

# On representations of locally inverse $*$ -semigroups<sup>1</sup>

Teruo IMAOKA (今岡 輝男)

Isamu INATA (稲田 勇)

Hiroaki YOKOYAMA (横山 浩明)

## Abstract

The purpose of this paper is to obtain an analogous representation of the Preston-Vagner Representation for locally inverse  $*$ -semigroups which is a generalization of [7].

Firstly, by introducing a concept of a  $\pi$ -set (which is slightly different from the one in [7]), we shall construct the  $\pi$ -symmetric locally inverse  $*$ -semigroup  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  on a  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$ , and show that  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  is a locally inverse  $*$ -semigroup and that any locally inverse  $*$ -semigroup can be embedded up to  $*$ -isomorphism in  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  on a  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$ . Moreover, we shall show that the wreath product (in the sense of Cowan[1]) of locally inverse  $*$ -semigroups is also a locally inverse  $*$ -semigroup.

## 1 Introduction

A semigroup  $S$  with a unary operation  $*$  :  $S \rightarrow S$  is called a *regular  $*$ -semigroup* if it satisfies

- (i)  $(x^*)^* = x$ ,
- (ii)  $(xy)^* = y^*x^*$ ,
- (iii)  $xx^*x = x$ .

Let  $S$  be a regular  $*$ -semigroup. An idempotent  $e$  in  $S$  is called a *projection* if it satisfies  $e^* = e$ . For any subset  $A$  of  $S$ , denote the sets of idempotents and projections of  $A$  by  $E(A)$  and  $P(A)$ , respectively. The following result is well-known, and we use it frequently throughout this paper.

---

<sup>1</sup>This is the abstract and the details will be published elsewhere. The results of § 2 and 4 were obtained after the conference.

**Result 1.1** (see [4]). *Let  $S$  be a regular  $*$ -semigroup. Then we have the followings:*

- (1)  $E(S) = P(S)^2$ , more precisely, for any  $e \in E(S)$ , there exist  $f, g \in P(S)$  such that  $f\mathcal{R}e\mathcal{L}g$  and  $e = fg$ ;
- (2) for any  $a \in S$  and  $e \in P(S)$ ,  $a*ea \in P(S)$ ;
- (3) each  $\mathcal{L}$ -class and each  $\mathcal{R}$ -class have one and only one projection.

A regular  $*$ -semigroup  $S$  is called a *locally inverse  $*$ -semigroup* if, for any  $e \in E(S)$ ,  $eSe$  is an inverse subsemigroup of  $S$ .

**Lemma 1.2** *A regular  $*$ -semigroup  $S$  is a locally inverse  $*$ -semigroup if and only if, for each  $e \in P(S)$ ,  $eSe$  is an inverse subsemigroup of  $S$ .*

A regular  $*$ -semigroup  $S$  is called a *generalized inverse  $*$ -semigroup* if  $E(S)$  forms a normal band, that is,  $E(S)$  satisfies the identity  $xyzx = xzyx$ . It is obvious that a generalized inverse  $*$ -semigroup is a locally inverse  $*$ -semigroup.

**Remark.** It is clear that a regular  $*$ -semigroup  $S$  is a generalized inverse  $*$ -semigroup if and only if it satisfies the following condition:

$$\text{for any } e, f, g, h \in P(S), e f g h = e g f h \text{ (in } S \text{)}.$$

However, we remark that even if a locally inverse  $*$ -semigroup  $S$  satisfies the condition

$$\text{for any } e, f, g \in P(S), e f g e = e g f e \text{ (in } S \text{)},$$

it is not always a generalized inverse  $*$ -semigroup. The second remark of [6] is its counterexample.

Let  $X$  be a set. If  $X = \bigcup \{X_i : i \in I\}$  is a partition of  $X$ , denote it by  $X = \sum \{X_i : i \in I\}$ . For a mapping  $\alpha : A \rightarrow B$ , denote the domain and the range of  $\alpha$  by  $d(\alpha)$  and  $r(\alpha)$ , respectively. For a subset  $C$  of  $A$ ,  $\alpha|_C$  means the restriction of  $\alpha$  to  $C$ .

Let  $\mathcal{I}_X$  be the symmetric inverse semigroup on a set  $X$ . For a subset  $A$  of  $X$ ,  $1_A$  means the identity mapping on  $A$ . Let  $\mathcal{A}$  be an inverse subsemigroup of  $\mathcal{I}_X$  and  $\theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  a mapping. Denote the image  $(\alpha, \beta)\theta$  of an ordered pair  $(\alpha, \beta)$  by  $\theta_{\alpha, \beta}$ . Set  $\mathcal{M} = \{\theta_{\alpha, \beta} : \alpha, \beta \in \mathcal{A}\}$ . If  $\mathcal{M}$  satisfies the following conditions:

- (C1)  $\theta_{\alpha, \beta}^{-1} = \theta_{\beta^{-1}, \alpha^{-1}},$
- (C2)  $\theta_{\alpha, \alpha^{-1}} = 1_{r(\alpha)},$
- (C3)  $\theta_{1_{d(\alpha)}, \alpha} = 1_{d(\alpha)},$
- (C4)  $\theta_{\alpha, \beta} \theta_{\beta, \gamma} \theta_{\alpha, \beta, \gamma} = \theta_{\alpha, \beta} \theta_{\beta, \gamma} \theta_{\alpha, \beta, \gamma},$

we call it *the structure sandwich set* of  $\mathcal{A}$  determined by  $\theta$ .

**Result 1.3** (see [7]) *Let  $\mathcal{A}$  be an inverse subsemigroup of the symmetric inverse semigroup  $\mathcal{I}_X$  on a set  $X$ , and  $\mathcal{M}$  the structure sandwich set of  $\mathcal{A}$  determined by a mapping  $\theta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ . Define a multiplication  $\circ$  and a unary operation  $*$  on  $\mathcal{A}$  as follows:*

$$\begin{aligned}\alpha \circ \beta &= \alpha \theta_{\alpha, \beta} \beta, \\ \alpha^* &= \alpha^{-1}\end{aligned}$$

*Then  $\mathcal{A}(\circ, *)$  becomes a regular  $*$ -semigroup.*

Hereafter, we call such a semigroup  $\mathcal{A}(\circ, *)$  a *regular  $*$ -semigroup of partial one-to-one mappings* determined by the *structure sandwich set*  $\mathcal{M}$ , and denote it by  $\mathcal{A}(\mathcal{M})$ . The notation and terminology are those of [3] and [4], unless otherwise stated.

In § 2, we shall firstly consider a  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$  which is a set  $X$  with a partition  $\pi' : X = \sum\{X_e : e \in \Lambda\}$ , a reflexive and symmetric relation  $\omega$  on  $\Lambda$  and a set of mappings  $\{\sigma_{e,f} : (e, f) \in \omega\}$ , where  $\sigma_{e,f}$  is a bijection of  $X_e$  onto  $X_f$ . We remark that a  $\pi$ -set, defined in this paper, is slightly different from the one in [7], which is called a *strong  $\pi$ -set* in this paper. The set  $\mathcal{LI}_{X(\pi')}$ , say, of all partial one-to-one  $\pi$ -mappings on  $X(\pi'; \omega; \{\sigma_{e,f}\})$  is an inverse subsemigroup of  $\mathcal{I}_X$ . By using  $\{\sigma_{e,f} : (e, f) \in \omega\}$ , we shall construct a structure sandwich set  $\mathcal{M}$ , and show that  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  is a locally inverse  $*$ -semigroup. We call such a semigroup  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  the  *$\pi$ -symmetric locally inverse  $*$ -semigroup* on a  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$  with the structure sandwich set  $\mathcal{M}$ .

In § 3, we shall show that any locally inverse  $*$ -semigroup is embedded up to  $*$ -isomorphism in the  $\pi$ -symmetric locally inverse  $*$ -semigroup  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  on a  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$ .

As a generalization of [2], Cowan [1] gave us the definition of the wreath product  $\text{Swr}T(X)$  of inverse semigroups  $S$  and  $T(X)$ , where  $T(X)$  is an inverse subsemigroup of  $\mathcal{I}_X$ . And he showed that the wreath product  $\text{Swr}T(X)$  is also an inverse semigroup. In § 4, we shall show that the wreath product of locally inverse  $*$ -semigroups  $S$  and  $T(X)$  ( $\subseteq \mathcal{LI}_{X(\pi')}$ ) is a locally inverse  $*$ -semigroup. Moreover, we shall obtain that the wreath product of generalized inverse  $*$ -semigroups is also a generalized inverse  $*$ -semigroup.

## 2 $\pi$ -Symmetric locally inverse $*$ -semigroups

Let  $X$  be a non-empty set. If there exist a partition  $X = \sum\{X_e : e \in \Lambda\}$  and a reflexive and symmetric relation  $\omega$  on  $\Lambda$  such that

- (i) for each  $(e, f) \in \omega$ , there exists a bijection  $\sigma_{e,f} : X_e \rightarrow X_f$ ,
- (ii) for all  $e \in \Lambda$ ,  $\sigma_{e,e} = 1_{X_e}$ ,
- (iii) for any  $(e, f) \in \omega$ ,  $\sigma_{f,e} = \sigma_{e,f}^{-1}$ ,

then  $X$  is called a  $\pi$ -set with a partition  $\pi' : X = \sum \{X_e : e \in \Lambda\}$ , a relation  $\omega$  and a set of mappings  $\{\sigma_{e,f} : (e, f) \in \omega\}$ , and denote it by  $X(\pi'; \omega; \{\sigma_{e,f}\})$ , or simply by  $X(\pi')$ . If a  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$  satisfies the following two conditions

- (iv)  $\omega$  is transitive, that is, it is an equivalence relation,
- (v) for  $(e, f), (f, g) \in \omega$ ,  $\sigma_{e,f}\sigma_{f,g} = \sigma_{e,g}$ ,

it is called a *strong  $\pi$ -set*.

Let  $X(\pi'; \omega; \{\sigma_{e,f}\})$  be a  $\pi$ -set. A subset  $A$  of  $X$  is called a  $\pi$ -single subset of  $X$  if for each  $e \in \Lambda$ , there exists at most one element  $f \in \Lambda$  such that  $X_f \cap A \neq \emptyset$  and  $(e, f) \in \omega$ . We consider the empty set as a  $\pi$ -single subset. Denote the family of all  $\pi$ -single subsets of  $X(\pi'; \omega; \{\sigma_{e,f}\})$  by  $\mathbf{T}$ .

A mapping  $\alpha$  in the symmetric inverse semigroup  $\mathcal{I}_X$  on  $X$  is called a *partial one-to-one  $\pi$ -mapping* of  $X(\pi'; \omega; \{\sigma_{e,f}\})$  if  $d(\alpha)$  and  $r(\alpha)$  are  $\pi$ -single subsets. Let  $\mathcal{LI}_{X(\pi')}$  be the set of all partial one-to-one  $\pi$ -mappings of  $X(\pi'; \omega; \{\sigma_{e,f}\})$ , that is,  $\mathcal{LI}_{X(\pi')} = \{\alpha \in \mathcal{I}_X : d(\alpha), r(\alpha) \in \mathbf{T}\}$ . The following lemma is clear.

**Lemma 2.1** *The set  $\mathcal{LI}_{X(\pi')}$ , defined above, is an inverse subsemigroup of  $\mathcal{I}_X$ .*

For  $A, B \in \mathbf{T}$ , define a mapping  $\theta_{A,B}$  as follows:

$$\begin{aligned}
 d(\theta_{A,B}) &= \{x \in A : \text{there exist } e, f \in \Lambda \text{ such that } (e, f) \in \omega, \\
 &\quad x \in X_e \text{ and } x\sigma_{e,f} \in B\}, \\
 r(\theta_{A,B}) &= \{y \in B : \text{there exist } e, f \in \Lambda \text{ such that } (e, f) \in \omega, \\
 &\quad y \in X_f \text{ and } y\sigma_{f,e} \in A\}, \\
 x\theta_{A,B} &= x\sigma_{e,f} \quad (x \in d(\theta_{A,B}) \cap X_e, (e, f) \in \omega).
 \end{aligned}
 \tag{3.1}$$

For any  $\alpha, \beta \in \mathcal{LI}_{X(\pi')}$ , define  $\theta_{\alpha,\beta} = \theta_{r(\alpha),d(\beta)}$ . Since a subset of  $\pi$ -single subset is also a  $\pi$ -single subset, we have that  $\theta_{\alpha,\beta} \in \mathcal{LI}_{X(\pi')}$  for all  $\alpha, \beta \in \mathcal{LI}_{X(\pi')}$ . Let  $\mathcal{M} = \{\theta_{\alpha,\beta} : \alpha, \beta \in \mathcal{LI}_{X(\pi')}\}$ .

**Lemma 2.2** *The set  $\mathcal{M}$ , defined above, is the structure sandwich set of  $\mathcal{LI}_{X(\pi')}$  determined by a mapping  $\theta : \mathcal{LI}_{X(\pi')} \times \mathcal{LI}_{X(\pi')} \rightarrow \mathcal{LI}_{X(\pi')}$   $((\alpha, \beta) \mapsto \theta_{\alpha,\beta})$ . Therefore,  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  is a regular  $*$ -semigroup.*

We call the set  $\mathcal{M}$ , defined above, the *structure sandwich set determined by a  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$* .

It is clear that each projection of  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  is the identity mapping  $1_A$  on a  $\pi$ -single subset  $A$ . Let  $1_A$  be any projection and let  $\alpha, \beta$  be any projections of  $1_A \circ \mathcal{LI}_{X(\pi')} \circ 1_A$ . There exist  $B, C \in \mathbf{T}$  such that  $\alpha = 1_A \circ 1_B \circ 1_A$  and  $\beta = 1_A \circ 1_C \circ 1_A$ . Then  $\alpha = \theta_{A,B}\theta_{A,B}^{-1} = 1_{d(\theta_{A,B})}$  and  $\beta = 1_{d(\theta_{A,C})}$ . Since  $d(\theta_{A,B}) \subseteq A$  and  $d(\theta_{A,C}) \subseteq A$ ,  $\theta_{1_{d(\theta_{A,B})}, 1_{d(\theta_{A,C})}} \subseteq \theta_{A,A} = 1_A$ .

Similarly  $\theta_{1_{\theta_{A,C}}, 1_{\theta_{A,B}}} \subseteq 1_A$ . Then  $\alpha \circ \beta = \beta \circ \alpha$ . Therefore,  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  is a locally inverse  $*$ -semigroup. We call it the  $\pi$ -symmetric locally inverse  $*$ -semigroup on  $X(\pi'; \omega; \{\sigma_{e,f}\})$  with the structure sandwich set  $\mathcal{M}$ . Now, we have the following theorem.

**Theorem 2.3** *Let  $X$  be a  $\pi$ -set with a partition  $\pi' : X = \sum\{X_e : e \in \Lambda\}$ , a relation  $\omega$  on  $\Lambda$  and a set of mappings  $\{\sigma_{e,f} : (e, f) \in \omega\}$ , and let  $\mathcal{M}$  be the structure sandwich set determined by  $X(\pi'; \omega; \{\sigma_{e,f}\})$ . Then  $\mathcal{LI}_{X(\pi')}$  is an inverse subsemigroup of  $\mathcal{I}_X$ . Moreover,  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  is a locally inverse  $*$ -semigroup.*

Let  $X(\pi'; \omega; \{\sigma_{e,f}\})$  be a strong  $\pi$ -set, where  $\pi' = \sum\{X_e : e \in \Lambda\}$ . Since  $\omega$  is an equivalence relation on  $\Lambda$ , there exists the partition  $\Lambda = \sum\{\Lambda_i : i \in I\}$  induced by  $\omega$ . For each  $i \in I$ , denote the subset  $\cup\{X_e : e \in \Lambda_i\}$  by  $\mathbf{X}_i$ .

**Lemma 2.4** *A subset  $A$  of  $X$  is a  $\pi$ -single subset if and only if it satisfies the condition that for any  $i \in I$ ,  $A \cap \mathbf{X}_i \neq \emptyset$  implies  $A \cap \mathbf{X}_i \subseteq Xe$  for some  $e \in \Lambda_i$ .*

Let  $\mathcal{M}$  be the structure sandwich set determined by  $X(\pi'; \omega; \{\sigma_{e,f}\})$ . By Theorem 3.6 of [7],  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  is a generalized inverse  $*$ -semigroup. We call such a semigroup the  $\pi$ -symmetric generalized inverse  $*$ -semigroup and denote it by  $\mathcal{GI}_{X(\pi')}(\mathcal{M})$  instead of  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ .

**Corollary 2.5** (Theorem 3.6 [7]) *Let  $X$  be a strong  $\pi$ -set with a partition  $\pi' : X = \sum\{X_e : e \in \Lambda\}$ , an equivalence relation  $\omega$  on  $\Lambda$  and a set of mappings  $\{\sigma_{e,f} : (e, f) \in \omega\}$ , and let  $\mathcal{M}$  be the structure sandwich set determined by  $X(\pi'; \omega)$ . Then  $\mathcal{GI}_{X(\pi')}(\mathcal{M})$  is a generalized inverse  $*$ -semigroup.*

### 3 Representations

Let  $S$  be a locally inverse  $*$ -semigroup and  $\mathcal{I}_S$  the symmetric inverse semigroup on  $S$ . In this section, denote  $E(S)$  and  $P(S)$  simply by  $E$  and  $P$ , respectively. Since each  $\mathcal{L}$ -class has one and only one projection,  $\pi' : S = \sum\{L_e : e \in P\}$  is a partition of  $S$ , where  $L_e$  denotes the  $\mathcal{L}$ -class containing  $e$ . Let  $\omega = \{(e, f) \in P \times P : e\mathcal{R}g\mathcal{L}f \text{ for some } g \in E\}$ . It is clear that  $\omega$  is a reflexive and symmetric relation on  $P$ . For  $(e, f) \in \omega$ , define  $\sigma_{e,f} : L_e \rightarrow L_f$  by  $x\sigma_{e,f} = xf$ . It follows from Green's Lemma that  $S(\pi'; \omega; \{\sigma_{e,f}\})$  is a  $\pi$ -set. Let  $\mathbf{T}$  be the set of all  $\pi$ -single subsets of  $S(\pi'; \omega; \{\sigma_{e,f}\})$  and  $\mathcal{M}$  the structure sandwich set determined by  $S(\pi'; \omega; \{\sigma_{e,f}\})$ . By Theorem 2.5,  $\mathcal{LI}_{S(\pi')}(\mathcal{M})$  is a locally inverse  $*$ -semigroup.

For any  $a \in S$ , let  $\rho_a : Sa^* \rightarrow Sa$  be a mapping defined by

$$x\rho_a = xa.$$

It is trivial that  $\rho_a$  and  $\rho_{a^*}$  are mutually inverse mappings of  $Sa^*$  and  $Sa$  onto each other, and hence  $\rho_a \in \mathcal{I}_S$ . A subset of  $S$  is said to be  $\mathcal{L}$ -full if it is a union of some  $\mathcal{L}$ -classes of  $S$ .

**Lemma 3.1** (1) For any  $a \in S$ ,  $\rho_a \in \mathcal{LI}_{S(\pi')}$ .

(2) For any  $a, b \in S$ ,  $\theta_{\rho_a, \rho_b} = \rho_a * abb^*$ . Therefore,  $\rho_a \circ \rho_b = \rho_a \rho_a * abb^* \rho_b$ .

By the lemma above and Theorem 2.2 of [5], we can easily see the following lemma.

**Lemma 3.2** Define a mapping  $\phi : S \rightarrow \mathcal{LI}_{S(\pi')}(\mathcal{M})$  by

$$a\phi = \rho_a.$$

Then  $\phi$  is a  $*$ -monomorphism.

Now, we have the main theorem.

**Theorem 3.3** A locally inverse  $*$ -semigroup can be embedded up to  $*$ -isomorphism in the  $\pi$ -symmetric locally inverse  $*$ -semigroup  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  on a  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$  with the structure sandwich set  $\mathcal{M}$  determined by  $X(\pi'; \omega; \{\sigma_{e,f}\})$ .

If  $S$  is a generalized inverse  $*$ -semigroup, then a  $\pi$ -set  $S(\pi'; \omega; \{\sigma_{e,f}\})$ , constructed above, is a strong  $\pi$ -set. For, let  $(e, f), (f, g) \in \omega$ . Then there exist  $h, k \in E(S)$  such that  $eRh\mathcal{L}f$  and  $fRk\mathcal{L}g$ . Since  $S$  is a generalized inverse  $*$ -semigroup,  $efg = eg \in E(S)$  and  $eReg\mathcal{L}g$ . In this case, it follows from [7] that  $\sigma_{e,f}\sigma_{f,g} = \sigma_{e,g}$ , and hence  $S(\pi'; \omega; \{\sigma_{e,f}\})$  is a strong  $\pi$ -set. Then we have the following corollary.

**Corollary 3.4** (Theorem 4.8 [7]). A generalized inverse  $*$ -semigroup can be embedded up to  $*$ -isomorphism in  $\mathcal{GI}_{X(\pi')}$  on a strong  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$ .

## 4 Wreath products

Let  $S$  and  $T$  be locally inverse  $*$ -semigroups. By Theorem 3.3,  $T$  can be embedded in the  $\pi$ -symmetric locally inverse  $*$ -semigroup  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$  on a  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$  with the structure sandwich set  $\mathcal{M}$  determined by  $X(\pi'; \omega; \{\sigma_{e,f}\})$ . In this case, we can consider  $T$  as a locally inverse  $*$ -subsemigroup of  $\mathcal{LI}_{X(\pi')}(\mathcal{M})$ , and so denote it by  $T(X)$ .

By  ${}^XS$ , denote the set of all mappings from the family  $\mathbf{T}$  of  $\pi$ -single subsets of  $X(\pi'; \omega; \{\sigma_{e,f}\})$  into  $S$ , and define a multiplication on  ${}^XS$  by

$$\begin{aligned} d(\psi_1\psi_2) &= d(\psi_1) \cap d(\psi_2), \\ x(\psi_1\psi_2) &= (x\psi_1)(x\psi_2). \end{aligned}$$

For any  $\alpha \in \mathcal{LI}_{X(\pi')}$  and  $\psi \in {}^X S$ , let us define  ${}^\alpha \psi (\in {}^X S)$  by

$${}^\alpha \psi = \alpha \theta_{\alpha, \psi} \psi,$$

where  $\theta_{\alpha, \psi} = \theta_{r(\alpha), d(\psi)} \in \mathcal{M}$ .

Let  $\psi \in {}^X S$  and  $\alpha \in \mathcal{LI}_{X(\pi')}$  such that  $d(\psi) = d(\alpha)$ . Define a mapping  $\psi_\alpha^* (\in {}^X S)$  by

$$\begin{aligned} d(\psi_\alpha^*) &= d(\alpha^{-1}), \\ x\psi_\alpha^* &= (x\alpha^{-1}\theta_{\alpha^{-1}, \psi}\psi)^*. \end{aligned}$$

Since  $r(\alpha^{-1}) = d(\alpha) = d(\psi)$ ,  $\theta_{\alpha^{-1}, \psi} = 1_{d(\alpha)}$  and hence  $x\psi_\alpha^* = (x\alpha^{-1}\psi)^*$  for all  $x \in d(\psi_\alpha^*)$ .

Now, we define the (right) wreath product  $\text{Swr}T(X)$  of  $S$  and  $T(X)$  as follows:

$$\begin{aligned} \text{Swr}T(X) &= \{(\psi, \alpha) \in {}^X S \times T(X) : d(\psi) = d(\alpha)\}, \\ (\psi, \alpha)(\varphi, \beta) &= (\psi {}^\alpha \varphi, \alpha \circ \beta), \\ (\psi, \alpha) &= (\psi_\alpha^*, \alpha^{-1}). \end{aligned}$$

Let  $(\psi, \alpha), (\varphi, \beta) \in \text{Swr}T(X)$ . Then

$$\begin{aligned} x \in d(\psi {}^\alpha \varphi) &\iff x \in d(\psi) \text{ and } x \in d({}^\alpha \varphi) = d(\alpha \theta_{\alpha, \varphi} \varphi) \\ &\iff x \in d(\alpha), x\alpha \in d(\theta_{\alpha, \varphi}) \text{ and } x\alpha \theta_{\alpha, \varphi} \in d(\varphi) \\ &\iff x \in d(\alpha), x\alpha \in d(\theta_{\alpha, \beta}) \text{ and } x\alpha \theta_{\alpha, \beta} \in d(\beta) \\ &\iff x \in d(\alpha \theta_{\alpha, \beta} \beta) = d(\alpha \circ \beta). \end{aligned}$$

Then  $(\psi, \alpha)(\varphi, \beta) = (\psi {}^\alpha \varphi) \in \text{Swr}T(X)$ , and hence  $\text{Swr}T(X)$  is closed under the multiplication. It immediately follows from the definition of  $\psi_\alpha^*$  that  $\text{Swr}T(X)$  is closed under the unary operation  $*$ .

**Theorem 4.1** *Let  $S$  and  $T(X)$  be locally inverse  $*$ -semigroups. Then  $\text{Swr}T(X)$  is a locally inverse  $*$ -semigroup. Moreover, we have*

$$\begin{aligned} P(\text{Swr}T(X)) &= \{(\psi, 1_A) \in \text{Swr}T(X) : A \in \mathbf{T} \text{ and } r(\psi) \subseteq P(S)\}, \\ E(\text{Swr}T(X)) &= \{(\psi, \alpha) \in \text{Swr}T(X) : \alpha \in E(T(X)) \text{ and } r(\psi) \subseteq E(S)\}. \end{aligned}$$

Next, we shall consider wreath products of generalized inverse  $*$ -semigroups. Let  $S$  and  $T(X) (\subseteq \mathcal{GI}_{X(\pi'; \omega; \{\sigma_{e,f}\})}(\mathcal{M}))$  be generalized inverse  $*$ -semigroups.

**Lemma 4.2** *Let  $A, B, C$  be a  $\pi$ -single subsets of a strong  $\pi$ -set  $X(\pi'; \omega; \{\sigma_{e,f}\})$ , and let  $\psi \in {}^X S$  such that  $d(\psi) = C$ . Then, for any  $x \in d(1_A \circ 1_B \circ 1_C)$ ,  $x {}^{1_A \circ 1_B} \psi = x {}^{1_A} \psi$ .*

By using the lemma above, we have the following theorem.

**Theorem 4.3** *Let  $S$  and  $T(X) (\subseteq \mathcal{GI}_{X(\pi')}(\mathcal{M}))$  be generalized inverse  $*$ -semigroups, then  $\text{Swr}T(X)$  is a generalized inverse  $*$ -semigroup.*

## References

- [1] D. F. COWAN, *A class of varieties of inverse semigroups*, J. Algebra **174** (1991), 115–142.
- [2] C. H. HOUGHTON, *Embedding inverse semigroups in wreath products*, Glasgow Math. J. **17** (1976), 77–82.
- [3] J. M. HOWIE, *An Introduction to Semigroup Theory*, Academic Press, London, 1976.
- [4] T. IMAOKA, *On fundamental regular  $*$ -semigroups*, Mem. Fac. Sci. Shimane Univ. **14** (1980), 19–23.
- [5] T. IMAOKA, *Prehomomorphisms on regular  $*$ -semigroups*, Mem. Fac. Sci. Shimane Univ. **15** (1981), 23–27.
- [6] T. IMAOKA, *Identities for Idempotents of generalized inverse  $[*]$ -semigroups*, Mem. Fac. Sci. Shimane Univ. **28** (1994), 9–11.
- [7] T. IMAOKA, *Representations of generalized inverse  $*$ -semigroups*, submitted.
- [8] G. B. PRESTON, *Representations of inverse semi-groups*, J. London Math. Soc. **29** (1954), 411–419.
- [9] V. V. VAGNER, *Generalized groups*, Doklady Akad. Nauk SSSR **84** (1952), 1119–1122 (Russian).

Department of Mathematics  
 Shimane University  
 Matsue, Shimane 690, Japan