

PROPERTIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. Let A be the class of functions $f(z)$ which are analytic in the open unit disk U with $f(0) = 0$ and $f'(0) = 1$. Two subclasses $H(\lambda, \mu)$ and $H_0(\lambda, \mu)$ of A with some inequalities for functions $f(z)$ are introduced. The object of the present paper is to consider some interesting properties for functions $f(z)$ belonging to these subclasses.

1. Introduction. Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disk $E = \{z: |z| < 1\}$. We denote by S the subclass of A consisting of functions which are univalent in E . A function $f(z) \in A$ is called starlike in $|z| < r$ ($0 < r \leq 1$) if it satisfies $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ for $|z| < r$.

For a function $f(z) \in A$, we say that $f(z)$ is in the class $H(\lambda, \mu)$ if and only if it satisfies the conditions $f(z)/z \neq 0$ for $z \in E$ and

$$\left| \frac{z^2 f'(z)}{f^2(z)} - \lambda z^2 \left(\frac{z}{f(z)} \right)'' - 1 \right| < \mu \quad (z \in E), \quad (1)$$

where λ is a complex number with $\operatorname{Re} \lambda \geq 0$ and μ is a positive real number. Also we define the class $H_0(\lambda, \mu)$ by

$$H_0(\lambda, \mu) = \{f(z) \in H(\lambda, \mu): f''(0) = 0\}.$$

In [2], Nunokawa, Obradovic and Owa proved that if $f(z) \in A$ with $f(z)/z \neq 0$ for $z \in E$ and $|(z/f(z))''| \leq 1$ in E , then $f(z) \in S$. Ozaki and Nunokawa [4] showed that $H(0, 1) \subset S$ and Obradovic et al. [3] considered the classes $H(0, 1)$ and $H_0(0, 1)$. In the present paper we investigate certain properties for the classes

$H(\lambda, \mu)$ and $H_0(\lambda, \mu)$. Our results generalize or improve the results obtained in [2], [3] and [4] and some other new results are also given.

2. Properties of the class $H(\lambda, \mu)$. Let $f(z)$ and $g(z)$ be analytic in E . Then we say that $f(z)$ is subordinate to $g(z)$ in E , written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in E such that $|w(z)| \leq |z|$ and $f(z) = g(w(z))$ for $z \in E$. If $g(z)$ is univalent in E , then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

We need the following lemma due to Miller and Mocanu [1].

Lemma. Let $h(z)$ be analytic and convex univalent in E , $h(0) = 1$, and let $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \dots$ ($n \in \mathbb{N}$) be analytic in E . If $p(z) + z p'(z)/c \prec h(z)$, where $c \neq 0$ and $\operatorname{Re} c \geq 0$, then

$$p(z) \prec \frac{c}{n} z^{-c/n} \int_0^z t^{c/n-1} h(t) dt.$$

For $\operatorname{Re} \lambda \geq 0$ and $\mu > 0$, it is easy to verify that the function

$$f(z) = \frac{z}{(1 + \sqrt{\mu/|1+2\lambda|} z)^2} \quad (2)$$

belongs to $H(\lambda, \mu)$ if and only if $\mu \leq |1+2\lambda|$. Applying the lemma, we derive

Theorem 1. Let $\operatorname{Re} \lambda \geq 0$ and $0 < \mu \leq |1+2\lambda|$. Then $H(\lambda, \mu) \subset S$.

Proof. Let

$$p(z) = \frac{z^2 f'(z)}{f^2(z)} = 1 + p_2 z^2 + \dots \quad (3)$$

for $f(z) \in H(\lambda, \mu)$. Then

$$z p'(z) = -z^2 \left(\frac{z}{f(z)} \right)''$$

and it follows from condition (1) that

$$p(z) + \lambda z p'(z) = \frac{z^2 f'(z)}{f^2(z)} - \lambda z^2 \left(\frac{z}{f(z)} \right)'' \prec 1 + \mu z,$$

For $\lambda \neq 0$, $\operatorname{Re} \lambda \geq 0$ and $\mu > 0$, an application of the lemma yields

$$p(z) < 1 + \frac{\mu}{1+2\lambda} z. \quad (4)$$

From (3), (4) and the Schwarz lemma, we have

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq \frac{\mu}{|1+2\lambda|} |z|^2 \quad (z \in E) \quad (5)$$

for $\operatorname{Re} \lambda \geq 0$ and $\mu > 0$. Hence

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < \frac{\mu}{|1+2\lambda|} \leq 1 \quad (z \in E) \quad (6)$$

for $\operatorname{Re} \lambda \geq 0$ and $0 < \mu \leq |1+2\lambda|$.

Now, using Theorem 2 in [4], from (6) we conclude that $f(z) \in S$.

If we let $\mu = |1+2\lambda|$ and $\lambda \rightarrow \infty$, then condition (1) can be written as

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq 2 \quad (z \in E) \quad (7)$$

by the Schwarz lemma. Thus we obtain an improvement of the main theorem in [2]. Namely, we have

Corollary 1. Let $f(z) \in A$ with $f(z)/z \neq 0$ for $z \in E$ and let $f(z)$ satisfy (7). Then $f(z) \in S$.

Remark 1. Recently, Yang and Liu [5] showed Corollary 1 by a different method.

Corollary 2. Let $\operatorname{Re} \lambda \geq 0$, $0 < \mu \leq |1+2\lambda|$ and

$$f(z) = \frac{z}{1 + \sum_{n=1}^{\infty} b_n z^n} \in A. \quad (8)$$

If

$$\sum_{n=2}^{\infty} (n-1) |1+n\lambda| |b_n| \leq \mu, \quad (9)$$

then $f(z) \in S$.

Proof. From (8) and (9) we have

$$\left| \frac{z^2 f'(z)}{f^2(z)} - \lambda z^2 \left(\frac{z}{f(z)} \right)'' - 1 \right| = \left| - \sum_{n=2}^{\infty} (n-1)(1+n\lambda) b_n z^n \right|$$

$$\leq \sum_{n=2}^{\infty} (n-1) |1+n\lambda| |b_n| \leq \mu$$

for $z \in E$. Therefore $f(z) \in H(\lambda, \mu) \subset S$ by using Theorem 1.

Theorem 2. Let $0 \leq \lambda_1 < \lambda_2$ and $\mu > 0$. Then $H(\lambda_2, \mu) \subset H(\lambda_1, \mu)$.

Proof. Let $f(z) \in H(\lambda_2, \mu)$. Then

$$\frac{z^2 f'(z)}{f^2(z)} - \lambda_2 z^2 \left(\frac{z}{f(z)} \right)'' < 1 + \mu z$$

and from (4) in the proof of Theorem 1 we obtain

$$\frac{z^2 f'(z)}{f^2(z)} < 1 + \frac{\mu}{1+2\lambda_2} z < 1 + \mu z.$$

Hence

$$\frac{z^2 f'(z)}{f^2(z)} - \lambda_1 z^2 \left(\frac{z}{f(z)} \right)'' = \frac{\lambda_1}{\lambda_2} \left\{ \frac{z^2 f'(z)}{f^2(z)} - \lambda_2 z^2 \left(\frac{z}{f(z)} \right)'' \right\} + \left(1 - \frac{\lambda_1}{\lambda_2} \right) \frac{z^2 f'(z)}{f^2(z)}$$

$$< 1 + \mu z$$

for $0 \leq \lambda_1 < \lambda_2$. This implies that $f(z) \in H(\lambda_1, \mu)$.

Theorem 3. Let $\operatorname{Re} \lambda \geq 0$, $\mu > 0$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H(\lambda, \mu)$. Then

$$\left| \frac{z}{f(z)} - 1 + a_2 z \right| \leq \frac{\mu}{|1+2\lambda|} |z|^2, \quad (10)$$

$$\left| \frac{z}{f(z)} - 1 \right| \leq |z| \left(|a_2| + \frac{\mu}{|1+2\lambda|} |z| \right), \quad (11)$$

$$1 - |z| \left(|a_2| + \frac{\mu}{|1+2\lambda|} |z| \right) \leq \operatorname{Re} \frac{z}{f(z)} \leq 1 + |z| \left(|a_2| + \frac{\mu}{|1+2\lambda|} |z| \right) \quad (12)$$

and

$$|f(z)| \geq \frac{|z|}{1 + |a_2| |z| + (\mu/|1+2\lambda|) |z|^2}. \quad (13)$$

Equalities in (10)-(13) are attained if we take

$$f(z) = \frac{z}{1 \pm bz + (\mu/(1+2\lambda))z^2} \in H(\lambda, \mu)$$

with $0 < \mu \leq |1+2\lambda|$ and $0 \leq b \leq 2\sqrt{\mu/|1+2\lambda|}$.

Proof. For $f(z) = z + a_2 z^2 + \dots \in H(\lambda, \mu)$, we find that

$$\int_0^z \left(\frac{f'(t)}{f^2(t)} - \frac{1}{t^2} \right) dt = \left(\frac{1}{t} - \frac{1}{f(t)} \right) \Big|_0^z = \frac{1}{z} - \frac{1}{f(z)} - a_2. \quad (14)$$

Using (5) in the proof of Theorem 1, it follows from (14) that

$$\left| \frac{1}{f(z)} - \frac{1}{z} + a_2 \right| \leq \int_0^{|z|} \left| \frac{f'(t)}{f^2(t)} - \frac{1}{t^2} \right| dt \leq \frac{\mu}{|1+2\lambda|} |z| \quad (z \in E),$$

which gives (10). In view of (10), we easily have (11), (12) and (13).

Remark 2. Taking $\lambda=0$ and $\mu=1$ in (11) and (12), we get the corresponding results in [3].

Theorem 4. Let $\operatorname{Re} \lambda \geq 0$, $\mu > 0$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H(\lambda, \mu)$. Then

$$|a_2^2 - a_3| \leq \mu/|1+2\lambda|. \quad (15)$$

The result is sharp for $0 < \mu \leq |1+2\lambda|$.

Proof. Since

$$\frac{z}{f(z)} - 1 + a_2 z = (a_2^2 - a_3) z^2 + \sum_{n=3}^{\infty} b_n z^n,$$

from (10) in Theorem 3 we deduce that

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{r e^{i\theta}}{f(r e^{i\theta})} - 1 + a_2 r e^{i\theta} \right|^2 d\theta &= |a_2^2 - a_3|^2 r^4 + \sum_{n=3}^{\infty} |b_n|^2 r^{2n} \\ &\leq \left(\frac{\mu}{|1+2\lambda|} \right)^2 r^4 \quad (0 < r < 1), \end{aligned}$$

which leads to (15).

It is easy to see that the estimate (15) is best possible for the function $f(z)$ given by (2) with $0 < \mu \leq |1+2\lambda|$.

3. Properties of the class $H_0(\lambda, \mu)$

Theorem 5. Let $\operatorname{Re} \lambda \geq 0$ and $f(z) \in H_0(\lambda, \mu)$.

(a) If $|1+2\lambda|/\sqrt{2} \leq \mu \leq |1+2\lambda|$, then $f(z)$ is starlike in $|z| < \sqrt{|1+2\lambda|/(\sqrt{2}\mu)}$;

(b) If $|1+2\lambda|/2 \leq \mu \leq |1+2\lambda|$, then $\operatorname{Re} f'(z) > 0$ for $|z| < \sqrt{|1+2\lambda|/(2\mu)}$.

Proof. We use a technique in [5]. Let $\operatorname{Re} \lambda \geq 0$ and $0 < \mu \leq |1+2\lambda|$. Then from (5) in the proof of Theorem 1 we have

$$\left| \arg \frac{z^2 f'(z)}{f^2(z)} \right| \leq \arcsin \left(\frac{\mu}{|1+2\lambda|} |z|^2 \right) \quad (z \in E). \quad (16)$$

Also it follows from (10) in Theorem 3 with $a_2 = 0$ that

$$\left| \arg \frac{z}{f(z)} \right| \leq \arcsin \left(\frac{\mu}{|1+2\lambda|} |z|^2 \right) \quad (z \in E). \quad (17)$$

(a) If $|1+2\lambda|/\sqrt{2} \leq \mu \leq |1+2\lambda|$, then from (16) and (17) we obtain

$$\begin{aligned} \left| \arg \frac{z f'(z)}{f(z)} \right| &\leq \left| \arg \frac{z^2 f'(z)}{f^2(z)} \right| + \left| \arg \frac{z}{f(z)} \right| \\ &\leq 2 \arcsin \left(\frac{\mu}{|1+2\lambda|} |z|^2 \right) < \frac{\pi}{2} \end{aligned}$$

for $|z| < r_1 = \sqrt{|1+2\lambda|/(\sqrt{2}\mu)} \leq 1$. This shows that $f(z)$ is starlike in $|z| < r_1$.

(b) If $|1+2\lambda|/2 \leq \mu \leq |1+2\lambda|$, then it follows from (16) and (17) that

$$\begin{aligned} \left| \arg f'(z) \right| &\leq \left| \arg \frac{z^2 f'(z)}{f^2(z)} \right| + 2 \left| \arg \frac{z}{f(z)} \right| \\ &\leq 3 \arcsin \left(\frac{\mu}{|1+2\lambda|} |z|^2 \right) < \frac{\pi}{2} \end{aligned}$$

for $|z| < r_2 = \sqrt{|1+2\lambda|/(2\mu)} \leq 1$. This implies that $\operatorname{Re} f'(z) > 0$ for $|z| < r_2$.

Remark 3. Letting $\lambda = 0$ and $\mu = 1$ in Theorem 5, (b) improves a result of [3] and (a) is the same as in [3].

Corollary 3. Let $\operatorname{Re} \lambda \geq 0$ and $f(z) \in H_0(\lambda, \mu)$.

(a) If $0 < \mu \leq |1 + 2\lambda|/\sqrt{2}$, then $f(z)$ is starlike in E ;

(b) If $0 < \mu \leq |1 + 2\lambda|/2$, then $\operatorname{Re} f'(z) > 0$ for $z \in E$.

Theorem 6. Let

$$f(z) = \frac{z}{1 + \sum_{n=2}^{\infty} b_n z^n}, \quad (18)$$

$f_1(z) = z$ and $f_m(z) = z/(1 + b_2 z^2 + \dots + b_m z^m)$ ($m \geq 2$). If $\operatorname{Re} \lambda \geq 0$, $0 < \mu \leq |1 + 2\lambda|$ and

$$\sum_{n=2}^{\infty} (n-1)|1 + n\lambda||b_n| \leq \mu, \quad (19)$$

then we have

(a) $f(z) \in H_0(\lambda, \mu) \subset S$;

(b) For $z \in E$,

$$\operatorname{Re} \frac{f_m(z)}{f(z)} > 1 - \frac{\mu}{m|1 + (m+1)\lambda|} \quad (20)$$

and

$$\operatorname{Re} \frac{f(z)}{f_m(z)} > \frac{m|1 + (m+1)\lambda|}{m|1 + (m+1)\lambda| + \mu}. \quad (21)$$

The results are sharp for each $m \in \mathbb{N}$.

Proof. Let $c_n = (n-1)|1 + n\lambda|/\mu$ ($n \geq 2$). Then

$$c_{n+1} > c_n \geq 1 \quad (n \geq 2) \quad (22)$$

for $\operatorname{Re} \lambda \geq 0$ and $0 < \mu \leq |1 + 2\lambda|$. From (19) and (22) we deduce that the function $f(z)$ given by (18) is analytic in E and

$$\sum_{n=2}^m |b_n| + c_{m+1} \sum_{n=m+1}^{\infty} |b_n| \leq \sum_{n=2}^{\infty} c_n |b_n| \leq 1 \quad (m \geq 2). \quad (23)$$

(a) Noting that $f(z) \in A$ and $f''(0) = 0$, from the proof of Corollary 2 we see that

$f(z) \in H_0(\lambda, \mu) \subset S$.

(b) Let

$$p_1(z) = c_{m+1} \left\{ \frac{f_m(z)}{f(z)} - \left(1 - \frac{1}{c_{m+1}} \right) \right\}.$$

Then

$$p_1(z) = 1 + \frac{c_{m+1} \sum_{n=m+1}^{\infty} b_n z^n}{1 + \sum_{n=2}^m b_n z^n}$$

and from (23) we deduce that

$$\begin{aligned} \left| \frac{p_1(z)-1}{p_1(z)+1} \right| &= \left| \frac{c_{m+1} \sum_{n=m+1}^{\infty} b_n z^n}{2 \left(1 + \sum_{n=2}^m b_n z^n \right) + c_{m+1} \sum_{n=m+1}^{\infty} b_n z^n} \right| \\ &\leq \frac{c_{m+1} \sum_{n=m+1}^{\infty} |b_n|}{2 - 2 \sum_{n=2}^m |b_n| - c_{m+1} \sum_{n=m+1}^{\infty} |b_n|} \\ &\leq 1 \quad (z \in E). \end{aligned}$$

Hence we conclude that $\operatorname{Re} \{ f_m(z)/f(z) \} > 1 - 1/c_{m+1}$ for $z \in E$. This proves (20) for $m \geq 2$.

If we take

$$f(z) = \frac{z}{1 + (\mu/(m+1 + (m+1)\lambda)) z^{m+1}}, \quad (24)$$

then $f_m(z) = z$ and $f_m(z)/f(z) \rightarrow 1 - \mu/(m+1 + (m+1)\lambda)$ as $z \rightarrow e^{i\pi/(m+1)}$. Hence the bound in (20) is best possible for each $m \geq 2$.

Similarly, if we put

$$p_2(z) = (1 + c_{m+1}) \left\{ \frac{f(z)}{f_m(z)} - \frac{c_{m+1}}{1 + c_{m+1}} \right\},$$

then it follows from (23) that

$$\begin{aligned}
\left| \frac{p_2(z)-1}{p_2(z)+1} \right| &= \left| \frac{-(1+c_{m+1}) \sum_{n=m+1}^{\infty} b_n z^n}{2 \left(1 + \sum_{n=2}^m b_n z^n\right) - (c_{m+1}-1) \sum_{n=m+1}^{\infty} b_n z^n} \right| \\
&\leq \frac{(1+c_{m+1}) \sum_{n=m+1}^{\infty} |b_n|}{2 - 2 \sum_{n=2}^m |b_n| - (c_{m+1}-1) \sum_{n=m+1}^{\infty} |b_n|} \\
&\leq 1 \quad (z \in E).
\end{aligned}$$

Now we easily have the inequality (21) for $m \geq 2$ and the bound in (21) is sharp for the function $f(z)$ given by (24).

Finally, (23) becomes

$$c_2 \sum_{n=2}^{\infty} |b_n| \leq \sum_{n=2}^{\infty} c_n |b_n| \leq 1$$

when $m=1$. By using the same way as in the above, the proof of the theorem is completed.

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