

ON SOME SUFFICIENT CONDITIONS FOR STARLIKENESS AND CONVEXITY

MILUTIN OBRADOVIĆ (ベオグラード大学)

SHIGEYOSHI OWA (近畿大・理工 尾和重義)

M. K. AOUF (マンスール大学)

ABSTRACT

For a function $f(z) = z + a_2 z^2 + \dots$ analytic in the unit disk, we consider the conditions of the form $|f'(z) + zf''(z) - 1| < j$ which imply starlikeness or convexity of it.

I. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denote the class of functions $f(z)$ analytic in the unit disk $\mathbb{U} = \{z: |z| < 1\}$ with the conditions $f(0) = f'(0) - 1 = 0$. As usual, we denote by $\mathcal{K}$, $\mathcal{S}^*$, and $\mathcal{C}$ the subclasses of $\mathcal{A}$ whose members are convex, starlike, and close-to-convex, respectively. All these classes are subclasses of univalent functions in the unit disk $\mathbb{U}$ (see, for example [1]).

Let $f(z)$ and $F(z)$ be analytic in the unit disk $\mathbb{U}$. Then we say that $f(z)$ is subordinate to $F(z)$, written by $f(z) \prec F(z)$ or $f \prec F$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$), and $f(z) = F(w(z))$. If $F(z)$ is univalent in $\mathbb{U}$, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

If for a function $f(z) \in \mathcal{A}$ we have

$$|f'(z) + zf''(z) - 1| < 2 \quad (z \in \mathbb{U}),$$

which is equivalent to

$$(zf'(z))' \prec 1 + 2z,$$

then by applying Lemma 1, given below, we get

$$(1) \quad f'(z) \prec 1 + z,$$

i.e. $\operatorname{Re}\{f'(z)\} > 0$ ($z \in \mathbb{U}$), and $f(z) \in \mathcal{C}$. But, as Mocanu [4] showed,

the condition (1) doesn't imply $f(z) \in S^*$.

In that sense, we may ask a question on a constant j , $j < 2$, such that the condition

$$|f'(z) + zf''(z) - 1| < j \quad (z \in U)$$

implies $f(z) \in S^*$ or $f(z) \in K$. It will be the object of this paper.

But previously, we cite the following lemmas that will be used further.

LEMMA 1 ([2]). Let $F(z)$ be a convex function in U (i.e. $F(z)$ is univalent and $F(U)$ is a convex domain). If $\operatorname{Re}\{\gamma\} > 0$ and $f(z)$ is analytic in U , then

$$f(z) \prec F(z) \Rightarrow \frac{1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt \prec \frac{1}{z^\gamma} \int_0^z F(t) t^{\gamma-1} dt.$$

LEMMA 2 ([4]). If $f(z) \in A$ and

$$|f'(z) - 1| < \frac{2}{\sqrt{5}} = 0.894... \quad (z \in U),$$

then $f(z) \in S^*$.

LEMMA 3 ([3]). Let Ω be a subset of the complex plane $\mathbb{C}$ and suppose that the function $\psi: \mathbb{C}^2 \times U \rightarrow \mathbb{C}$ satisfies the condition $\psi(ix, y; z) \notin \Omega$, for all real $x, y \leq -(1+x^2)/2$ and all $z \in U$. If the function $p(z)$ is analytic in U , $p(0) = 1$ and $\psi(p(z), zp'(z); z) \in \Omega$ ($z \in U$), then $\operatorname{Re}\{p(z)\} > 0$.

2. RESULTS AND CONSEQUENCES

We start with the following statement which easily follows from Lemma 1 and Lemma 2.

THEOREM I. Let $f(z) \in A$ satisfy the condition

$$(2) \quad |f'(z) + zf''(z) - 1| < \frac{4}{\sqrt{5}} = 1.788... \quad (z \in U),$$

then $f(z) \in S^*$.

PROOF. Since the condition (2) may be rewritten in the form

$$(zf'(z))' \prec 1 + \frac{4}{\sqrt{5}} z,$$

an application of Lemma 1 gives that $f'(z) \prec 1 + (2/\sqrt{5})z$, i.e.

$$|f'(z) - 1| < \frac{2}{\sqrt{5}} \quad (z \in \mathbb{U}).$$

Therefore, by using Lemma 2, we have $f(z) \in \mathcal{S}^*$.

But we can get the precise result for a stronger condition than (2).
Namely, we have

THEOREM 2. If $f(z) \in \mathcal{A}$ satisfies

$$(3) \quad |f'(z) + zf''(z) - 1| < \frac{3}{2} \quad (z \in \mathbb{U}),$$

then $f(z) \in \mathcal{S}^*$ and

$$(4) \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathbb{U}).$$

PROOF. At the begining, we note that we use the method given by Mocanu [4]. From (3) we have $(zf'(z))' \prec 1 + (3/2)z$, and by using Lemma 1, $f'(z) \prec 1 + (3/4)z$, and so (by using the same lemma once again)

$$(5) \quad \frac{f(z)}{z} \prec 1 + \frac{3}{8} z.$$

If we put

$$(6) \quad \frac{zf'(z)}{f(z)} = \frac{2p(z)}{p(z) + 1} \quad \text{and} \quad g(z) = \frac{f(z)}{z},$$

then the inequality (3) is equivalent to

$$(7) \quad \left| g(z) \frac{2zp'(z) + 4p(z)^2}{(p(z) + 1)^2} - 1 \right| < \frac{3}{2} \quad (z \in \mathbb{U}).$$

To prove (4) from (7), it is enough to prove that $\operatorname{Re}\{p(z)\} > 0$ ($z \in \mathbb{U}$).

By Lemma 3, it is sufficient to prove that

$$(8) \quad \left| g(z) - \frac{2y + 4(ix)^2}{(ix + 1)^2} - 1 \right| \geq \frac{3}{2}$$

for all real x , $y \leq -(1 + x^2)/2$ and all $z \in U$. Later, if we let $g(z) = u + iv$, then (8) is equivalent to

$$(9) \quad 16(u^2 + v^2)y^2 - 16\{(4(u^2 + v^2) - u)x^2 + 2vx + u\}y \\ + \{64(u^2 + v^2) - 32u - 5\}x^4 + 64vx^3 + (32u - 10)x^2 - 5 \geq 0.$$

From the relation (5), we have

$$(10) \quad u^2 + v^2 - 2u + \frac{55}{64} < 0.$$

Also, from (10), we easily obtain the following inequalities which we will be used:

$$(11) \quad 4(u^2 + v^2) - u > 0, \quad 10(u^2 + v^2) + 3u > 0, \\ 20(u^2 + v^2) - 8u - 1 > 0, \quad \frac{5}{8} < u < \frac{11}{8}.$$

By using (11), we deduce that

$$\{4(u^2 + v^2) - u\}x^2 + 2vx + u > 0$$

for all real x . Therefore, if we denote by L the left-side of (9) and if we use $y \leq -(1 + x^2)/2$, then we obtain

$$(12) \quad L \geq Ax^4 + 2Bx^3 + cx^2 + 2Dx + E \equiv M(x),$$

where $A = 5\{20(u^2 + v^2) - 8u - 1\}$, $B = 40v$, $C = 40(u^2 + v^2) + 32u - 10$, $D = 8v$, and $E = 4(u^2 + v^2) + 8u - 5$.

If we write $M(x) = x^2M_1(x) + M_2(x)$, where $M_1(x) = Ax^2 + 2Bx + C_1$, $M_2(x) = C_2x^2 + 2Dx + E$, $C_1 = 40(u^2 + v^2) + 12u = 4\{10(u^2 + v^2) + 3u\}$, and $C_2 = 10(2u - 1)$, then we shall prove that $M_1(x) \geq 0$ and $M_2(x) \geq 0$ for all real x . First, from (11) we get $A > 0$, $C_2 > 0$, and after that if we put $u^2 + v^2 = 2u - a$ with $55/64 < a \leq 1$, then we have

$$B^2 - AC_1 = -20\{816u^2 - (183 + 780a)u + 90 + 200a^2\} < 0$$

and

$$D^2 - C_2E = -2\{192u^2 - 2(97 + 20a)u + 52a + 25\} < 0.$$

Hence we have $M_1(x) > 0$ and $M_2(x) > 0$, and we conclude that $M(x) > 0$.

REMARK I. In the paper [4], Mocanu has proved that if $\alpha \geq 1/2$, $f(z) \in A$ and

$$|f'(z) + \alpha z f''(z) - 1| < 1 \quad (z \in U),$$

then $f(z) \in S^*$ and $|zf'(z)/f(z) - 1| < 1$ ($z \in U$). This means that we improved Mocanu's result for $\alpha = 1$.

The following theorem gives the results on convexity problem.

THEOREM 3. Let $f(z)$ be in the class A .

(i) If $f(z)$ satisfies

$$(13) \quad |f'(z) + z f''(z) - 1| < \frac{2}{\sqrt{5}} \quad (z \in U),$$

then $f(z) \in K$.

(ii) If $f(z)$ satisfies

$$(14) \quad |f'(z) + z f''(z) - 1| < j \text{ and } |\arg(f'(z))| \leq \arctg \left(\frac{\sqrt{1-j^2}}{j} \right)$$

for some $2/\sqrt{5} < j \leq 1$ and for all $z \in U$, then also $f(z) \in K$.

PROOF. (i) The condition (13) is equivalent to

$$|(zf'(z))' - 1| < \frac{2}{\sqrt{5}} \quad (z \in U),$$

and by Lemma 2 we obtain $zf'(z) \in S^*$, that is, $f(z) \in K$.

(ii) Let $f(z) \in A$ satisfies the conditions in (14) and let $2/\sqrt{5} < j < 1$. Then from the first condition in (14) we have

$$(15) \quad \left| f'(z) \left(1 + \frac{z f''(z)}{f'(z)} \right) - 1 \right| < j \quad (z \in U).$$

If we put $g(z) = f'(z)$ and $p(z) = 1 + z f''(z)/f'(z)$, then we write

$|g(z)p(z) - 1| < j$ ($z \in U$) instead of (15). To prove that $\operatorname{Re}\{p(z)\} > 0$ ($z \in U$), by Lemma 3, it is enough to prove that

$$|g(z)(ix) - 1|^2 - j^2 \geq 0.$$

for given j ($2/\sqrt{5} < j < 1$), for all real x , $y \leq -(1+x^2)/2$, and for all $z \in \mathbb{U}$. In that sense, let's put $g(z) = u + iv$. Then from the second inequality of (14), we have $u/v \leq \sqrt{1-j^2}/j$ ($z \in \mathbb{U}$), and so

$$\begin{aligned} |g(z)(ix) - 1|^2 - j^2 &= |(u + iv)(ix) - 1|^2 - j^2 \\ &= (u^2 + v^2)x^2 + 2vix + 1 - j^2 \\ &\geq 0. \end{aligned}$$

Then, by Lemma 3, we obtain $\operatorname{Re}\{p(z)\} > 0$ ($z \in \mathbb{U}$), i.e. $f(z) \in \mathbb{K}$. For $j = 1$ from (14), we get $\arg(f'(z)) = 0$ ($z \in \mathbb{U}$), which gives $f'(z) = \text{const.}$, i.e. $f(z) \equiv z$.

COROLLARY 1. Let $f(z) \in \mathbb{A}$ satisfy

$$|f'(z) + zf''(z) - 1| < j \text{ and } |f'(z) - 1| < \sqrt{1-j^2}$$

for some j ($2/\sqrt{5} < j \leq 1$) and for all $z \in \mathbb{U}$. Then $f(z) \in \mathbb{K}$.

REMARK 2. If we consider the functions

$$f(z) = z + \frac{j}{n^2} z^n \quad (n > 2, 2/\sqrt{5} < j \leq n/\sqrt{n^2+1}),$$

then we have that

$$|f'(z) + zf''(z) - 1| = j|z|^{n-1} < j \quad (z \in \mathbb{U}),$$

and by Part (i) we don't conclude that $f(z) \in \mathbb{K}$. But since

$$|f'(z) - 1| = \frac{j}{n} |z|^{n-1} < \frac{j}{n} \leq \sqrt{1-j^2} \quad (z \in \mathbb{U}),$$

by Corollary 1, we have $f(z) \in \mathbb{K}$. In that sense, the part (ii) of Theorem 3 is justified.

Finally, if we make a summary of all the previous results and considerations we can easily derive

COROLLARY 2. Let $f(z)$ be in the class $\mathbb{A}$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$

Then the following implications are true:

$$(i) \quad \sum_{n=2}^{\infty} n^2 |a_n| \leq 2 \implies \operatorname{Re}\{f'(z)\} > 0 \quad (z \in U), \text{ i.e. } f(z) \in C;$$

$$(ii) \quad \sum_{n=2}^{\infty} n^2 |a_n| \leq \frac{4}{\sqrt{5}} \implies f(z) \in S^*;$$

$$(iii) \quad \sum_{n=2}^{\infty} n^2 |a_n| \leq \frac{3}{2} \implies f(z) \in S^* \quad \text{and} \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in U);$$

$$(iv) \quad \sum_{n=2}^{\infty} n^2 |a_n| \leq j \quad \text{and} \quad \sum_{n=2}^{\infty} n |a_n| \leq \sqrt{1 - j^2} \quad \left(\frac{2}{\sqrt{5}} < j \leq 1 \right) \\ \implies f(z) \in K;$$

$$(v) \quad \sum_{n=2}^{\infty} n^2 |a_n| \leq \frac{2}{\sqrt{5}} \implies f(z) \in K.$$

ACKNOWLEDGMENT

This work of the first author was supported by the Science Fund of Serbia under Grant No.0401A through Matematički institut.

REFERENCES

- [1] P. L. Duren, Univalent Functions, Springer-Verlag, New York, Berlin, Heidelberg, and Tokyo, 1983.
- [2] D. J. Hallenbeck and St. Ruscheweyh, Subordination by convex functions, Proc. Amer. Math. Soc. 52(1975), 191 - 195.
- [3] S. S. Miller and P. T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J. 28(1981), 167 - 171.
- [4] P. T. Mocanu, Some starlikeness conditions for analytic functions, Rev. Roumanie Math. Pures Appl. 33(1988), 117 - 124.

M. Obradović: Department of Mathematics
Faculty of Technology and Metallurgy
University of Belgrade
4 Karnegieva Street
11000 Belgrade
Yugoslavia

S. Owa: Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan

M. K. Aouf: Department of Mathematics
Faculty of Science
University of Mansoura
Mansoura
Egypt