \end{pmatrix} \begin{pmatrix} Z_{n,0}\left(\frac{n}{2}-s\right) \\ Z_{n,1}\left(\frac{n}{2}-s\right) \\ \vdots \\ Z_{n,n}\left(\frac{n}{2}-s\right) \end{pmatrix}$$

$$= \left(\frac{\pi}{2}\right)^{-(n/2-s)} \Gamma\left(\frac{n}{2}-s\right) \frac{1}{\sqrt{2^{n-k}}} \sum_{j=0}^{n} W_j^{\perp} Z_{n,j}\left(\frac{n}{2}-s\right)$$

$$= \left(\frac{\pi}{2}\right)^{-(n/2-s)} \Gamma\left(\frac{n}{2}-s\right) \frac{1}{\sqrt{2^{n-k}}} Z_{\Lambda(C^{\perp})}\left(\frac{n}{2}-s\right). \tag{4.7}$$

The residue of $Z_{\Lambda(C)}(s)$ at the pole $s = n/2$ is equal to the sum of those of $Z_{n,j}(s)$. Therefore the residue of $Z_{\Lambda(C)}(s)$ at the pole $s = n/2$ is $2^k (\pi/4)^{n/2} / \Gamma(n/2)$. In particular, if in Eq. (4.7) $C = C^{\perp}$, the equation becomes

$$\left(\frac{\pi}{2}\right)^{-s} \Gamma(s) Z_{\Lambda(C)}(s) = \left(\frac{\pi}{2}\right)^{-(n/2-s)} \Gamma\left(\frac{n}{2}-s\right) Z_{\Lambda(C)}\left(\frac{n}{2}-s\right).$$

Put $k = n/2$ into $2^k (\pi/4)^{n/2} / \Gamma(n/2)$, then the residue of $Z_{\Lambda(C)}(s)$ at the pole $s = n/2$ is $(\pi/2)^{n/2} / \Gamma(n/2)$. □

Example 4.6. The Hamming weight enumerator of C_7 is $W_{C_7}(x_0, x_1) = x_0^7 + 7x_0^4x_1^3 + 7x_0^3x_1^4 + x_1^7$. Then, for all $s \in \mathbb{C}$ with $\Re s > 7/2$, the partial Epstein zeta function on C_7 is

$$Z_{\Lambda(C_7)}(s) = Z_{7,0}(s) + 7Z_{7,3}(s) + 7Z_{7,4}(s) + Z_{7,7}(s).$$

On the other hand, the partial Epstein zeta function on the dual code $C_7^\perp$ is

$$Z_{\Lambda(C_7^\perp)}(s) = Z_{7,0}(s) + 7Z_{7,4}(s),$$

because the weight enumerator of $C_7^\perp$ is $W_{C_7^\perp}(x_0, x_1) = x_0^7 + 7x_0^3x_1^4$. Two zeta functions $Z_{\Lambda(C_7)}(s)$ and $Z_{\Lambda(C_7^\perp)}(s)$ satisfy the equation

$$\left(\frac{\pi}{2}\right)^{-s} \Gamma(s) \frac{1}{\sqrt{2^4}} Z_{\Lambda(C_7)}(s) = \left(\frac{\pi}{2}\right)^{-(7/2-s)} \Gamma\left(\frac{7}{2} - s\right) \frac{1}{\sqrt{2^3}} Z_{\Lambda(C_7^\perp)}\left(\frac{7}{2} - s\right).$$

Example 4.7. The Hamming weight enumerator of the self-dual code C_8 is $W_{C_8}(x_0, x_1) = x_0^8 + 14x_0^4x_1^4 + x_1^8$. Then, for all $s \in \mathbb{C}$ with $\Re s > 4$, the partial Epstein zeta function on C_8 is

$$Z_{\Lambda(C_8)}(s) = Z_{8,0}(s) + 14Z_{8,4}(s) + Z_{8,8}(s).$$

This zeta function satisfies the functional equation

$$\left(\frac{\pi}{2}\right)^{-s} \Gamma(s)\, Z_{\Lambda(C_8)}(s) = \left(\frac{\pi}{2}\right)^{-(4-s)} \Gamma(4-s)\, Z_{\Lambda(C_8)}(4-s).$$

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