

ENDO-TRIVIAL MODULES AND WEAK HOMOMORPHISMS

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ABSTRACT. This is a report on some recent joint work with Jon F. Carlson, which appears in [CT4]. It is an expanded version of a talk given at the RIMS workshop *Cohomology of finite groups and related topics*, February 18–20, 2015.

1. INTRODUCTION

Endo-trivial modules for a finite group G over a field k of prime characteristic p play a significant role in modular representation theory. They have been classified in case G is a p -group [CT2, CT3] and various results have appeared since for some specific families of groups [CHM, CMN1, CMN2, CMN3, CMT1, CMT2, CMT3, LM2, Ma, MT, NR]. Another line of research is concerned with the classification of all endo-trivial modules which are simple [Ro, LMS, LM1].

The abelian group $T(G)$ is finitely generated and its torsion-free part is essentially known (see [CMT3] for details). The question remains of describing its torsion subgroup, written $TT(G)$. Let S be a Sylow p -subgroup of G , which we suppose nontrivial. The subgroup

$$K(G) = \text{Ker} \{ \text{Res}_S^G : T(G) \longrightarrow T(S) \}$$

is easily seen to be finite and it is known to be equal to the whole torsion subgroup $TT(G)$ in most cases. Specifically, this happens whenever S is not cyclic, generalized quaternion, or semi-dihedral (because, if we exclude these three cases, then $T(S)$ is torsion-free by [CT3]). The excluded cases are treated in [CMT2], [Ka] and [MT].

The problem is now to describe $K(G)$ and this uses a new approach, introduced by Balmer in [Ba]. He shows that $K(G)$ is isomorphic to the group $A(G)$ of all weak homomorphisms $G \rightarrow k^\times$ (defined below in Section 3). One can then use $A(G)$ instead of $K(G)$. Let $N := N_G(S)$ be the normalizer of S . It is not hard to show that $A(G)$ embeds via restriction into $A(N)$ and the problem is to describe its image. Moreover, because N has a nontrivial normal p -subgroup, it turns out that every weak homomorphism $N \rightarrow k^\times$ is actually an ordinary group homomorphism, so that $A(N) = \text{Hom}(N, k^*)$, the abelian group of all one-dimensional representations of N .

The main difficulty is to describe the image of the injective map

$$\text{Res}_N^G : A(G) \longrightarrow A(N) = \text{Hom}(N, k^*).$$

By duality of abelian groups, this image is of the form $\text{Hom}(N/J, k^*)$ for some normal subgroup J such that N/J is abelian and of order prime to p . In other words, the image is the dual group $(N/J)^*$ of the abelian group N/J . So the problem of describing $K(G)$ comes down to the following specific question : What is J ?

A system of subgroups $\{\rho^i(S)\}$ was introduced in [CMN3] (for solving the problem when G is a general linear group) and was further considered in [CT4] for an arbitrary finite group G (see Section 4 for the definition). We have a nested sequence of subgroups

$$S \subseteq \rho^1(S) \subseteq \rho^2(S) \subseteq \dots \subseteq N = N_G(S),$$

and we let $\rho^\infty(S)$ be the limit of the system, namely the union of all $\rho^i(S)$.

The following theorem was proved in [CT4] :

Theorem 1.1. *With the notation above, $\rho^\infty(S) \subseteq J$.*

The question of equality is open and is stated as a conjecture in [CT4] :

Conjecture 1.2. $J = \rho^\infty(S)$.

Finally, the main theorem of [CT4] gives a positive answer in a special case :

Theorem 1.3. *If a Sylow p -subgroup S of G is abelian, then $J = \rho^2(S) = \rho^\infty(S)$. In other words, $K(G)$ is isomorphic to the dual group $(N_G(S)/\rho^2(S))^*$ of the abelian group $N_G(S)/\rho^2(S)$.*

In view of the direct description of the subgroup $\rho^2(S)$, this theorem completes the classification of all torsion endo-trivial modules when a Sylow p -subgroup is abelian.

2. RESTRICTION OF ENDO-TRIVIAL MODULES

Let k denote an algebraically closed field of prime characteristic p and let G be a finite group. We assume that G has order divisible by p and we let S be a Sylow p -subgroup of G . Recall that a kG -module M is *endo-trivial* if its endomorphism algebra $\text{End}_k(M)$ is isomorphic (as a kG -module) to the direct sum of the trivial module k and a projective kG -module. In other words, M is endo-trivial if and only if $\text{Hom}_k(M, M) \cong M^\vee \otimes M \cong k \oplus (\text{proj})$, where $M^\vee$ denotes the k -dual of M . Any endo-trivial module M splits as the direct sum $M = M_0 \oplus (\text{proj})$ for an indecomposable endo-trivial kG -module M_0 , which is unique up to isomorphism. We let $T(G)$ be the set of equivalence classes of endo-trivial kG -modules for the equivalence relation

$$M \sim L \iff M_0 \cong L_0.$$

The tensor product induces an abelian group structure on the set $T(G)$. The identity element is the class of the trivial module, while the inverse of the class of a module M is the class of the dual module $M^\vee$. By a theorem of Puig, the group $T(G)$ is known to be a finitely generated abelian group.

A useful fact is the following.

Lemma 2.1. *Let M be a kG -module.*

- (a) *M is endo-trivial if and only if $M \downarrow_E^G$ is endo-trivial for every elementary abelian p -subgroup E of G .*
- (b) *If M satisfies the condition $M \downarrow_S^G \cong k \oplus (\text{proj})$, where S is a Sylow p -subgroup of G , then M is endo-trivial.*

The proof of (a) uses Chouinard's theorem. It appears in Lemma 2.9 of [CT1] for p -groups, but the proof is the same for any finite group. Statement (b) follows immediately from (a).

We now let $K(G)$ be the kernel of the restriction map $\text{Res}_S^G : T(G) \rightarrow T(S)$, where S is a Sylow p -subgroup of G . In other words, the class of an endo-trivial kG -module M belongs to $K(G)$ if and only if M has *trivial Sylow restriction*, that is, $M \downarrow_S^G \cong k \oplus (\text{proj})$. This implies in particular that, if M is indecomposable, then M has vertex S and trivial source. Note also that if a kG -module M satisfies the condition $M \downarrow_S^G \cong k \oplus (\text{proj})$, then M is necessarily endo-trivial by Lemma 2.1, and its class lies in $K(G)$.

Lemma 2.2. *Let $K(G)$ be the kernel of the restriction map $\text{Res}_S^G : T(G) \rightarrow T(S)$.*

- (a) *$K(G)$ is a finite subgroup of $T(G)$.*
- (b) *$K(G)$ is the entire torsion subgroup $TT(G)$ of $T(G)$, provided S is not cyclic, generalized quaternion, or semi-dihedral.*

This is proved in Lemma 2.3 of [CMT1]. The first part is easy and is due to the fact that there are finitely many indecomposable kG -modules with trivial source. The second statement is much deeper and depends on the fact that, by the main result of [CT3], $T(S)$ is torsion-free if S is not cyclic, generalized quaternion, or semi-dihedral.

We now introduce some notation. For any finite group H , we let H' be the smallest normal subgroup of H such that H/H' is an abelian p' -group. In other words $H' = [H, H]S$ is the subgroup of H generated by the commutator subgroup $[H, H]$ and by a Sylow p -subgroup S of H . Clearly, H' is in the kernel of any group homomorphism $H \rightarrow k^\times$, where $k^\times$ denotes the group of nonzero elements of k . Since k contains all p' -roots of unity (because k is algebraically closed by assumption), H' is actually the intersection of the kernels of all group homomorphisms $H \rightarrow k^\times$. In other words, $\text{Hom}(H, k^\times) \cong (H/H')^*$, the dual group of the abelian group H/H' .

The next result is a straightforward application of the Mackey formula. The details appear in Lemma 2.6 of [MT].

Lemma 2.3. *Suppose that a finite group H has a nontrivial normal p -subgroup. If M is an indecomposable kH -module with trivial Sylow intersection (so that M is endo-trivial and its class is in $K(H)$), then M has dimension one. In other words $K(H) \cong \text{Hom}(H, k^\times) \cong (H/H')^*$.*

Our next result is an easy application of the Green correspondence. For details, see Proposition 2.6 in [CMN1].

Lemma 2.4. *Let S be a Sylow p -subgroup of G and let $N = N_G(S)$.*

- (a) *The restriction map $\text{Res}_N^G : T(G) \rightarrow T(N)$ is injective, induced by the Green correspondence.*
- (b) *In particular, the restriction map $\text{Res}_N^G : K(G) \rightarrow K(N)$ is injective.*

By Lemma 2.3, we know that $K(N)$ consists of the classes of all one-dimensional representations of N . The main problem is to know which of them are in the image

of the restriction map from $K(G)$. In other words, given a one-dimensional kN -module U , we need to know when its Green correspondent M is endo-trivial.

Definition 2.5. We define J to be the intersection of the kernels of all the one-dimensional kN -modules U such that the Green correspondent of U is an endo-trivial kG -module. Thus J is a subgroup of N , which can also be characterized as the intersection of the kernels of all the one-dimensional kN -modules U whose class lies in the image of the restriction $\text{Res}_N^G : T(G) \rightarrow T(N)$.

In other words, we obtain the following.

Lemma 2.6. The image of the restriction map $\text{Res}_N^G : K(G) \rightarrow K(N)$ is equal to $(N/J)^* \cong \text{Hom}(N/J, k^\times)$, as a subgroup of $\text{Hom}(N, k^\times) \cong K(N)$. In other words, $K(G) \cong (N/J)^* \cong \text{Hom}(N/J, k^\times)$.

We see that the problem of characterizing the group $K(G)$ is equivalent to the question of finding the subgroup J .

3. WEAK HOMOMORPHISMS

In [Ba], Balmer provided a new characterization of the group $K(G)$ in terms of the group of weak homomorphisms. As above, S denotes a Sylow p -subgroup of G .

Definition 3.1. A map $\chi : G \rightarrow k^\times$ is called a weak homomorphism if it satisfies the following three conditions:

- (a) If $s \in S$, then $\chi(s) = 1$.
- (b) If $g \in G$ and $S \cap {}^gS = \{1\}$, then $\chi(g) = 1$.
- (c) If $a, b \in G$ and if $S \cap {}^aS \cap {}^{ab}S \neq \{1\}$, then $\chi(ab) = \chi(a)\chi(b)$.

The set $A(G)$ of all weak homomorphisms is an abelian group under the usual product of maps.

Theorem 3.2. (Balmer [Ba]) The groups $K(G)$ and $A(G)$ are isomorphic.

Balmer's isomorphism is explicit and is described in [Ba]. Things become easy for a group H having a nontrivial normal p -subgroup, thanks to the third condition in Definition 3.1.

Lemma 3.3. Suppose that a finite group H has a nontrivial normal p -subgroup. Then every weak homomorphism $\chi : H \rightarrow k^\times$ is a group homomorphism.

Consequently, we obtain Balmer's isomorphism in this special case :

$$A(H) \cong \text{Hom}(H, k^\times) \cong (H/H')^* \cong K(H).$$

In view of Balmer's isomorphism, Lemma 2.4 and Lemma 2.6 can now be restated.

Lemma 3.4. Let S be a Sylow p -subgroup of G and let $N = N_G(S)$.

- (a) The restriction map $\text{Res}_N^G : A(G) \rightarrow A(N)$ is injective.
- (b) The image of the restriction map $\text{Res}_N^G : A(G) \rightarrow A(N)$ is equal to $(N/J)^* = \text{Hom}(N/J, k^\times)$, as a subgroup of $\text{Hom}(N, k^\times) = A(N)$.

It would be interesting to have a direct proof of (a), using only the definition of weak homomorphisms. Note that we still face the problem of finding what is the subgroup J .

4. A SYSTEM OF LOCAL SUBGROUPS

For any nontrivial subgroup Q of a Sylow p -subgroup S , we define a sequence of subgroups $\{\rho^i(Q) \mid i \geq 1\}$ inductively as follows :

$$\rho^1(Q) := N_G(Q)'.$$

As before, $N_G(Q)'$ is the product of the commutator subgroup of $N_G(Q)$ and a Sylow p -subgroup of $N_G(Q)$. Note that $Q \subseteq \rho^1(Q) \subseteq N_G(Q)$. For $i \geq 2$, we let

$$\rho^i(Q) := \langle N_G(Q) \cap \rho^{i-1}(R) \mid \{1\} \neq R \subseteq S \rangle,$$

the subgroup generated by all the subgroups $N_G(Q) \cap \rho^{i-1}(R)$, for all nontrivial subgroups R of S . This contains $\rho^{i-1}(Q)$, so we have a nested sequence of subgroups

$$Q \subseteq \rho^1(Q) \subseteq \rho^2(Q) \subseteq \rho^3(Q) \subseteq \dots \subseteq N_G(Q).$$

Since G is finite, the sequence eventually stabilizes and we let $\rho^\infty(Q)$ be the limit subgroup of the sequence $\{\rho^i(Q) \mid i \geq 1\}$, namely their union.

The following observation was made in [CMN3].

Proposition 4.1. *Let $\chi : G \rightarrow k^\times$ be a weak homomorphism. If $x \in \rho^i(Q)$ for some $i \geq 1$ and for some nontrivial subgroup $Q \subseteq S$, then $\chi(x) = 1$.*

In the case that $i = 1$, the statement is a trivial consequence of Lemma 3.3 applied to $H = N_G(Q)$. Then the proposition is proved by induction (see Proposition 4.1 in [CT4] for details).

Applying this to the case of the Sylow p -subgroup S , we now obtain the following theorem about the subgroup J defined in 2.5.

Theorem 4.2. $\rho^\infty(S) \subseteq J$.

Proof. Recall that J is the intersection of the kernels of all the one-dimensional kN -modules U such that the Green correspondent of U is an endo-trivial kG -module. Let U be one of them and let M be its Green correspondent. Then M is endo-trivial and $M \downarrow_N^G \cong U \oplus (\text{proj})$. In order to translate this information in terms of weak homomorphisms, we let $\chi : G \rightarrow k^\times$ be the weak homomorphism corresponding to the class of M under Balmer's isomorphism $K(G) \cong A(G)$. Then $\chi|_N : N \rightarrow k^\times$ is a homomorphism (by Lemma 3.3) and this corresponds to the one-dimensional kN -module U under Balmer's isomorphism $K(N) \cong \text{Hom}(N, k^\times) = A(N)$. By Proposition 4.1 above, χ vanishes on $\rho^\infty(S)$ and therefore

$$\rho^\infty(S) \subseteq \text{Ker}(\chi|_N) = \text{Ker}(U).$$

This holds for every U as above, so $\rho^\infty(S)$ is contained in the intersection of the corresponding kernels, that is, $\rho^\infty(S) \subseteq J$. $\square$

Theorem 4.2 is essentially proved in [CT4], except that the result is stated in the following equivalent form (see Theorem 4.3 in [CT4]).

Theorem 4.3. *Suppose that M is a kG -module with trivial Sylow restriction, i.e. $M\downarrow_S^G \cong k \oplus (\text{proj})$. Then $M\downarrow_{\rho^\infty(S)}^G \cong k \oplus (\text{proj})$.*

Proof. Since $M\downarrow_S^G \cong k \oplus (\text{proj})$, the module M must be endo-trivial, by Lemma 2.1. Then we have $M\downarrow_N^G \cong U \oplus (\text{proj})$ for some indecomposable endo-trivial kN -module U . By Lemma 2.3, U has dimension 1. By Theorem 4.2, $\rho^\infty(S) \subseteq \text{Ker}(U)$ and therefore $U\downarrow_{\rho^\infty(S)}^N \cong k$. It follows that

$$M\downarrow_{\rho^\infty(S)}^G = M\downarrow_N^G \downarrow_{\rho^\infty(S)}^N \cong (U \oplus (\text{proj}))\downarrow_{\rho^\infty(S)}^N \cong k \oplus (\text{proj}),$$

as required. $\square$

5. A CONJECTURE

We conjecture that the inclusion in Theorem 4.2 is an equality. This is Conjecture 5.5 in [CT4].

Conjecture 5.1. $J = \rho^\infty(S)$.

Evidence for this conjecture is based on numerous examples (see Section 8 in [CT4]), as well as on the following positive answer in the special case when G has an abelian Sylow p -subgroup.

Theorem 5.2. *Suppose that a Sylow p -subgroup S of G is abelian. Let $N = N_G(S)$.*

- (a) *The image of the restriction map $\text{Res}_N^G : A(G) \rightarrow A(N)$ consists exactly of all group homomorphisms $N_G(S) \rightarrow k^\times$ having $\rho^2(S)$ in their kernel.*
- (b) $K(G) \cong A(G) \cong (N_G(S)/\rho^2(S))^*$.
- (c) $J = \rho^2(S) = \rho^\infty(S)$.

This theorem is proved in [CT4]. Note that (b) and (c) follow immediately from (a), using Lemma 2.6 and Lemma 3.4. Thus the important part is the proof of (a). Starting from a group homomorphism $N_G(S) \rightarrow k^\times$ having $\rho^2(S)$ in its kernel, one has to extend it to a weak homomorphism $G \rightarrow k^\times$. Thanks to the assumption that S is abelian, this is made possible by using an explicit form of the fact that $N_G(S)$ controls fusion (Burnside's theorem). The property that S is abelian is also used in the fact that, for any subgroup Q of S , the group S is contained in $N_G(Q)$, allowing for a Frattini argument. We refer to [CT4] for more details.

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