

## ON SPIRALLIKE FUNCTIONS AND ROBERTSON FUNCTIONS

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**ABSTRACT.** In this short survey paper it is shown that several crucial theorems for the theory of Löwner chains whose first coefficients are normalized by  $e^t$  can be extended for Löwner chains with complex first coefficient. As a consequence of the above consideration, several new univalence and quasiconformal extension criteria for the class of spirallike functions and Robertson functions are derived.

### 1. INTRODUCTION

Let  $\mathbb{C}$  denotes the complex plane and  $\mathbb{D}$  the open unit disk  $\{z \in \mathbb{C} : |z| < 1\}$  in the complex plane. Let  $f_t(z) = f(z, t) = \sum_{n=1}^{\infty} a_n(t)z^n$ ,  $a_1(t) \neq 0$ , be a function defined on  $\mathbb{D} \times [0, \infty)$  and analytic in  $\mathbb{D}$  for each  $t \in [0, \infty)$ , where  $a_1(t)$  is a complex-valued function on  $[0, \infty)$ . Then  $f_t(z)$  is said to be a *Löwner chain* if  $f_t(z)$  has the following properties;

1.  $f_t(z)$  is univalent in  $\mathbb{D}$  for each  $t \in [0, \infty)$ ,
2.  $a_1(t)$  is locally absolutely continuous on  $[0, \infty)$ ,  $|a_1(t)|$  is strictly increasing on  $t \in [0, \infty)$  and  $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$ ,
3.  $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$  for  $0 \leq s < t < \infty$ .

It is known that if  $f_t(z)$  is a Löwner chain then  $f_{t_n}(\mathbb{D}) \rightarrow f_{t_0}(\mathbb{D})$  if  $t_n \rightarrow t_0 \in [0, \infty)$  and  $f_{t_n}(\mathbb{D}) \rightarrow \mathbb{C}$  if  $t_n \rightarrow \infty$  in the sense of kernel convergence with respect to the origin. However the converse is not true in general. Also, if  $f_t$  is a Löwner chain then a strict inclusion relationship of the expanding image domains (i.e.  $f_s(\mathbb{D}) \subsetneq f_t(\mathbb{D})$  for  $0 \leq s < t < \infty$ ) holds. We can adopt this as the definition of Löwner chains instead of “ $|a_1(t)|$  is strictly increasing on  $[0, \infty)$ ” in the condition 2. For the precise proofs of these arguments, the reader is referred to e.g. [4, pp.136–138] or [5, pp.94–97]. If  $a_1(t) = e^t$ , then we shall say that  $f_t(z)$  is a *standard Löwner chain*. In this case the above condition 2 is superfluous.

Standard Löwner chains and several related theorems due to Pommerenke [9] and Becker [1] play a crucial role in the theory of univalent functions. In this short survey we show that those theorems work well without normalization of Löwner chains. As a

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consequence of this argument, several new univalence and quasiconformal extension criteria are obtained, which concerning with the typical subclasses of univalent functions, for instance, spirallike functions and Robertson functions.

## 2. LÖWNER CHAINS WITH COMPLEX FIRST COEFFICIENT

Unless otherwise noted, in this section we will denote a standard Löwner chain by  $h(z, t)$  and a general Löwner chain by  $f(z, t)$  for convenience.

The following necessary and sufficient condition for a standard Löwner chain has been derived by Pommerenke;

**Theorem A** ([9, 10]). *Let  $0 < r_0 \leq 1$ . Let  $h(z, t) = e^t z + \sum_{n=2}^{\infty} c_n(t) z^n$  be a function defined on  $\mathbb{D} \times [0, \infty)$ . Then the function  $h(z, t)$  is a standard Löwner chain if and only if the following two conditions are satisfied;*

- (i) *The function  $h(z, t)$  is analytic in  $z \in \mathbb{D}_{r_0}$  for each  $t \in [0, \infty)$ , absolutely continuous in  $t \in [0, \infty)$  for each  $z \in \mathbb{D}_{r_0}$  and satisfies*

$$|h(z, t)| \leq K_0 e^t \quad (z \in \mathbb{D}_{r_0}, t \in [0, \infty))$$

*for some positive constants  $K_0$ .*

- (ii) *There exists a function  $p(z, t)$  analytic in  $z \in \mathbb{D}$  for each  $t \in [0, \infty)$  and measurable in  $t \in [0, \infty)$  for each  $z \in \mathbb{D}$  satisfying*

$$\operatorname{Re} p(z, t) > 0 \quad (z \in \mathbb{D}, t \in [0, \infty))$$

*such that*

$$\dot{h}(z, t) = zh'(z, t)p(z, t) \quad (z \in \mathbb{D}_{r_0}, \text{ almost every } t \in [0, \infty))$$

*where  $\dot{h} = \partial h / \partial t$  and  $h' = \partial h / \partial z$ .*

Theorem A can be generalized for a Löwner chain which has the complex-valued first coefficient as the following form;

**Theorem 1** ([6]). *Let  $0 < r_1 \leq 1$ . Let  $f(z, t) = \sum_{n=1}^{\infty} a_n(t) z^n$ ,  $a_1(t) \neq 0$ , be a function defined on  $\mathbb{D} \times [0, \infty)$ , where  $a_1(t)$  is a complex-valued function on  $[0, \infty)$ . Then the function  $f(z, t)$  is a Löwner chain if and only if the following conditions are satisfied;*

- (i') *The function  $f(z, t)$  is analytic in  $\mathbb{D}_{r_1}$  for each  $t \in [0, \infty)$ , locally absolutely continuous in  $[0, \infty)$  for each  $z \in \mathbb{D}_{r_1}$  and satisfies  $\lim_{t \rightarrow \infty} |a_1(t)| = \infty$  and*

$$|f(z, t)| \leq K_1 |a_1(t)| \quad (z \in \mathbb{D}_{r_1}, \text{ a.e. } t \in [0, \infty))$$

*for some positive constants  $K_1$ .*

(ii') *There exists a function  $p(z, t)$  analytic in  $\mathbb{D}$  for each  $t \in [0, \infty)$  and measurable in  $[0, \infty)$  for each  $z \in \mathbb{D}$  satisfying*

$$\operatorname{Re} p(z, t) > 0 \quad (z \in \mathbb{D}, t \in [0, \infty))$$

*such that*

$$\dot{f}(z, t) = z f'(z, t) p(z, t) \quad (z \in \mathbb{D}_{r_1}, \text{ almost every } t \in [0, \infty)) \quad (1)$$

*where  $\dot{f} = \partial f / \partial t$  and  $f' = \partial f / \partial z$ .*

**Sketch of the proof of Theorem 1.** Let  $f(z, t) = \sum_{n=1}^{\infty} a_n(t) z^n$  be a Löwner chain, where  $a_1(t) \neq 0$  is a complex-valued locally absolutely continuous function on  $[0, \infty)$ . Set  $\lambda(t) := -\arg a_1(t)$  (here  $\lambda(t)$  is absolutely continuous on  $t \in [0, \infty)$ ). Let us define  $g(z, t)$  and  $h(z, t)$  as follows;

$$g(z, t) = \sum_{n=1}^{\infty} b_n(t) z^n := f(e^{i\lambda(t)} z, t) \quad (2)$$

and

$$h(z, t) := \frac{1}{|a_1(0)|} g(z, b_1^{-1}(|a_1(0)| e^t)). \quad (3)$$

We can easily see that  $f(z, t)$  is a Löwner chain if and only if  $h(z, t)$  is a standard Löwner chain. Also it can be proved that the reparametrization (2) and (3) preserve the other properties of the sufficient part of Theorem 1 and Theorem A with  $K_1 = K_0/|a_1(0)|^2$  and  $r_0 = r_1$ .  $\square$

For the precise proof, see [6]. The reader can refer also [2] which contains similar arguments as above.

We shall also see that a quasiconformal extension criterion for a standard Löwner chain due to Becker [1] is extended for a Löwner chain with the complex-valued first coefficient as following. Here, a sense-preserving homomorphism  $f$  of  $G \subset \mathbb{C}$  is called *k-quasiconformal* if  $f_z$  and  $f_{\bar{z}}$ , the partial derivatives in  $z$  and  $\bar{z}$  in the distributional sense, are locally integrable on  $G$  and satisfy  $|f_{\bar{z}}| \leq k|f_z|$  almost everywhere in  $G$ , where  $k \in [0, 1)$ .

**Theorem B** ([2, 3, 6]). *Suppose that  $f_t(z) = f(z, t)$  is a Löwner chain for which  $p(z, t)$  in (1) satisfies the condition*

$$p(z, t) \in U(k) := \left\{ w \in \mathbb{C} : \left| \frac{w-1}{w+1} \right| \leq k \right\}$$

*for all  $z \in \mathbb{D}$  and almost all  $t \in [0, \infty)$ . Then  $f_t(z)$  admits a continuous extension to  $\overline{\mathbb{D}}$  for each  $t \geq 0$  and the map  $\hat{f}$  defined by*

$$\hat{f}(z) = \begin{cases} f(z, 0), & \text{if } |z| < 1, \\ f\left(\frac{z}{|z|}, \log |z|\right), & \text{if } |z| \geq 1, \end{cases}$$

is a  $k$ -quasiconformal extension of  $f_0$  to  $\mathbb{C}$ .

### 3. SPIRALLIKE FUNCTIONS

A function  $f \in \mathcal{A}$  is called  $\gamma$ -spirallike and known to be univalent if  $f$  satisfies

$$\operatorname{Re} \left\{ e^{-i\gamma} \frac{zf'(z)}{f(z)} \right\} > 0$$

for a real number  $\gamma \in (-\pi/2, \pi/2)$  in  $\mathbb{D}$ . We denote such a class of functions by  $\mathcal{SP}(\gamma)$ . If  $\gamma = 0$  then  $\mathcal{SP}(0)$  is the well known class of starlike functions. By constructing a Löwner chain without normalization on the first derivative and applying Theorem 1 and Theorem B we have the following new quasiconformal extension criterion;

**Theorem 2** ([6]). *Let  $\gamma \in (-\pi/2, \pi/2)$  and  $k \in [|\tan(\gamma/2)|, 1)$ . For  $f \in \mathcal{A}$ , if*

$$e^{-i\gamma} \frac{zf'(z)}{f(z)} \in U(k)$$

*for all  $z \in \mathbb{D}$ , then  $f$  has a  $k$ -quasiconformal extension to  $\mathbb{C}$ , where  $U(k)$  is a disk defined in Theorem B.*

*Proof.* Let  $c$  be a complex constant with  $\operatorname{Re} c > 0$ . If we set

$$f_t(z) = e^{ct} f(z), \quad (4)$$

then  $\lim_{t \rightarrow \infty} |f'_t(0)| = \lim_{t \rightarrow \infty} |e^{ct}| = \infty$  since  $\operatorname{Re} c > 0$ . Therefore we obtain our theorem if we put  $c = e^{i\gamma}$  and apply Theorem 1 and Theorem B to (4).  $\square$

It is known [10] that a standard Löwner chain which corresponds to  $\gamma$ -spirallike functions is

$$h_t(z) = e^{(1-i\tan\gamma)t} f(e^{i\tan\gamma t} z). \quad (5)$$

The standard chain (5) with Theorem A and Theorem B derive another quasiconformal extension criterion from Theorem 2;

**Proposition 3** ([6]). *Let  $\gamma \in (-\pi/2, \pi/2)$  and  $k \in [0, 1)$ . For  $f \in \mathcal{A}$ , if*

$$\frac{zf'(z)}{f(z)} \in U(\gamma, k)$$

*for all  $z \in \mathbb{D}$ , then  $f$  has a  $k$ -quasiconformal extension to  $\mathbb{C}$ , where  $U(\gamma, k)$  is the hyperbolic disk in the tilted half plane  $\{z \in \mathbb{C} : \operatorname{Re} e^{-i\gamma} z > 0\}$  centered at 1 with radius  $\operatorname{arctanh} k$ ,  $0 \leq k < 1$ , i.e.,*

$$U(\gamma, k) = \left\{ w \in \mathbb{C} : \left| w - \frac{1 + e^{2i\gamma} k^2}{1 - k^2} \right| \leq \frac{2k \cos \gamma}{1 - k^2} \right\}.$$

## 4. ROBERTSON FUNCTIONS

A function  $f \in \mathcal{A}$  is said to be  $\gamma$ -Robertson ( $\gamma \in (-\pi/2, \pi/2)$ ) if  $f$  satisfies

$$\operatorname{Re} \left\{ e^{-i\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} > 0$$

for all  $z \in \mathbb{D}$ . This class of functions was introduced by Robertson [11] in 1969. Let  $\mathcal{R}(\gamma)$  be the set of those functions. The definition of  $\gamma$ -Robertson functions shows immediately that  $\mathcal{R}(0)$  is precisely the class of convex functions. The class  $\mathcal{R}(\lambda)$  has been investigated by various authors. The reader can be referred to e.g. [7] and the references therein.

In contrast to the case of spirallike functions, the class  $\mathcal{R}(\gamma)$  is not always contained in  $\mathcal{S}$  for any  $\gamma$ ,  $|\gamma| \in [0, \pi/2)$ . In fact, if  $0 < \cos \lambda \leq 1/2$  or  $\cos \lambda = 1$  then  $\mathcal{R}(\lambda) \subset \mathcal{S}$  and otherwise  $\mathcal{R}(\gamma) \not\subset \mathcal{S}$  ([11, 8]). By using the theory of Löwner chains it can be obtained another proof for univalence of  $\mathcal{R}(\gamma)$ . A Löwner chain for  $\gamma$ -Robertson functions is given by

$$f_t(z) = f(e^{-t}z) - e^{-2i\gamma}(e^{2t} - 1)e^{-t}zf'(e^{-t}z)$$

and Theorem 1 shows that if  $f \in \mathcal{R}(\gamma)$  with  $\gamma = 0$  or  $\gamma \in [\pi/3, \pi/2)$  then  $f$  is univalent ([7]).

Lastly, we shall introduce the following quasiconformal extension criterion which is proved by essentially making use of Löwner chains with complex-valued first coefficient;

**Theorem 4** ([7]). *Let  $f \in \mathcal{A}$ ,  $k \in [0, 1)$  and  $\gamma \in (-\pi/2, \pi/2)$ ,  $q > -1$  be related by*

$$0 < \cos \gamma \leq \begin{cases} k/2, & \text{if } -1 < q \leq 0, \\ k/(2 + 4q), & \text{if } 0 < q. \end{cases}$$

*If  $f$  satisfies*

$$\operatorname{Re} \left\{ e^{-i\gamma} \left( 1 + \frac{zf''(z)}{f'(z)} + q \frac{zf'(z)}{f(z)} \right) \right\} > 0$$

*for all  $z \in \mathbb{D}$ , then  $f \in \mathcal{S}(k)$ . If, in addition,  $f''(0) = 0$ , (4) can be replaced by*

$$0 < \cos \gamma \leq \begin{cases} k, & \text{if } -1 < q \leq 0, \\ k/(1 + 2q), & \text{if } 0 < q. \end{cases}$$

We note that the case  $q = 0$  claims a quasiconformal extension criterion for  $\gamma$ -Robertson functions.

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