

On the domain of a Schrödinger operator with complex potential – Old and New –

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The aim of this talk is to review and compare the spectral properties of (the closed extension of) $-\Delta + U$ ($U \geq 0$) and $-\Delta + iV$ in $L^2(\mathbb{R}^d)$ for C^∞ potentials U or V with polynomial behavior.

The case with magnetic field is also considered. More precisely, we would like to compare the criteria for:

- essential selfadjointness (**esa**) or maximal accretivity (**maxacc**)
- Compactness of the resolvent.
- Maximal inequalities,

for these operators.

By L^2 -maximal inequalities, we mean the existence of $C > 0$ s. t.

$$\|u\|_{H^2}^2 + \|Uu\|_{L^2}^2 \leq C \left(\|(-\Delta + U)u\|_{L^2}^2 + \|u\|_{L^2}^2 \right), \quad \forall u \in C_0^\infty(\mathbb{R}^d), \quad (0.1)$$

or

$$\|u\|_{H^2}^2 + \|Vu\|_{L^2}^2 \leq C \left(\|(-\Delta + iV)u\|_{L^2}^2 + \|u\|_{L^2}^2 \right), \quad \forall u \in C_0^\infty(\mathbb{R}^d). \quad (0.2)$$

We will also discuss the magnetic case:

$$P_{\mathbf{A},V} = -\Delta_A + W := \sum_{j=1}^d (D_{x_j} - A_j(x))^2 + W(x),$$

(with $W = U + iV$) and the notion of maximal regularity is expressed in terms of the magnetic Sobolev spaces:

$$\begin{aligned} & \| (D - \mathbf{A})u \|_{L^2(\mathbb{R}^d, \mathbb{C}^d)}^2 \\ & + \sum_{j,\ell} \| (D_j - A_j)(D_\ell - A_\ell)u \|_{L^2(\mathbb{R}^d)}^2 \\ & + \| |W|u \|_{L^2(\mathbb{R}^d)}^2 \\ & \leq C \left(\|P_{\mathbf{A},W}u\|_{L^2(\mathbb{R}^d)}^2 + \|u\|_{L^2(\mathbb{R}^d)}^2 \right), \end{aligned} \quad (0.3)$$

The question of analyzing $-\Delta + iV$ or more generally $P_{\mathbf{A},iV} := -\Delta_A + iV$ appears in many situations:

- Time dependent Ginzburg-Landau theory leads for example to the spectral analysis of

$$D_x^2 + (D_y - \frac{x^2}{2})^2 + iy$$

Here $\text{curl } \mathbf{A} = x$ vanishes along a line.

- Control theory
- Bloch-Torrey (complex Airy) equation

$$-\Delta + ix$$

- Spectral analysis of the complex harmonic oscillator.

Moreover, in some of the applications, V does not satisfy necessarily a sign condition $V \leq 0$ as for dissipative systems.

After reviewing all the main results devoted to this question in the selfadjoint case, we will show that similar results can be proved in the case of a complex potential. These recent results have been obtained in collaboration with Y. Almog and J. Nourrigat.

Below, we give a selected non exhaustive bibliography.

References

- [1] Y. Almog, D. Grebenkov, and B. Helffer. On a Schrödinger operator with a purely imaginary potential in the semiclassical limit. ArXiv (2017).
- [2] Y. Almog and B. Helffer. On the spectrum of non-selfadjoint Schrödinger operators with compact resolvent. Comm. in PDE 40 (8) (2015), pp. 1441–1466.
- [3] P. Auscher and B. Ben Ali. Maximal inequalities and Riesz transform on L^p space for Schrödinger operators with non negative potentials. Ann. Inst. Fourier 57 (6) (2007) 1975–2013.
- [4] Y. Avron, I. Herbst, and B. Simon. Schrödinger operators with magnetic fields I. General interactions. Duke Math. J. 45 (4), p. 847–883 (1978).
- [5] D. Guibourg. Inégalités maximales pour l'opérateur de Schrödinger. CRAS 316 (1993), 249–252.
- [6] B. Helffer and A. Mohamed. *Sur le spectre essentiel des opérateurs de Schrödinger avec champ magnétique.* Ann. Inst. Fourier 38(2), pp. 95–113 (1988).
- [7] B. Helffer and J. Nourrigat. *Hypoellipticité Maximale pour des Opérateurs Polynômes de Champs de Vecteurs.* Progress in Mathematics, Birkhäuser, Vol. 58 (1985).
- [8] B. Helffer and J. Nourrigat. On the domain of a magnetic Schrödinger operator with complex electric potential. In preparation.

- [9] T. Ikebe and T. Kato. Uniqueness of the self-adjoint extension of singular elliptic differential operators. Arch. for Rat. Mech. and Anal. 9 (1962), 77–92.
- [10] A. Iwatsuka. Magnetic Schrödinger operators with compact resolvent. J. Math. Kyoto Univ. 26, p. 357-374 (1986).
- [11] Z. Shen. Estimates in L^p for magnetic Schrödinger operators. Indiana Univ. Math. J., 45 (1996), pp. 817-841.
- [12] B. Simon. Tosio Kato's work on non-relativistic quantum mechanics. This conference.