

AROUND THE FURUTA INEQUALITY

フルタ不等式の周辺の不等式

前橋工科大学 亀井栄三郎 (EIZABURO KAMEI)

MAEBASHI INSTITUTE OF TECHNOLOGY

1. Chaotic order と Furuta inequality A と B は Hilbert space 上の positive operator とする. A が positive (resp. positive invertible) operator のとき $A \geq 0$ (resp. $A > 0$) と表す. 久保安藤 [18] によって導入された A と B の α -power mean は次のように与えられる.

$$A \sharp_{\alpha} B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}} \quad \text{for } 0 \leq \alpha \leq 1.$$

Furuta inequality [6] はこの α -power mean を用いることで次のように表すことができる ([2],[3],[12],[13],[14]).

Furuta inequality: If $A \geq B \geq 0$, then

$$(F) \quad A^u \sharp_{\frac{1-u}{p-u}} B^p \leq A \quad \text{and} \quad B \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

holds for $u \leq 0$ and $1 \leq p$.

この Furuta inequality は Löwner-Heinz inequality の歴史的拡張と絶賛されている.

$$(LH) \quad \text{If } A \geq B \geq 0, \text{ then } A^{\alpha} \geq B^{\alpha} \text{ for } 0 \leq \alpha \leq 1.$$

我々は [12] (cf.[7]) において (F) を一行にまとめて表せることを示した. これを satellite theorem of the Furuta inequality と呼ぶ:

If $A \geq B \geq 0$, then

$$(SF) \quad A^u \sharp_{\frac{1-u}{p-u}} B^p \leq B \leq A \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

holds for all $u \leq 0$ and $p \geq 1$.

$A, B > 0$ に対し, $\log A \geq \log B$ の時 $A \gg B$ と表し chaotic order ([3],[16],[17]) と呼んでいる. 次は chaotic order の特徴づけであるが, 今後 chaotic order の議論における出発点となるものであることより, これを chaotic Furuta inequality [3] と命名しておく.

If $A \gg B$, then

$$(CF) \quad A^u \sharp_{\frac{1-u}{p-u}} B^p \leq I \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

for any $p \geq 0$ and $u \leq 0$.

satellite theorem (SF) は通常の順序 $A \geq B$ における Furuta inequality (F) と chaotic order $A \gg B$ の違いをはっきりと示している. 実際, 次のような結果を得ることができる ([16],[17]).

If $A \gg B$, then

$$(SCF) \quad A^u \sharp_{\frac{1-u}{p-u}} B^p \leq B \quad \text{and} \quad A \leq B^u \sharp_{\frac{1-u}{p-u}} A^p$$

holds for any $p \geq 1$ and $u \leq 0$.

(CF) と (SCF) は更に次のように一般化することができる [16].

Theorem A. For $A, B > 0$, if $A \gg B$, then the following (1) and (2) hold.

$$(1) \quad A^u \#_{\frac{u}{p-u}} B^p \leq B^\delta \text{ and } A^\delta \leq B^u \#_{\frac{u}{p-u}} A^p \text{ for } u \leq 0 \text{ and } 0 \leq \delta \leq p$$

$$(2) \quad A^u \#_{\frac{u}{p-u}} B^p \leq A^\alpha \text{ and } B^\alpha \leq B^u \#_{\frac{u}{p-u}} A^p \text{ for } u \leq \alpha \leq 0 \text{ and } 0 \leq p.$$

2. Furuta inequality の一般化, Grand Furuta inequality. 古田は Furuta inequality の一般化を次のような形で示した [8]. これは 安藤-日合が与えた log majorization についての主結果と同等な不等式 [1] と Furuta inequality を繋ぐものとなっており, 我々はこれを grand Furuta inequality と呼んでいる [4],[5],[15].

The grand Furuta inequality: If $A \geq B \geq 0$ and A is invertible, then for each $1 \leq p$ and $0 \leq t \leq 1$,

$$(GF) \quad A^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A \text{ and } B \leq B^{-r+t} \#_{\frac{1-t+r}{(p-t)s+r}} (B^t \natural_s A^p)$$

holds for $t \leq r$ and $1 \leq s$.

これらの指数 $\frac{1-t+r}{(p-t)s+r}$ が best possible であるということについては棚橋によって示されている [19]. (GF) における指数 s を $\frac{\beta-t}{p-t}$ for $1 \leq p \leq \beta$ と置き換え, α -power mean を用いることで (F) の場合と同様に次のような satellite form を与えることができる [15].

If $A \geq B > 0$, then the following (SGF) holds for $0 \leq t \leq 1 \leq p \leq \beta$ and $u \leq 0$.

$$A^u \#_{\frac{1-u}{\beta-u}} (A^t \natural_{\frac{\beta-t}{\beta-t}} B^p) \leq (A^t \natural_{\frac{\beta-t}{\beta-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A \leq (B^t \natural_{\frac{\beta-t}{\beta-t}} A^p)^{\frac{1}{\beta}} \leq B^u \#_{\frac{1-u}{\beta-u}} (B^t \natural_{\frac{\beta-t}{\beta-t}} A^p).$$

上で使われた記号 $\natural$ は α -power mean を $\alpha \in R$ にまで拡張したものであり $\alpha \in [0, 1]$ に於いては $\#$ と一致するがそれ以外では作用素平均とはならない.

(SGF) における中心部分の不等式は [4], [5] において示しているように (SF) の視点からすれば (GF) の本質的な性質であろう. そこで我々はこの進化型を次のように与えておく.

Theorem 1. Let $A \geq B > 0$ and $0 \leq t \leq 1 \leq p$. Then

$$H(\beta) = (A^t \natural_{\frac{\beta-t}{\beta-t}} B^p)^{\frac{1}{\beta}}$$

is a decreasing function with $\beta \geq p$ and in particular $H(\beta) \leq B^p$ for $\beta \geq p$.

Proof. First of all, suppose that $1 \leq \frac{\beta-t}{p-t} \leq 2$. Then

$$A^t \natural_{\frac{\beta-t}{\beta-t}} B^p = B^p \natural_{\frac{\beta-t}{\beta-t}} A^t = B^p (B^{-p} \#_{\frac{\beta-t}{\beta-t}} A^{-t}) B^p \leq B^p (B^{-p} \#_{\frac{\beta-t}{\beta-t}} B^{-t}) B^p = B^\beta$$

By (LH), we have $(A^t \natural_{\frac{\beta-t}{\beta-t}} B^p)^{\frac{1}{\beta}} \leq B^p$.

Next we assume that $H(\beta) \leq B^p$ for a given $\beta \geq p$. Since $p \geq 1$, we have $B_1 = (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{1}{\beta}} \leq B \leq A$. If we take β_1 with $1 \leq \frac{\beta_1-t}{\beta-t} \leq 2$, then the preceding argument ensures that

$$A^t \natural_{\frac{\beta_1-t}{p-t}} B^p = A^t \natural_{\frac{\beta_1-t}{p-t}} (A^t \natural_{\frac{\beta-t}{p-t}} B^p) = A^t \natural_{\frac{\beta_1-t}{p-t}} B_1^{\beta} \leq B_1^{\beta_1},$$

that is, $A^t \natural_{\frac{\beta_1-t}{p-t}} B^p \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\beta_1}{\beta}}$. So it follows from (LH) that

$$(A^t \natural_{\frac{\beta_1-t}{p-t}} B^p)^{\frac{\beta_1}{\beta}} \leq (A^t \natural_{\frac{\beta-t}{p-t}} B^p)^{\frac{\beta}{\beta}} \leq B^p,$$

which shows the monotonicity of $H(\beta)$.

3. 内山による Furuta innequality の一般化の試み.

最近, 内山は (GF) の一般化の方向として次のような形を与えた [20].

If $A \geq B \geq C > 0$, then for each $0 \leq t \leq 1 \leq p$

$$(U) \quad A^{1-t} \geq A^{-r} \natural_{\frac{1-t+r}{(p-t)s+r}} (B^{-\frac{1}{s}} C^p B^{-\frac{1}{s}})^s$$

holds for $r \geq t$ and $s \geq 1$.

これに関して我々は (SGF) の観点からすれば「歪み」があるように感じ, 次のような形を提案した [5].

If $A, B, C > 0$ satisfy $A \gg B$ and $B \geq C$, then for each $0 \leq t \leq 1$

$$B \geq C \geq (B^t \natural_s C^p)^{\frac{1}{(p-t)s+t}} \geq A^{-r+t} \natural_{\frac{1+r-t}{(p-t)s+r}} (B^t \natural_s C^p)$$

holds for all $p \geq 1$, $s \geq 1$ and $r \geq t$.

この不等式に於いて $A \geq B = C$ ならば (F) を得, $A = B \geq C$ とすれば (GF) となることは明らかであろう.

ところが古田は [9] (cf.[10]) において [11] を用いることで (U) のより一般的な結果が得られることを示した. このことについては Theorem 4 で触れることにして, ここでは古田の結果を chaotic order の観点から見直してみる.

Theorem 2. For fixed $A, B, C > 0$ and $0 \leq t \leq 1 \leq p$, if $A \gg D = (B^{-\frac{1}{s}} C^p B^{-\frac{1}{s}})^{\frac{1}{p-t}}$ is satisfied, then (1) holds for $\beta \geq p$ and $r \geq t$.

$$(1) \quad B^{\frac{1}{\beta}} A^{-t} B^{\frac{1}{\beta}} \natural_{\frac{1}{\beta}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \geq B^{\frac{1}{\beta}} A^{-r} B^{\frac{1}{\beta}} \natural_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

Additionally, if $A \geq B$, then (2) holds.

$$(2) \quad B^{\frac{1}{\beta}} A^{-r} B^{\frac{1}{\beta}} \natural_{\frac{1-t+r}{\beta-t+r}} C^p \geq B^{\frac{1}{\beta}} A^{-r} B^{\frac{1}{\beta}} \natural_{\frac{1-t+r}{\beta-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

Proof. Since $A \gg D$, (CF) implies that

$$(\dagger) \quad (A^{\frac{1}{\beta}} D^{\beta-t} A^{\frac{1}{\beta}})^{\frac{1}{\beta}} \leq A^t$$

and so $(A^{\frac{1}{2}}D^{\beta-t}A^{\frac{1}{2}})^{\frac{1}{\beta}} \ll A$. Therefore it follows from (SCF) that

$$A^{-r+t} \#_{\frac{1-(t-r)}{\beta-(t-r)}} \{(A^{\frac{1}{2}}D^{\beta-t}A^{\frac{1}{2}})^{\frac{1}{\beta}}\}^{\beta} \leq (A^{\frac{1}{2}}D^{\beta-t}A^{\frac{1}{2}})^{\frac{1}{\beta}},$$

namely

$$A^{-r} \#_{\frac{1-t+r}{\beta-t+r}} D^{\beta-t} \leq A^{-t} \#_{\frac{1}{\beta}} D^{\beta-t}.$$

Since $B^{\frac{1}{2}}D^{\beta-t}B^{\frac{1}{2}} = B^t \#_{\frac{\beta-t}{\beta-t}} C^p$, we have (1) by multiplying $B^{\frac{1}{2}}$ on both sides.

(2) is also shown as follows: Since $A^t \gg (A^{\frac{1}{2}}D^{\beta-t}A^{\frac{1}{2}})^{\frac{1}{\beta}}$ as in above, Theorem A (1) implies that

$$(A^t)^{-\frac{r-t}{\beta-t}} \#_{\frac{\frac{r}{\beta} + \frac{r-t}{\beta-t}}{\frac{\beta}{\beta-t} + \frac{r-t}{\beta-t}}} (A^{\frac{1}{2}}D^{\beta-t}A^{\frac{1}{2}})^{\frac{1}{\beta} \frac{\beta}{\beta-t}} \leq (A^{\frac{1}{2}}D^{\beta-t}A^{\frac{1}{2}})^{\frac{1}{\beta} \frac{\beta}{\beta-t}},$$

that is,

$$A^{-r+t} \#_{\frac{\beta-t+r}{\beta-t+r}} A^{\frac{1}{2}}D^{\beta-t}A^{\frac{1}{2}} \leq (A^{\frac{1}{2}}D^{\beta-t}A^{\frac{1}{2}})^{\frac{\beta}{\beta-t}}.$$

Multiplying $A^{-\frac{1}{2}}$ from the both sides of the above, we have

$$A^{-r} \#_{\frac{\beta-t+r}{\beta-t+r}} D^{\beta-t} \leq A^{-t} \#_{\frac{1}{\beta}} D^{\beta-t} \leq B^{-t} \#_{\frac{1}{\beta}} D^{\beta-t} = B^{-\frac{1}{2}} (B^t \#_{\frac{\beta-t}{\beta-t}} C^p)^{\frac{\beta}{\beta-t}} B^{-\frac{1}{2}} \leq B^{-\frac{1}{2}} C^p B^{-\frac{1}{2}},$$

where the final inequality follows from Theorem 1. Again multiplying $B^{\frac{1}{2}}$ to each sides of this formula, we have

$$B^{\frac{1}{2}}A^{-r}B^{\frac{1}{2}} \#_{\frac{\beta-t+r}{\beta-t+r}} (B^t \#_{\frac{\beta-t}{\beta-t}} C^p) \leq C^p.$$

Hence it follows that

$$\begin{aligned} & B^{\frac{1}{2}}A^{-r}B^{\frac{1}{2}} \#_{\frac{1-t+r}{\beta-t+r}} (B^t \#_{\frac{\beta-t}{\beta-t}} C^p) \\ &= B^{\frac{1}{2}}A^{-r}B^{\frac{1}{2}} \#_{\frac{1-t+r}{\beta-t+r}} \{B^{\frac{1}{2}}A^{-r}B^{\frac{1}{2}} \#_{\frac{\beta-t+r}{\beta-t+r}} (B^t \#_{\frac{\beta-t}{\beta-t}} C^p)\} \\ &\leq B^{\frac{1}{2}}A^{-r}B^{\frac{1}{2}} \#_{\frac{1-t+r}{\beta-t+r}} C^p. \end{aligned}$$

次に Theorem A の応用として上と同様な不等式を与えておく.

Theorem 3. If $A, B, C > 0$ satisfy $A \gg D = (B^{-\frac{1}{2}}C^pB^{-\frac{1}{2}})^{\frac{1}{\beta-t}}$ for some $0 \leq t \leq 1 \leq p$, the the following (1) and (2) hold for $\beta \geq p$ and $r \geq t$.

$$(1) \quad B^t \#_{\frac{1-t}{\beta-t}} C^p \geq B^{\frac{1}{2}}A^{-t}B^{\frac{1}{2}} \#_{\frac{1}{\beta}} (B^t \#_{\frac{\beta-t}{\beta-t}} C^p) \geq B^{\frac{1}{2}}A^{-r}B^{\frac{1}{2}} \#_{\frac{1-t+r}{\beta-t+r}} (B^t \#_{\frac{\beta-t}{\beta-t}} C^p)$$

$$(2) \quad B^t \#_{\frac{1-t}{\beta-t}} C^p \geq B^{\frac{1}{2}}A^{-t}B^{\frac{1}{2}} \#_{\frac{1}{\beta}} C^p \geq B^{\frac{1}{2}}A^{-r}B^{\frac{1}{2}} \#_{\frac{1-t+r}{\beta-t+r}} C^p \geq B^{\frac{1}{2}}A^{-r}B^{\frac{1}{2}} \#_{\frac{1-t+r}{\beta-t+r}} (B^t \#_{\frac{\beta-t}{\beta-t}} C^p)$$

Proof. (1) follows from Theorem A (2) and (1). Actually we have

$$\begin{aligned} & A^{-r} \#_{\frac{1-t+r}{\beta-t+r}} D^{\beta-t} = D^{\beta-t} \#_{\frac{\beta-1}{\beta-t+r}} A^{-r} \\ &= D^{\beta-t} \#_{\frac{\beta-1}{\beta}} \{D^{\beta-t} \#_{\frac{\beta}{\beta-t+r}} A^{-r}\} \\ &= D^{\beta-t} \#_{\frac{\beta-1}{\beta}} \{A^{-r} \#_{\frac{-t+r}{\beta-t+r}} D^{\beta-t}\} \\ &\leq D^{\beta-t} \#_{\frac{\beta-1}{\beta}} A^{-t} = A^{-t} \#_{\frac{1}{\beta}} D^{\beta-t} \\ &= A^{-t} \#_{\frac{1-t+r}{\beta-t+r}} (B^{-\frac{1}{2}}C^pB^{-\frac{1}{2}})^{\frac{\beta-t}{\beta-t}} \leq (B^{-\frac{1}{2}}C^pB^{-\frac{1}{2}})^{\frac{1-t}{\beta-t}} \end{aligned}$$

So the conclusion is obtained by multiplying $B^{\frac{1}{2}}$ both sides of each term.

In addition, (2) except the final part is obtained by taking $\beta = p$ in (1). Moreover we have

$$A^{-r} \#_{\frac{1-t+r}{p-t+r}} D^{\beta-t} = A^{-r} \#_{\frac{1-t+r}{p-t+r}} \{A^{-r} \#_{\frac{p-t+r}{p-t+r}} D^{\beta-t}\} \leq A^{-r} \#_{\frac{1-t+r}{p-t+r}} D^{p-t}$$

by Theorem A (2). Multiplying $B^{\frac{1}{2}}$ to each term from both sides, we attain the conclusion.

最後に古田 [9] による結果が上の Theorem 2 および Theorem 3 から簡単に導かれることを示そう。ここでも s を $\frac{\beta-t}{p-t}$ for $\beta \geq p$ と置き換えておく。

Theorem 4 (Furuta). *If $A \geq B \geq C > 0$ and $0 \leq t \leq 1 \leq p$, then*

$$(1) \quad B \geq C \geq (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{p}} \geq B^{\frac{1}{2}} A^{-t} B^{\frac{1}{2}} \#_{\frac{1}{2}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \geq B^{\frac{1}{2}} A^{-r} B^{\frac{1}{2}} \#_{\frac{1-t+r}{p-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

and

$$(2) \quad B \geq C \geq B^{\frac{1}{2}} A^{-r} B^{\frac{1}{2}} \#_{\frac{1-t+r}{p-t+r}} C^p \geq B^{\frac{1}{2}} A^{-r} B^{\frac{1}{2}} \#_{\frac{1-t+r}{p-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)$$

hold for $\beta \geq p$ and $r \geq t$.

Proof. First of all, the assumption $B \geq C > 0$ ensures $(B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{p}} \leq B$ by (SGF). As in the proof of Theorem 2, (†) is the essential point, which is shown as follows:

Let $D = (B^{-\frac{1}{2}} C^p B^{-\frac{1}{2}})^{\frac{1}{p-t}}$ be as in Theorem 2. Then

$$A^{-t} \#_{\frac{1}{2}} D^{\beta-t} \leq B^{-t} \#_{\frac{1}{2}} D^{\beta-t} = B^{-\frac{1}{2}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{p}} B^{-\frac{1}{2}} \leq B^{-\frac{1}{2}} B^t B^{-\frac{1}{2}} = I.$$

Since (†) is shown, (1) connects with Theorem 2 (1). Namely we have

$$\begin{aligned} B^{\frac{1}{2}} A^{-r} B^{\frac{1}{2}} \#_{\frac{1-t+r}{p-t+r}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) &\leq B^{\frac{1}{2}} A^{-t} B^{\frac{1}{2}} \#_{\frac{1}{2}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) \\ &\leq B^{\frac{1}{2}} B^{-t} B^{\frac{1}{2}} \#_{\frac{1}{2}} (B^t \natural_{\frac{\beta-t}{p-t}} C^p) = (B^t \natural_{\frac{\beta-t}{p-t}} C^p)^{\frac{1}{p}} \leq C \leq B. \end{aligned}$$

Next we show (2). For this, we have only to check $B^{\frac{1}{2}} A^{-r} B^{\frac{1}{2}} \#_{\frac{1-t+r}{p-t+r}} C^p \leq C \leq B$ by (†) and Theorem 2 (2). Fortunately, it is obtained by taking $\beta = p$ in the former (1).

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