

GLOBAL SOLUTIONS FOR THE ROTATING NAVIER-STOKES EQUATIONS

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1. INTRODUCTION

This paper is survey of the paper [14]. We consider the Cauchy problems for the Navier-Stokes equations with the Coriolis force

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \Omega e_3 \times u + (u \cdot \nabla)u + \nabla p = 0 & t > 0, x \in \mathbb{R}^3, \\ \operatorname{div} u = 0 & t > 0, x \in \mathbb{R}^3, \\ u(0, x) = u_0(x) & x \in \mathbb{R}^3, \end{cases} \quad (\text{NSC})$$

where $u = u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ and $p = p(t, x)$ denote the unknown velocity field and the unknown pressure of the fluid at the point $(t, x) \in (0, \infty) \times \mathbb{R}^3$, respectively, while $u_0 = u_0(x) = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$ denotes the given initial velocity field satisfying the compatibility condition $\operatorname{div} u_0 = 0$. Here, $\Omega \in \mathbb{R}$ is the speed of rotation around the vertical unit vector $e_3 = (0, 0, 1)$.

For (NSC) with $\Omega = 0$, there are a lot of results for the existence of global solutions. It is known that global smooth solutions are obtained for small initial data in some scaling invariant function spaces. Kato [8] studied the existence of global solutions for small initial data in the Lebesgue space $L^3(\mathbb{R}^3)$. Here, the space $L^3(\mathbb{R}^3)$ is scaling invariant to the equation (NSC) with $\Omega = 0$. In fact, for a solution u let u_λ be defined by $u_\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x)$ for $\lambda > 0$. Then, u_λ is also a solution to (NSC) with $\Omega = 0$ and we have the following norm invariant property in the Lebesgue space:

$$\|u_\lambda(0)\|_{L^p(\mathbb{R}^3)} = \|u(0)\|_{L^p(\mathbb{R}^3)} \text{ for any } \lambda > 0 \text{ if } p = 3.$$

On the results for small initial data in such scaling invariant function spaces, Kozono-Yamazaki [20] studied in the Besov spaces $\dot{B}_{p,\infty}^{-1+\frac{n}{p}}(\mathbb{R}^3)$ with $3 < p < \infty$, Koch-Tataru [18] studied in the class of bounded mean oscillation $BMO^{-1}(\mathbb{R}^3)$.

In this paper, we take the speed $|\Omega|$ large and show the existence of global solutions to (NSC) for large initial data in $\dot{H}^s(\mathbb{R}^3)$ with $s \geq 1/2$. In particular, we give a sufficient condition on the norm of initial data and the speed Ω for the existence of global solutions. For the existence of global solutions to (NSC), Chemin-Desjardins-Gallagher-Grenier [6, 7] proved that for any initial data $u_0 \in L^2(\mathbb{R}^2)^2 + H^{\frac{1}{2}}(\mathbb{R}^3)^3$, there exists a positive parameter Ω_0 such that for every $\Omega \in \mathbb{R}$ with $|\Omega| \geq \Omega_0$ there exists a unique global solution. Babin-Mahalov-Nicolaenko [2, 3, 4] showed the existence of global solutions and the regularity of the solutions to (NSC) for the periodic initial data with large $|\Omega|$. On the other hand, Giga-Inui-Mahalov-Saal [11] showed the existence of global solutions for small initial data $u_0 \in FM_0^{-1}(\mathbb{R}^3)$ which smallness is independent of $\Omega \in \mathbb{R}$. On such other results of global

solutions for small initial data, Hieber-Shibata [12] studied in the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^3)$, Konieczny-Yoneda [19] studied in the Fourier-Besov space $F\dot{B}_{p,\infty}^{2-\frac{3}{p}}(\mathbb{R}^3)$ with $1 < p \leq \infty$. We note that the spaces $FM_0^{-1}(\mathbb{R}^3)$, $H^{\frac{1}{2}}(\mathbb{R}^3)$ and $F\dot{B}_{p,\infty}^{2-\frac{3}{p}}(\mathbb{R}^3)$ are scaling invariant spaces to (NSC) with $\Omega = 0$.

We consider the following integral equation:

$$u(t) = T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla \cdot (u \otimes u)d\tau, \quad (\text{IE})$$

where $\mathbb{P} = (\delta_{ij} + R_i R_j)_{1 \leq i,j \leq 3}$ denotes the Helmholtz projection onto the divergence-free vector fields and $T_\Omega(\cdot)$ denotes the semigroup corresponding to the linear problem of (NSC), which is given explicitly by

$$T_\Omega(t)f = \mathcal{F}^{-1} \left[\cos \left(\Omega \frac{\xi_3}{|\xi|} t \right) e^{-t|\xi|^2} I \hat{f}(\xi) + \sin \left(\Omega \frac{\xi_3}{|\xi|} t \right) e^{-t|\xi|^2} R(\xi) \hat{f}(\xi) \right]$$

for $t \geq 0$ and divergence-free vector fields f . Here, I is the identity matrix in $\mathbb{R}^3$, R_j ($j = 1, 2, 3$) is the Riesz transform and $R(\xi)$ is the skew-symmetric matrix symbol related to the Riesz transform, which is defined by

$$R(\xi) := \frac{1}{|\xi|} \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix} \quad \text{for } \xi \in \mathbb{R}^3 \setminus \{0\}.$$

We refer to Babin-Mahalov-Nikolaenko [1, 2, 3], Giga-Inui-Mahalov-Saal [10] and Hieber-Shibata [12] for the derivation of the explicit form of $T_\Omega(\cdot)$.

We consider the initial data $u_0 \in \dot{H}^s(\mathbb{R}^3)$ with $1/2 \leq s < 3/4$ to establish the existence theorem on global solutions. In the case $s > 1/2$, the sufficient speed Ω is characterized by the norm of initial data $\|u_0\|_{\dot{H}^s}$. In the case $s = 1/2$, the sufficient speed Ω is characterized by each precompact set $K \subset \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$, which the initial data belongs to. Our theorem for $s > 1/2$ is the following.

Theorem 1.1. *Let $\Omega \in \mathbb{R} \setminus \{0\}$, and let s, p and θ satisfy*

$$\frac{1}{2} < s < \frac{3}{4}, \quad \frac{1}{3} + \frac{s}{9} < \frac{1}{p} < \frac{2}{3} - \frac{s}{3}, \quad (1.1)$$

$$\frac{s}{2} - \frac{1}{2p} < \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4}, \quad \frac{3}{4} - \frac{3}{2p} \leq \frac{1}{\theta} < 1 - \frac{2}{p}. \quad (1.2)$$

Then, there exists a positive constant $C = C(s, p, \theta) > 0$ such that for any initial velocity field $u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ with

$$\|u_0\|_{\dot{H}^s} \leq C|\Omega|^{\frac{s}{2}-\frac{1}{4}} \quad \text{and} \quad \operatorname{div} u_0 = 0, \quad (1.3)$$

there exists a unique global solution $u \in C([0, \infty), \dot{H}^s(\mathbb{R}^3))^3 \cap L^\theta(0, \infty; \dot{H}_p^s(\mathbb{R}^3))^3$ to (NSC).

Remark 1.2. The existence of global solutions for small initial data $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$ were shown by Hieber-Shibata [12]. The size condition (1.3) on initial data can be regarded as a continuous extension of that in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$. Indeed, Hieber-Shibata [12] assumed the smallness condition $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leq \delta$ for some $\delta > 0$, which corresponds to our condition (1.3) with $s = 1/2$.

Remark 1.3. The space $L^{\theta_0}(0, \infty; \dot{H}_{p_0}^{s_0}(\mathbb{R}^3))$ is scaling invariant to (NSC) in the case $\Omega = 0$ if θ_0, s_0 and p_0 satisfy

$$\frac{2}{\theta_0} + \frac{3}{p_0} = 1 + s_0. \quad (1.4)$$

On the first condition of (1.2), we see that

$$\frac{2}{\theta} + \frac{3}{p} < \frac{5}{4} + \frac{s}{2} < 1 + s \quad \text{if } s > \frac{1}{2}.$$

Therefore, the space $L^\theta(0, \infty; \dot{H}_p^s(\mathbb{R}^3))$ in Theorem 1.1 includes more regular functions than those in the scaling invariant spaces.

In the case $s = 1/2$, it seems difficult to obtain the sufficient condition on the size of initial data and the speed for the existence of global solutions since $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is scaling invariant to (NSC) with $\Omega = 0$. Then, we introduce precompact sets in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ to obtain the following result.

Theorem 1.4. *Let K be an arbitrary precompact set in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$. Then, there exists $\omega(K) > 0$ such that for any $\Omega \in \mathbb{R}$ with $|\Omega| > \omega(K)$ and for any $u_0 \in K$ with $\operatorname{div} u_0 = 0$, there exists a unique global solution u to (NSC) in $C([0, \infty), \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))^3 \cap L^4(0, \infty; \dot{H}_3^{\frac{1}{2}}(\mathbb{R}^3))^3$.*

Remark 1.5. The space $L^4(0, \infty; \dot{H}_3^{\frac{1}{2}}(\mathbb{R}^3))$ in Theorem 1.4 is scaling invariant space in the case $\Omega = 0$ since $\theta_0 = 4$, $s_0 = 1/2$ and $p_0 = 3$ satisfy (1.4).

Remark 1.6. For the original Navier-Stokes equations

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla)u + \nabla p = 0 & t > 0, x \in \mathbb{R}^3, \\ \operatorname{div} u = 0 & t > 0, x \in \mathbb{R}^3, \\ u(0, x) = u_0(x) & x \in \mathbb{R}^3, \end{cases} \quad (\text{NS})$$

it is known by the results of Brezis [5], Giga [9] and Kozono [16] that the existence time T of local solutions for initial data in $L^r(\mathbb{R}^3)$ ($3 < r < \infty$) and $L^3(\mathbb{R}^3)$ is determined by the each bounded set B in $L^r(\mathbb{R}^3)$ ($3 < r < \infty$) and the each precompact set K in $L^3(\mathbb{R}^3)$, respectively. Note that the space $L^3(\mathbb{R}^3)$ is a scaling critical space to (NS). On the other hand, the sufficient speed Ω to obtain global solutions is determined by the bounded sets and precompact sets in Theorem 1.1 and Theorem 1.4, respectively. Therefore, our theorems can be regarded as a counterpart of such results from the viewpoint of the Coriolis parameter Ω for the existence of global solutions.

In this paper, we prove Theorem 1.1 only. For the proof of Theorem 1.4, see [14]. In Section 2, we introduce propositions to prove Theorem 1.1 which are on linear estimates for the semigroup $T_\Omega(\cdot)$ and the bilinear estimate. In Section 3, we prove Theorem 1.1.

2. PRELIMINARIES

In what follows, we denote by $C > 0$ various constants and by $0 < c < 1$ various small constants. In order to introduce propositions to prove theorems, let us recall the definition of the homogeneous Besov spaces in brief. Let ϕ be a radial smooth function satisfying

$$\operatorname{supp} \widehat{\phi} \subset \{\xi \in \mathbb{R}^3 \mid 2^{-1} \leq |\xi| \leq 2\}, \quad \sum_{j \in \mathbb{Z}} \widehat{\phi}(2^{-j}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be defined by

$$\phi_j(x) := 2^{3j} \phi(2^j x) \quad \text{for } j \in \mathbb{Z}, x \in \mathbb{R}^3.$$

Then, for $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, the homogeneous Besov space $\dot{B}_{p,q}^s(\mathbb{R}^3)$ is defined by the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ with

$$\|f\|_{\dot{B}_{p,q}^s} := \left\| \left\{ 2^{sj} \|\phi_j * f\|_{L^p(\mathbb{R}^3)} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} < \infty.$$

Lemma 2.1. [13] *Let $2 \leq p \leq \infty$. There exists $C > 0$ such that*

$$\|\mathcal{F}^{-1} e^{\pm i \frac{\xi_3}{|\xi|} \Omega t} \mathcal{F} f\|_{\dot{B}_{p,2}^0} \leq C \left\{ \frac{\log(e + |\Omega|t)}{1 + |\Omega|t} \right\}^{\frac{1}{2}(1-\frac{2}{p})} \|f\|_{\dot{B}_{\frac{p}{p-1},2}^{3(1-\frac{2}{p})}} \quad (2.1)$$

for all $\Omega \in \mathbb{R}$, $t > 0$, $f \in \dot{B}_{\frac{p}{p-1},2}^{3(1-\frac{2}{p})}(\mathbb{R}^3)$.

Lemma 2.2. *Let $1 < q \leq 2 \leq p < \infty$ satisfy $1/q \geq 1 - 1/p$. Then, there exists $C > 0$ such that*

$$\|T_\Omega(t)f\|_{L^p} \leq C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} \left\{ \frac{\log(e + |\Omega|t)}{1 + |\Omega|t} \right\}^{\frac{1}{2}(1-\frac{2}{p})} \|f\|_{L^q} \quad (2.2)$$

for all $\Omega \in \mathbb{R}$, $t > 0$, $f \in L^q(\mathbb{R}^3)$.

Proof. By the continuous embedding $\dot{B}_{p,2}^0(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ and (2.1), we have

$$\|T_\Omega(t)f\|_{L^p} \leq C \|T_\Omega(t)f\|_{\dot{B}_{p,2}^0} \leq C \left\{ \frac{\log(e + |\Omega|t)}{1 + |\Omega|t} \right\}^{\frac{1}{2}(1-\frac{2}{p})} \|e^{t\Delta} f\|_{\dot{B}_{\frac{p}{p-1},2}^{3(1-\frac{2}{p})}}.$$

And we have from Lemma 2.2 in [17] and the continuous embedding $L^q(\mathbb{R}^3) \hookrightarrow \dot{B}_{q,2}^0(\mathbb{R}^3)$

$$\|e^{t\Delta} f\|_{\dot{B}_{\frac{p}{p-1},2}^{3(1-\frac{2}{p})}} \leq C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} \|f\|_{\dot{B}_{q,2}^0} \leq C t^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})} \|f\|_{L^q(\mathbb{R}^3)}.$$

Therefore, we obtain (2.2). □

Proposition 2.3. [13] *Let $2 < p < 6$, $2 < \theta < \infty$ satisfy*

$$\frac{3}{4} - \frac{3}{2p} \leq \frac{1}{\theta} < 1 - \frac{2}{p}.$$

Then, there exists $C > 0$ such that

$$\|T_\Omega(\cdot)f\|_{L^\theta(0,\infty;L^p)} \leq C |\Omega|^{-\frac{1}{\theta} + \frac{3}{4}(1-\frac{2}{p})} \|f\|_{L^2}$$

for all $\Omega \in \mathbb{R} \setminus \{0\}$, $f \in L^2(\mathbb{R}^3)$.

Proposition 2.4. [14] *For every $f \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, it holds that*

$$\lim_{|\Omega| \rightarrow \infty} \|T_\Omega(\cdot)f\|_{L^4(0,\infty;\dot{H}_3^{\frac{1}{2}})} = 0. \quad (2.3)$$

Remark 2.5. The space $L^4(0, \infty; \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ is scaling invariant function space to (NSC) with $\Omega = 0$. (2.3) is proved by Proposition 2.3 and the approximation of functions in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ by smooth functions. (2.3) is used for the proof of Theorem 1.4.

Proposition 2.6. *Let $2 < p < 3$ and $6/5 < q < 2$ satisfy*

$$1 - \frac{1}{p} \leq \frac{1}{q} < \frac{1}{3} + \frac{1}{p}, \quad (2.4)$$

$$\max \left\{ 0, \frac{1}{2} - \frac{3}{2} \left(\frac{1}{q} - \frac{1}{p} \right) - \frac{1}{2} \left(1 - \frac{2}{p} \right) \right\} < \frac{1}{\theta} \leq \frac{1}{2} - \frac{3}{2} \left(\frac{1}{q} - \frac{1}{p} \right). \quad (2.5)$$

Then, there exists $C > 0$ such that

$$\left\| \int_0^t T_\Omega(t - \tau) \mathbb{P} \nabla f(\tau) d\tau \right\|_{L^\theta(0, \infty; \dot{H}_p^s)} \leq C |\Omega|^{-\{\frac{1}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{\theta}\}} \|f\|_{L^{\frac{q}{2}}(0, \infty; \dot{H}_q^s)} \quad (2.6)$$

for all $s \in \mathbb{R}$, $\Omega \in \mathbb{R} \setminus \{0\}$, $f \in L^{\frac{q}{2}}(0, \infty; \dot{H}_q^s(\mathbb{R}^3))$.

Proof. We only consider the case $s = 0$ for simplicity since the case $s \neq 0$ is treated similarly. By Lemma 2.2, we have

$$\begin{aligned} & \left\| \int_0^t T_\Omega(t - \tau) \mathbb{P} \nabla f(\tau) d\tau \right\|_{L^\theta(0, \infty; L^p)} \\ & \leq C \left\| \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{p})} \left\{ \frac{\log(e + |\Omega||t - \tau|)}{1 + |\Omega||t - \tau|} \right\}^{\frac{1}{2}(1 - \frac{2}{p})} \|f(\tau)\|_{L^q} d\tau \right\|_{L^\theta(0, \infty)}. \end{aligned}$$

In the case $1/\theta = 1/2 - 3(1/q - 1/p)/2$, we have from Hardy-Littlewood-Sobolev's inequality

$$\begin{aligned} & \left\| \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{p})} \left\{ \frac{\log(e + |\Omega||t - \tau|)}{1 + |\Omega||t - \tau|} \right\}^{\frac{1}{2}(1 - \frac{2}{p})} \|f(\tau)\|_{L^q} d\tau \right\|_{L^\theta(0, \infty)} \\ & \leq \left\| \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{p})} \|f(\tau)\|_{L^q} d\tau \right\|_{L^\theta(0, \infty)} \\ & \leq C \|f\|_{L^{\frac{q}{2}}(0, \infty; L^q)}. \end{aligned}$$

In the case $1/\theta < 1/2 - 3(1/q - 1/p)/2$, we have from Hausdorff-Young's inequality with $1/\theta = 2/\theta + 1/r - 1$

$$\begin{aligned} & \left\| \int_0^t (t - \tau)^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{p})} \left\{ \frac{\log(e + |\Omega||t - \tau|)}{1 + |\Omega||t - \tau|} \right\}^{\frac{1}{2}(1 - \frac{2}{p})} \|f(\tau)\|_{L^q} d\tau \right\|_{L^\theta(0, \infty)} \\ & \leq \left\| t^{-\frac{1}{2} - \frac{3}{2}(\frac{1}{q} - \frac{1}{p})} \left\{ \frac{\log(e + |\Omega|t)}{1 + |\Omega|t} \right\}^{\frac{1}{2}(1 - \frac{2}{p})} \right\|_{L^r(0, \infty)} \|f\|_{L^{\frac{q}{2}}(0, \infty; L^q)} \\ & = C |\Omega|^{\frac{1}{\theta} - \frac{1}{2} + \frac{3}{2}(\frac{1}{q} - \frac{1}{p})} \|f\|_{L^{\frac{q}{2}}(0, \infty; L^q)}. \end{aligned}$$

Therefore, we obtain (2.6). □

Lemma 2.7. *Let s, p satisfy*

$$0 \leq s < 3, \quad \frac{s}{3} < \frac{1}{p} < \frac{1}{2} + \frac{s}{6},$$

and let q satisfy

$$\frac{1}{q} = \frac{2}{p} - \frac{s}{3}.$$

Then, there exists $C > 0$ such that

$$\|fg\|_{\dot{H}_q^s} \leq C \|f\|_{\dot{H}_p^s} \|g\|_{\dot{H}_p^s}. \quad (2.7)$$

Proof. Let r satisfy $1/q = 1/p + 1/r$. In the Sobolev spaces, it is known that

$$\|fg\|_{\dot{H}_q^s} \leq C\|f\|_{\dot{H}_p^s}\|g\|_{L^r} + C\|f\|_{L^r}\|g\|_{\dot{H}_p^s}.$$

By the continuous embedding $\dot{H}_p^s(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$, we obtain (2.7). $\square$

3. PROOF OF THEOREM 1.1

Since the assumption on θ and p in Proposition 2.3 is satisfied by (1.1) and (1.2), there exists $C_0 > 0$ such that

$$\|T_\Omega(\cdot)u_0\|_{L^\theta(0,\infty;\dot{H}_p^s)} \leq |\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}C_0\|u_0\|_{\dot{H}^s}.$$

Let $\Psi(u)$ and Y be defined by

$$\Psi(u)(t) := T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla \cdot (u \otimes u)(\tau)d\tau, \quad (3.1)$$

$$Y := \{u \in L^\theta(0,\infty;\dot{H}_p^s(\mathbb{R}^3))^3 \mid \|u\|_{L^\theta(0,\infty;\dot{H}_p^s)} \leq 2C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot{H}^s}, \operatorname{div} u = 0\},$$

$$d(u, v) := \|u - v\|_{L^\theta(0,\infty;\dot{H}_p^s)}.$$

Let q satisfy $1/q = 2/p - s/3$. Since the assumptions on s, p, q and θ in Proposition 2.6 and Lemma 2.7 are satisfied by (1.1) and (1.2), for any $u, v \in Y$, we have from Proposition 2.3, Proposition 2.6 and Lemma 2.7

$$\begin{aligned} \|\Psi(u)\|_{L^\theta(0,\infty;\dot{H}_p^s)} &\leq C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot{H}^s} + C|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|u \otimes u\|_{L^{\frac{\theta}{2}}(0,\infty;\dot{H}_q^s)} \\ &\leq C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot{H}^s} + C|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|u\|_{L^\theta(0,\infty;\dot{H}_p^s)}^2 \\ &\leq C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot{H}^s} + C_1|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})+2\{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})\}}\|u_0\|_{\dot{H}^s}^2 \\ &\leq C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot{H}^s} + C_1|\Omega|^{-\frac{s}{2}+\frac{1}{4}}|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot{H}^s}^2, \\ \|\Psi(u) - \Psi(v)\|_{L^\theta(0,\infty;\dot{H}_p^s)} &= \left\| \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla \cdot \{u \otimes (u-v)(\tau) + (u-v) \otimes v(\tau)\}d\tau \right\|_{L^\theta(0,\infty;\dot{H}_p^s)} \\ &\leq C|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|u \otimes (u-v) + (u-v) \otimes v\|_{L^{\frac{\theta}{2}}(0,\infty;\dot{H}_q^s)} \\ &\leq C|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}(\|u\|_{L^\theta(0,\infty;\dot{H}_p^s)} + \|v\|_{L^\theta(0,\infty;\dot{H}_p^s)})\|u-v\|_{L^\theta(0,\infty;\dot{H}_p^s)} \\ &\leq C_2|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot{H}^s}\|u-v\|_{L^\theta(0,\infty;\dot{H}_p^s)} \\ &= C_2|\Omega|^{\frac{1}{4}+\frac{3}{2q}-\frac{3}{p}}\|u_0\|_{\dot{H}^s}\|u-v\|_{L^\theta(0,\infty;\dot{H}_p^s)} \\ &= C_2|\Omega|^{-\frac{s}{2}+\frac{1}{4}}\|u_0\|_{\dot{H}^s}\|u-v\|_{L^\theta(0,\infty;\dot{H}_p^s)}. \end{aligned} \quad (3.2)$$

If Ω, u_0 satisfy

$$C_1|\Omega|^{-\frac{s}{2}+\frac{1}{4}}\|u_0\|_{\dot{H}^s} \leq C_0, \quad C_2|\Omega|^{-\frac{s}{2}+\frac{1}{4}}\|u_0\|_{\dot{H}^s} \leq \frac{1}{2},$$

then, it is possible to apply Banach's fixed point theorem in Y and we obtain $u \in Y$ with

$$u(t) = T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla \cdot (u \otimes u)d\tau.$$

Here, we show that the solution $u \in Y$ satisfies $u(t) \in \dot{H}^s(\mathbb{R}^3)^3$ for all $t \geq 0$. On the linear part, it is easy to see that $T_\Omega(t)u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ for any $t \geq 0$. On the nonlinear part, let $1/q = 2/p - s/3$ and we have from Lemma 2.2, Lemma 2.7 and Hölder's inequality

$$\begin{aligned}
\left\| \int_0^t T_\Omega(t-\tau) \mathbb{P} \nabla \cdot (u \otimes u)(\tau) d\tau \right\|_{\dot{H}^s} &\leq C \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})} \|(u \otimes u)(\tau)\|_{\dot{H}_q^s} d\tau \\
&\leq C \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})} \|u(\tau)\|_{\dot{H}_p^s}^2 d\tau \\
&\leq C \left\| (t-\cdot)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})} \right\|_{L^{\frac{\theta}{\theta-2}}(0<\tau<t)} \left\| \|u(\tau)\|_{\dot{H}_p^s}^2 \right\|_{L^{\frac{\theta}{2}}(0,\infty)} \\
&\leq C t^{\frac{\theta-2}{\theta} [1-\frac{\theta}{\theta-2} \{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})\}]} \|u\|_{L^\theta(0,\infty;\dot{H}_p^s)}^2.
\end{aligned} \tag{3.3}$$

Here, we note on the integrability at $\tau = t$ that

$$\frac{\theta}{\theta-2} \left\{ \frac{1}{2} + \frac{3}{2} \left(\frac{1}{q} - \frac{1}{2} \right) \right\} < 1 \quad \text{if and only if} \quad \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4}.$$

Therefore, we obtain $u(t) \in \dot{H}^s(\mathbb{R}^3)^3$ and we also see $u \in C([0, \infty), \dot{H}^s(\mathbb{R}^3)^3)$. $\square$

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