

On The Symplectic Lazard Ring

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0. Introduction

In 'Elementary proofs of some results of cobordism theory using Steenrod operations' (Advances in Math., 7 (1971), 29-56.), D.Quillen determined complex cobordism ring MU_* using the formal group theory. This method is not applicable directly for the symplectic case.

However there are some works along ~~in~~ⁱⁿ this line. Espetially, Buhstaber-Novikov studied two-valued formal groups and gave some applications to symplectic cobordism ring MSp_* .

We will define symplectic formal system using formal power series like as (two-valued) formal group, and construct a geometrical example of symplectic formal system. To construct this geometrical example, We need some stable maps between the complex (or symplectic) projective and quasiprojective space.

Moreover, we can construct a ring associated with symplectic formal system. We denote the symplectic Lazard ring $LMSp$ as the associated ring for the universal symplectic formal system.

Then, we can construct a homomorphism $\theta: LMSp \rightarrow MSp_*/\text{Torsion}$. By some calculations and the result of R.Ōkita ('On the MSp Hattori-Stong problem', Osaka J. math. 13 (1976), 547-566.), we can conclude that if we apply the rational indecomposable functor $Q()$, then $Q(\theta)$ is an isomorphism.

1. Stable maps

There is a symplectification map $q : \mathbb{CP}^\infty \longrightarrow \mathbb{HP}^\infty$.

Since q is a fibre bundle whose fibre is S^2 , there is a Becker-Gottlieb transfer $t : \mathbb{HP}_+^\infty \xrightarrow{(s)} \mathbb{CP}_+^\infty$.

Let F be $\mathbb{C}$ or $\mathbb{H}$ and S_F^n unit sphere in F^n .

Let $G_n(\mathbb{C}) = U(n)$ and $G_n(\mathbb{H}) = \text{Sp}(n)$.

The quasiprojective space $Q_n(F)$ is defined to be the space of generalized reflections, that is, the image of

$$\phi : S_F^n \times S_F^1 \longrightarrow G_n(F)$$

where $\phi(u, q)$ is the automorphism which leaves v fixed if $\langle u, v \rangle = 0$ and sends u to uq .

We may define $Q_n(F)$ as the space obtained $S_F^n \times S_F^1$ by imposing the equivalence relation $(u, q) \sim (ug, g^{-1}qg)$ ($g \in S_F^1$), and collapsing $S_F^n \times 1$ to a point.

By the second definition, we can easily show that $Q_n(\mathbb{C}) \approx \Sigma(\mathbb{CP}_+^{n-1})$.

We put $\widehat{\mathbb{CP}}^n = Q_n(\mathbb{C})$ and $\widehat{\mathbb{HP}}^n = Q_n(\mathbb{H})$. Clearly

we have a symplectification map $\tilde{q} : \widehat{\mathbb{CP}}^\infty \longrightarrow \widehat{\mathbb{HP}}^\infty$.

Now we construct a map from $\widehat{\mathbb{HP}}^n$ to $\widehat{\mathbb{CP}}^{2n}$.

Let $z \in \mathbb{H}^n$ and $z = x + jy$ where $x, y \in \mathbb{C}^n$.

We denote complexification map $c : \mathbb{H}^n \longrightarrow \mathbb{C}^{2n}$ by setting $c(z) = x \oplus y \in \mathbb{C}^{2n}$.

Let $q = a + jb \in \mathbb{H}$ where $a, b \in \mathbb{C}$. Since S_C^1 is a maximal torus of S_H^1 , there is a $g \in S_H^1$ such that $g^{-1}qg \in S_C^1$. If $g^{-1}qg = e^{i\pi t}$, where $-1 < t < 0$, then $(gj)^{-1}qgj = e^{-i\pi t}$.

Thus there is a $g \in S_H^1$ such that $g^{-1}qg = e^{i\pi t}$ where $0 \leq t \leq 1$.

So a representative element of $\widetilde{HP}^n$ can be taken as $(x + jy, e^{i\pi t})$ where $x, y \in \mathbb{C}^n$ and $0 \leq t \leq 1$.

We define $\tilde{t}_n : \widetilde{HP}^n \rightarrow \widetilde{CP}^{2n}$ by the equation $\tilde{t}_n[(x + jy, e^{i\pi t})] = [(x \oplus y, e^{2i\pi t})]$.

Then the following proposition holds.

Proposition. The diagram

$$\begin{array}{ccc} \widetilde{HP}^n & \xrightarrow{\tilde{t}_n} & \widetilde{CP}^{2n} \\ \downarrow j & & \downarrow j \\ SP(n) & \xrightarrow{c} & U(2n) \end{array} \quad \text{commutes up to homotopy.}$$

By the theorem of Becker-Segal,

$Q(HP^\infty) \simeq BSp \times F$ ~~as an infinite loop space~~ where $Q(\)$ is a stabilize functor $\varinjlim_n \Omega^n S^n(\)$.

So we have a map $r : \widetilde{\Sigma HP}^\infty \rightarrow Q(HP^\infty)$ such that the diagram

$$\begin{array}{ccc} \widetilde{\Sigma HP}^\infty & \xrightarrow{j} & \Sigma Sp \\ r \downarrow & & \downarrow \wr \\ Q(HP^\infty) & \xrightarrow{j} & BSp \end{array} \quad \text{commutes up to homotopy.}$$

We may regard r as a stable map $r : \widetilde{\Sigma HP}^\infty \xrightarrow{(s)} HP^\infty$.

We put $\overline{HP}^\infty = \widetilde{\Sigma}^{-1} \widetilde{HP}^\infty$, $\overline{q} = \widetilde{\Sigma}^{-1} \widetilde{q}$ and $\overline{t} = \widetilde{\Sigma}^{-1} \widetilde{t}$.

Then we have following stable maps :

$$\begin{array}{l} CP_+^\infty \xrightarrow{q} HP_+^\infty \xrightarrow{t} CP_+^\infty, \\ CP_+^\infty \xrightarrow{\overline{q}} \overline{HP}_+^\infty \xrightarrow{\overline{t}} CP_+^\infty \quad \text{and} \\ \widetilde{\Sigma}^2 \overline{HP}^\infty \xrightarrow{r} HP^\infty. \end{array}$$

We can easily calculate the homomorphisms induced by these maps on the ordinaly

homology theory.

Let y^{MSP} be the euler class of MSP, y the class of ordinaly homology.

Then in H MSP-theory, we have $y^{\text{MSP}} = h(y) = \sum_{i \geq 0} h_i y^{i+1}$.

Let x be the complex euler class of the ordinaly homology H.

Now we can define the symplectic formal system.

Let R be a commutative ring with unit and $R[[X, \bar{X}, Y, \bar{Y}]]$ formal power series ring with four variables $X, \bar{X}, Y$ and $\bar{Y}$.

Definition 4.1. A symplectic formal system is a set of formal power series $E(X), F_k(X, \bar{X}, Y, \bar{Y})$ and $G_k(X, \bar{X}, Y, \bar{Y})$ (for $k \geq 1$) such that satisfy

$$\begin{aligned} (i) \quad E(X) &= \sum_{i \geq 1} a_i X^i, \\ F_k(X, \bar{X}, Y, \bar{Y}) &= \sum_{i, j \geq 0} b_{i, j}^{(k)} X^i \cdot Y^j + \sum_{i, j \geq 1} c_{i, j}^{(k)} \bar{X} \cdot X^{i-1} \bar{Y} \cdot Y^{j-1}, \\ G_k(X, \bar{X}, Y, \bar{Y}) &= \sum_{i, j \geq 0} d_{i, j}^{(k)} (\bar{X} \cdot X^{i-1} \bar{Y}^j + \bar{Y} \cdot Y^{i-1} X^j) \end{aligned}$$

and under $\bar{X}^2 = E(X), \bar{Y}^2 = E(Y)$, satisfy also

$$(ii) \text{ (unitary relation) } b_{1,0}^{(1)} = d_{1,0}^{(1)} = 1, \quad b_{n,0}^{(1)} = d_{n,0}^{(1)} = 0 \text{ for } n \neq 1,$$

(iii) (associative relation)

$$\begin{aligned} &D(F_1(X, \bar{X}, Y, \bar{Y}), G_1(X, \bar{X}, Y, \bar{Y}), Z, \bar{Z}) \\ &= D(X, \bar{X}, F_1(Y, \bar{Y}, Z, \bar{Z}), G_1(Y, \bar{Y}, Z, \bar{Z})) \quad \text{for } D = F_1 \text{ or } G_1, \end{aligned}$$

$$(iv) \text{ (commutative relation) } b_{i,j}^{(1)} = b_{j,i}^{(1)}, \quad c_{i,j}^{(1)} = c_{j,i}^{(1)},$$

$$(v) \text{ (differntial relation) } c_{1,1}^{(1)} = -2, \quad c_{1,n}^{(1)} = c_{n,1}^{(1)} = 0 \text{ for } n \neq 1,$$

$$(vi) \text{ (power relation) } F_k(X, \bar{X}, Y, \bar{Y}) = (F_1(X, \bar{X}, Y, \bar{Y}))^k,$$

$$G_k(X, \bar{X}, Y, \bar{Y}) = G_1(X, \bar{X}, Y, \bar{Y}) F_{k-1}(X, \bar{X}, Y, \bar{Y}) \quad \text{and}$$

$$(vii) \text{ (squar relation) } (G_1(X, \bar{X}, Y, \bar{Y}))^2 = E(F_1(X, \bar{X}, Y, \bar{Y})).$$

Definition 4.2.

Let $\Gamma = \{E, F_k, G_k\}$ be a symplectic formal system over R .

An associated symplectic ring for Γ , R_Γ , is the subring of R which is generated by the elements $8a_i, 4b_{i,j}^{(2k-1)}, 2b_{i,j}^{(2k)}, c_{i,j}^{(k)}, 4d_{i,j}^{(k)}$ and 1 .

Now we can define symplectic Lazard ring $LMSp$ as follows.

Let S be $Z[a_i, b_{i,j}^{(k)}, c_{i,j}^{(k)}, d_{i,j}^{(k)}]$ where $a_i, b_{i,j}^{(k)}, c_{i,j}^{(k)}$ and $d_{i,j}^{(k)}$ are variables and I the ideal of relations that appear in (i) ~ (vii) of (4.1).

Then we get a universal symplectic formal system over S/I .

We denote Γ_{univ} as this system over S/I and do $LMSp$ as $(S/I)_{\Gamma_{univ}}$.

Next we want to construct a symplectic formal system over $H_*(M\text{Sp})$.

For simplicity, we denote $f(x)$ and $\bar{f}(x)$ as $h(-x^2)$ and $\frac{1}{2} \frac{d}{dx} h(-x^2)$

$H_*(M\text{Sp})[[x]]$ where $h(-x^2)$ is as previous.

We denote a symplectic formal system Γ_H by setting,

$$E^H(f(x)) = (\bar{f}(x))^2,$$

$$F_k^H(f(x), \bar{f}(x), f(y), \bar{f}(y)) = (f(x+y))^k \quad \text{and}$$

$$G_k^H(f(x), \bar{f}(x), f(y), \bar{f}(y)) = \bar{f}(x+y) \cdot (f(x+y))^{k-1} \quad \text{for } k \geq 1.$$

Then the relations (i) ~ (vii) except (v) are almost trivial.

Proposition 4.4. In Γ_H , differential relation holds.

We have a ring homomorphism $\theta': LMSp \rightarrow H_*(M\text{Sp})_{\Gamma_H}$ by the universality.

Theorem. $\text{Im}(\theta') \subseteq \text{Im}(\text{Hurewicz homomorphism} : M\text{Sp}_* \rightarrow H_*(M\text{Sp}))$.