

Convergence rates of asymptotic solutions to Hamilton-Jacobi equations in Euclidean n space

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This is a survey of the paper [F]. Let us consider the Cauchy problem for the Hamilton-Jacobi equation

$$(0.1) \quad u_t(x, t) + H(x, Du(x, t)) = 0 \quad \text{in } \mathbb{R}^n \times (0, \infty),$$

$$(0.2) \quad u(\cdot, 0) = u_0 \quad \text{in } \mathbb{R}^n.$$

By [FIL, I], under suitable assumptions on H, u_0 , it is shown that the Cauchy problem (0.1)-(0.2) admits a unique solution $u \in C(\mathbb{R}^n \times [0, \infty))$ and there is a pair $(c, v) \in \mathbb{R} \times C(\mathbb{R}^n)$ such that

$$(0.3) \quad \lim_{t \rightarrow \infty} (u(x, t) + ct) = v(x) \quad \text{locally uniformly in } \mathbb{R}^n.$$

Furthermore, v is a solution of

$$(0.4) \quad H(x, Dv(x)) = c \quad \text{in } \mathbb{R}^n.$$

In this talk, we are interested in rates of convergence of (0.3). We assume the following:

$$(A1) \quad H \in C(\mathbb{R}^n \times \mathbb{R}^n).$$

$$(A2) \quad \text{For each } x \in \mathbb{R}^n, H(x, \cdot) \text{ is convex in } \mathbb{R}^n.$$

$$(A3) \quad \lim_{r \rightarrow \infty} \inf \{H(x, p) \mid x \in B(0, R), p \in \mathbb{R}^n \setminus B(0, r)\} = \infty \quad \text{for } R > 0.$$

$$(A4) \quad \text{There exists a pair } (\theta_0, c, w^+, w^-) \in (0, \infty) \times \mathbb{R} \times C(\mathbb{R}^n) \times C^1(\mathbb{R}^n) \text{ such that } w^+ \text{ and } w^- \text{ are, respectively, a subsolution and a supersolution of (0.4) and}$$

$$(0.5) \quad 0 \leq w^+(x) - w^-(x) \leq \frac{1}{\theta_0} (c - H(x, Dw^-(x))) \quad \text{in } \mathbb{R}^n.$$

If (A1)-(A4) is fulfilled, then we define the function $\hat{w}$ by

$$(0.6) \quad \hat{w}(x) = \sup \{w(x) \mid w \in W\} \quad \text{in } \mathbb{R}^n,$$

where W is the set of all $w \in C(\mathbb{R}^n)$ such that $w^- + w$ is a viscosity subsolution of (0.4)

*"Viscosity Solution Theory of Differential Equations and its Developments", 2006.05.31-06.02, Research Institute for Mathematical Sciences Kyoto University

for the constant c of (A4), and w satisfies the inequality

$$(0.7) \quad 0 \leq w(x) \leq w^+(x) - w^-(x) \quad \text{in } \mathbb{R}^n.$$

Remark 1. $0 \leq \hat{w}(\cdot) \in \text{Lip}_{loc}(\mathbb{R}^n)$, and $w^- + \hat{w}$ is a solution of (0.4). $\square$

Besides (A1)-(A4), we assume:

(A5) There exists a function $\varphi \in C(\mathbb{R}^n)$ such that $\inf\{w^-(x) + \hat{w}(x) + \varphi(x) | x \in \mathbb{R}^n\} > -\infty$ and the following comparison principle holds: Let

$$\Phi := \left\{ u \in C(\mathbb{R}^n \times [0, \infty)) \mid \inf\{u(x, t) + \varphi(x) \mid x \in \mathbb{R}^n, t \in [0, T]\} > -\infty \text{ for } T > 0 \right\}.$$

If $u_1 \in C(\mathbb{R}^n \times [0, \infty))$ and $u_2 \in \Phi$ are, respectively, a subsolution and a supersolution of (0.1) and satisfy $u_1(\cdot, 0) \leq u_2(\cdot, 0)$ in $\mathbb{R}^n$, then $u_1 \leq u_2$ in $\mathbb{R}^n \times [0, \infty)$.

(A6) $u_0 \in C(\mathbb{R}^n)$, and there exists a pair $(K, F) \in \mathbb{R} \times C([0, \infty))$ such that

$$(0.8) \quad F \geq 0 \quad \text{in } [0, \infty), \quad \limsup_{s \searrow 0} \frac{F(s)}{s} < \infty,$$

$$(0.9) \quad K + w^-(x) \leq u_0(x) \leq K + w^-(x) + F(\hat{w}(x)) \quad \text{in } \mathbb{R}^n.$$

Remark 2. We give a sufficient condition for (A5). Assume that (A1), (A3) and (A4) hold and that $H(x, \cdot)$ is strictly convex in $\mathbb{R}^n$ for each $x \in \mathbb{R}^n$ instead of (A2). Furthermore, assume that there exist functions $\psi_i \in \text{Lip}_{loc}(\mathbb{R}^n)$ and $\sigma_i \in C(\mathbb{R}^n)$, with $i = 0, 1$, such that for $i = 0, 1$,

$$(0.10) \quad \begin{cases} \lim_{|x| \rightarrow \infty} \sigma_i(x) = \infty, & \inf\{w^-(x) + \hat{w}(x) + \psi_0(x) \mid x \in \mathbb{R}^n\} > -\infty, \\ H(x, -D\psi_i(x)) \leq -\sigma_i(x) \text{ almost every } x \in \mathbb{R}^n. \\ \lim_{|x| \rightarrow \infty} (\psi_1(x) - \psi_0(x)) = \infty. \end{cases}$$

Then (A5) holds for $\varphi = \psi_0$ by [I, Theorem 4.1]. As for examples satisfying these conditions, we give in our talk. $\square$

Lemma 1. Let F be the function of (A6). Then, there exists a function $G \in C([0, \infty)) \cap C^1((0, \infty))$ such that

$$(0.11) \quad G(0) = 0, \quad s + F(s) \leq G(s) \leq sG'(s) \quad \text{in } (0, \infty). \quad \square$$

In the following, we assume (A1)-(A6). We define the constant $\theta \in (0, \infty]$ by

$$(0.12) \quad \theta = \sup\{\theta_0 \mid \theta_0 \text{ fulfills (0.5)}\}.$$

Theorem 1. Assume (A1)-(A6).

- (i) $\theta = \infty$ if and only if w^- is a solution of (0.4).
- (ii) Let $u \in \Phi$ be a solution of the Cauchy problem (0.1)-(0.2).
 - (a) If $\theta = \infty$, then $\hat{w} = 0$ in $\mathbb{R}^n$ and $u(x, t) + ct = K + w^-(x)$ in $\mathbb{R}^n \times [0, \infty)$.
 - (b) If $\theta < \infty$, then

$$(0.13) \quad -\hat{w}(x)e^{-\theta t} \leq u(x, t) + ct - (K + w^-(x) + \hat{w}(x)) \leq [G(\hat{w}(x)) - \hat{w}(x)]e^{-\theta t} \\ \text{in } \mathbb{R}^n \times [0, \infty),$$

where G is the function of Lemma 1. $\square$

Next, we give an example such that even if (A1)-(A5) hold, the rate of convergence in (0.3) is just equal to t^{-1} as $t \rightarrow \infty$ provided (A6) is violated. For $a, b \in \mathbb{R}$, let $a^+ = \max\{a, 0\}$, $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

Example 1. For $\alpha > 0$, let

$$H(x, p) = \alpha x \cdot p + \frac{1}{2}|p|^2 - \frac{\alpha^2}{2}(1 - |x|^2)^+ \quad \text{in } \mathbb{R}^n \times \mathbb{R}^n, \\ u_0(x) = \frac{\alpha}{2} \quad \text{in } \mathbb{R}^n.$$

Then, we have:

- (i) The assumptions (A1)-(A5) hold for the constants $c = -\alpha^2/2$, $\theta_0 = \alpha$ and the functions

$$w^+(x) = \zeta_1(x), \quad w^-(x) = \zeta_k(x) \quad (k \in (0, 1)), \quad \varphi(x) = \frac{\alpha}{2}|x|^2 \quad \text{in } \mathbb{R}^n,$$

where

$$\zeta_\ell(x) = \frac{\alpha}{2}(1 - |x|^2) + \alpha\ell \int_1^{|x| \vee 1} \sqrt{r^2 - 1} dr \quad \text{for } x \in \mathbb{R}^n, \ell \in (0, 1].$$

However, there is no pair $(K, F) \in \mathbb{R} \times C([0, \infty))$ for which (A6) holds.

- (ii) The Cauchy problem (0.1)-(0.2) admits a unique solution $u \in \Phi$ given by

$$\frac{\alpha}{2(\alpha t + 1)}(|x|^2 \wedge 1) \leq u(x, t) - \frac{\alpha^2}{2}t - \zeta_1(x) \leq \frac{\alpha}{2(\alpha t + 1)}|x|^2 \quad \text{in } \mathbb{R}^n \times [0, \infty),$$

where Φ is the set of (A5) for φ of (i). $\square$

Finally, we give an example such that the precise rate of convergence in (0.3) is obtained by our sufficient condition.

Example 2. For $\alpha, \beta > 0$, let

$$H(x, p) = \alpha x \cdot p + \frac{1}{2}|p|^2 - \frac{\beta}{2}|x|^2 \quad \text{in } \mathbb{R}^n \times \mathbb{R}^n.$$

Assume that $u_0 \in C(\mathbb{R}^n)$ and that there is a constant $\ell \in (1, \infty)$ such that

$$(0.14) \quad \frac{A}{2}|x|^2 \leq u_0(x) \leq \frac{A\ell}{2}|x|^2 \quad \text{in } \mathbb{R}^n,$$

where $A = \sqrt{\alpha^2 + \beta} - \alpha$. Then, we have:

(i) Let $k \in (0, 1)$. The assumptions (A1)-(A6) hold for

$$\begin{aligned} c &= 0, \quad \theta_0 \in (0, A(1+k) + 2\alpha] \\ w^+(x) &= \frac{A}{2}|x|^2, \quad w^-(x) = \frac{Ak}{2}|x|^2, \quad \varphi(x) = \frac{\alpha}{2}|x|^2 \quad \text{in } \mathbb{R}^n, \\ K &= 0, \quad F(s) = \frac{\ell - k}{1 - k}s \quad \text{in } [0, \infty). \end{aligned}$$

In this case, θ is equal to $A(1+k) + 2\alpha (=:\theta_k)$, and $\hat{w}(x) = A(1-k)|x|^2/2$ in $\mathbb{R}^n$.

(ii) Let Φ be the set of (A5) which is defined for φ of (i). Then, the Cauchy problem (0.1)-(0.2) admits a unique solution u in Φ . By letting $G(s) = F(s) + s$ in $[0, \infty)$, Theorem 1 leads to

$$-\frac{A(1-k)}{2}|x|^2 e^{-\theta_k t} \leq u(x, t) - \frac{A}{2}|x|^2 \leq \frac{A(\ell - k)}{2}|x|^2 e^{-\theta_k t} \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

In particular, letting $k \nearrow 1$, we obtain

$$0 \leq u(x, t) - \frac{A}{2}|x|^2 \leq \frac{A(\ell - 1)}{2}|x|^2 e^{-\lambda t} \quad \text{in } \mathbb{R}^n \times [0, \infty),$$

where $\lambda = 2\sqrt{\alpha^2 + \beta}$.

(iii) When $u_0(x) = A\ell|x|^2/2$ in $\mathbb{R}^n$, a unique solution $u \in \Phi$ is given by

$$u(x, t) = \frac{A|x|^2}{2} \left(1 + \frac{\lambda(\ell - 1)}{A(\ell - 1)(e^{\lambda t} - 1) + \lambda e^{\lambda t}} \right) \quad \text{in } \mathbb{R}^n \times [0, \infty).$$

Hence, the rate $e^{-\lambda t}$ which is obtained in (ii) is optimal in this case. $\square$

References

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