

New Family of Integral Operators of Meromorphic Functions

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Abstract. We define here an integral operator $I_n(f_i, g_i)(z)$ for meromorphic functions in the punctured open unit disk. Some properties for this operator are derived.

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1 Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \quad (1.1)$$

which are analytic in the punctured open unit disk

$$U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = U \setminus \{0\}, \quad (1.2)$$

where U is the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

We say that a function $f \in \Sigma$ is meromorphic starlike of order δ ($0 \leq \delta < 1$), and belongs to the class $\Sigma^*(\delta)$, if it satisfies the inequality

$$-\Re \left(\frac{zf'(z)}{f(z)} \right) > \delta. \quad (1.3)$$

A function $f \in \Sigma$ is a meromorphic convex function of order δ ($0 \leq \delta < 1$), if f satisfies the following inequality

$$-\Re \left(1 + \frac{zf''(z)}{f'(z)} \right) > \delta, \quad (1.4)$$

and we denote this class by $\Sigma_k(\delta)$.

For $f \in \Sigma$, Wang et al. [13] (see also [14]) introduced and studied the subclass $\Sigma_N(\lambda)$ of Σ consisting of functions $f(z)$ satisfying

$$-\Re \left(\frac{zf''(z)}{f'(z)} + 1 \right) < \lambda \quad (\lambda > 1, z \in \mathbb{U}).$$

In the literature, several integral operators of meromorphic functions in the punctured open unit disk have been investigated and studied by many authors (cf., e.g., [1-11]).

For $i = 1, 2, \dots, n$, $c > 0$, and $\alpha_i, \gamma_i \geq 0$, we now, introduce a generalized integral operator $I_n(f_i, g_i)(z) : \Sigma^n \rightarrow \Sigma$ as follows

$$I_n(f_i, g_i)(z) = \frac{c}{z^{c+1}} \int_0^z u^{c-1} \prod_{i=1}^n (uf_i(u))^{\alpha_i} (-u^2 g_i'(u))^{\gamma_i} du, \quad (1.5)$$

where $f_i, g_i \in \Sigma$. Indeed, by varying the parameters c , α_i and γ_i , the operator $I_n(f_i, g_i)$ reduces to the following well-known integral operators.

(i) for $\gamma_i = 0$, we obtain the integral operator

$$H(z) = I_n(f_i)(z) = \frac{c}{z^{c+1}} \int_0^z u^{c-1} \prod_{i=1}^n (uf_i(u))^{\alpha_i} du, \quad (1.6)$$

introduced by Frasin [8].

(ii) For $c = 1$ and $\gamma_i = 0$, we obtain the integral operator

$$\mathcal{H}_n(z) = I_n(f_i)(z) = \frac{1}{z^2} \int_0^z \prod_{i=1}^n (uf_i(u))^{\alpha_i} du, \quad (1.7)$$

introduced by Mohammed and Darus [9].

(iii) For $c = 1$ and $\alpha_i = 0$, we obtain the integral operator

$$\mathcal{H}_{\gamma_1, \dots, \gamma_n}(z) = I_n(g_i)(z) = \frac{1}{z^2} \int_0^z \prod_{i=1}^n (-u^2 g_i'(u))^{\gamma_i} du, \quad (1.8)$$

introduced by Mohammed and Darus [10]

(iv) If $n = 1$, $\alpha_1 = 1$, $f_1 = f$ and $\gamma_1 = 0$ we have the integral operator

$$I_c(f)(z) = \frac{c}{z^{c+1}} \int_0^z u^{c-1} f(u) du,$$

which was studied by many authors (cf., e.g., [1, 2, 6]).

For the starlikeness of the integral operator $I_n(f_i, g_i)$, we have to recall here the following Lemma.

Lemma 1.1([12]). *Suppose that the function $\Psi : \mathbb{C}^2 \rightarrow \mathbb{C}$ satisfies the following condition:*

$$\Re\{\Psi(is, t)\} \leq 0, \quad \left(s, t \in \mathcal{R}; t \leq \frac{-(1+s^2)}{2}\right).$$

If the function $p(z) = 1 + p_1z + \dots$, is analytic in $\mathbb{U}$ and

$$\Re\{\Psi(p(x), zp'(x))\} > 0, \quad (z \in \mathbb{U}),$$

then

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}).$$

2 Main Results

In the next theorem, we place conditions for the meromorphically starlikeness of the integral operator $I_n(f_i, g_i)(z)$ which is defined in (1.5).

Theorem 2.1. *For $i = 1, 2, \dots, n$, let $f_i, g_i \in \Sigma$, $\alpha_i, \gamma_i \geq 0$ and let $c > 0$. If $f_i \in \Sigma^*$, $g_i \in \Sigma_k$, and $\sum_{i=1}^n (\alpha_i + \gamma_i) = 1$, then the general integral operator $I_n(f_i, g_i)(z)$ belongs to the meromorphic starlike function class.*

Proof. From (1.5) it follows that

$$z^2 I_n'(f_i, g_i)(z) + (c+1)z I_n(f_i, g_i)(z) = c \prod_{i=1}^n (zf_i(z))^{\alpha_i} \left(-z^2 g_i'(z)\right)^{\gamma_i} \quad (2.1)$$

Differentiating both sides of (2.1) logarithmically and multiplying by z , we obtain

$$\begin{aligned} & \frac{z^2 I_n''(f_i, g_i)(z) + (c+3)z I_n'(f_i, g_i)(z) + (c+1)I_n(f_i, g_i)(z)}{z I_n'(f_i, g_i)(z) + (c+1)I_n(f_i, g_i)(z)} \\ &= \sum_{i=1}^n \alpha_i \frac{zf_i'(z)}{f_i(z)} + \sum_{i=1}^n \gamma_i \left(\frac{zg_i''(z)}{g_i'(z)} + 1 \right) + \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \gamma_i. \end{aligned} \quad (2.2)$$

Which is equivalent to

$$\frac{z^2 I_n''(f_i, g_i)(z) + (c+2)z I_n'(f_i, g_i)(z)}{z I_n'(f_i, g_i)(z) + (c+1)I_n(f_i, g_i)(z)}$$

$$= \sum_{i=1}^n \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^n \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^n (\alpha_i + \gamma_i). \quad (2.3)$$

We can write (2.3) as the following

$$\begin{aligned} & \frac{\frac{zI_n'(f_i, g_i)(z)}{I_n(f_i, g_i)(z)} \left(\frac{zI_n''(f_i, g_i)(z)}{I_n'(f_i, g_i)(z)} + c + 2 \right)}{\frac{zI_n'(f_i, g_i)(z)}{I_n(f_i, g_i)(z)} + c + 1} \\ &= \sum_{i=1}^n \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^n \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^n (\alpha_i + \gamma_i). \end{aligned} \quad (2.4)$$

We define the regular function p in U by

$$p(z) = -\frac{zI_n'(f_i, g_i)(z)}{I_n(f_i, g_i)(z)}, \quad (2.5)$$

and $p(0) = 1$. Differentiating $p(z)$ logarithmically, we obtain

$$-p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zI_n''(f_i, g_i)(z)}{I_n'(f_i, g_i)(z)}. \quad (2.6)$$

From (2.4), (2.5) and (2.6) we obtain

$$p(z) + \frac{zp'(z)}{-p(z) + c + 1} = \sum_{i=1}^n \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^n \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^n (\alpha_i + \gamma_i). \quad (2.7)$$

Let us put

$$\Psi(u, v) = u + \frac{v}{-u + c + 1}. \quad (2.8)$$

From the hypotheses of Theorem 2.1, (2.7) and (2.8) we obtain

$$\begin{aligned} \Re\{\Psi(p(z), zp'(z))\} &= \sum_{i=1}^n \alpha_i \left(-\Re \frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^n \gamma_i \left\{ \Re \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) \right\} + 1 - \sum_{i=1}^n (\alpha_i + \gamma_i) \\ &> 1 - \sum_{i=1}^n (\alpha_i + \gamma_i) = 0. \end{aligned} \quad (2.9)$$

Now we proceed to show that

$$\Re\{\Psi(is, t)\} \leq 0, \quad \left(s, t \in \mathcal{R}; t \leq \frac{-(1+s^2)}{2} \right).$$

Indeed, from (2.8), we have

$$\Re\{\Psi(is, t)\} = \Re \left\{ is + \frac{t}{-is + c + 1} \right\} = \frac{t(c+1)}{s^2 + (c+1)^2} \leq -\frac{(1+s^2)(c+1)}{2[s^2 + (c+1)^2]} < 0. \quad (2.10)$$

Thus, from (2.9), (2.10) and by using Lemma 1.1, we conclude that $\Re\{p(z)\} > 0$, and so

$$-\Re \left\{ \frac{zI_n'(f_i, g_i)(z)}{I_n(f_i, g_i)(z)} \right\} > 0.$$

that is $I_n(f_i, g_i)(z)$ is starlike.

Next, we place conditions for the integral operator $I_n(f_i, g_i)$ to be in the class $\Sigma_N(\lambda)$.

Theorem 2.2. For $i = 1, 2, \dots, n$, let $f_i, g_i \in \Sigma$, $\alpha_i, \gamma_i \geq 0$ and let $c > 0$. If $f_i \in \Sigma^*(\delta)$, $g_i \in \Sigma_k(\delta)$, and

$$\sum_{i=1}^n (\alpha_i + \gamma_i) > \frac{c+1}{1-\delta}, \quad (2.11)$$

then $I_n(f_i, g_i)(z) \in \Sigma_N(\lambda)$, $\lambda > 1$.

Proof. Equivalently, (2.3) can be written as

$$\frac{-\left(\frac{zI_n''(f_i, g_i)(z)}{I_n'(f_i, g_i)(z)} + 1\right) - c - 1}{\frac{(c+1)I_n(f_i, g_i)(z)}{zI_n'(f_i, g_i)(z)} + 1} = \sum_{i=1}^n \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^n \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + 1 - \sum_{i=1}^n (\alpha_i + \gamma_i). \quad (2.12)$$

Therefore

$$\begin{aligned} -\left(\frac{zI_n''(f_i, g_i)(z)}{I_n'(f_i, g_i)(z)} + 1\right) &= \frac{(c+1)I_n(f_i, g_i)(z)}{zI_n'(f_i, g_i)(z)} \left[\sum_{i=1}^n \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^n \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + 1 - \sum_{i=1}^n (\alpha_i + \gamma_i) \right] \\ &\quad + \sum_{i=1}^n \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^n \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + c + 2 - \sum_{i=1}^n (\alpha_i + \gamma_i). \end{aligned} \quad (2.13)$$

Taking real part of both sides of (2.13), we obtain

$$\begin{aligned} -\Re \left(\frac{zI_n''(f_i, g_i)(z)}{I_n'(f_i, g_i)(z)} + 1 \right) &= \Re \left\{ \frac{(c+1)I_n(f_i, g_i)(z)}{zI_n'(f_i, g_i)(z)} \left[\sum_{i=1}^n \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^n \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + 1 \right. \right. \\ &\quad \left. \left. - \sum_{i=1}^n (\alpha_i + \gamma_i) \right] \right\} + \sum_{i=1}^n \alpha_i \left(-\Re \frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^n \gamma_i \Re \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + c + 2 \\ &\quad - \sum_{i=1}^n (\alpha_i + \gamma_i) \\ &\leq \left| \frac{(c+1)I_n(f_i, g_i)(z)}{zI_n'(f_i, g_i)(z)} \left[\sum_{i=1}^n \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^n \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + 1 \right. \right. \\ &\quad \left. \left. - \sum_{i=1}^n (\alpha_i + \gamma_i) \right] \right| + \sum_{i=1}^n \alpha_i \left(-\Re \frac{zf_i'(z)}{f_i(z)}\right) + \sum_{i=1}^n \gamma_i \Re \left(-\frac{zg_i''(z)}{g_i'(z)} - 1\right) + c + 2 \\ &\quad - \sum_{i=1}^n (\alpha_i + \gamma_i). \end{aligned} \quad (2.14)$$

Let

$$\lambda = \left| \frac{(c+1)I_n(f_i, g_i)(z)}{zI_n'(f_i, g_i)(z)} \left[\sum_{i=1}^n \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^n \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^n (\alpha_i + \gamma_i) \right] \right| \\ + \sum_{i=1}^n \alpha_i \left(-\Re \frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^n \gamma_i \Re \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + c + 2 - \sum_{i=1}^n (\alpha_i + \gamma_i).$$

Since $\left| \frac{(c+1)I_n(f_i, g_i)(z)}{zI_n'(f_i, g_i)(z)} \left[\sum_{i=1}^n \alpha_i \left(-\frac{zf_i'(z)}{f_i(z)} \right) + \sum_{i=1}^n \gamma_i \left(-\frac{zg_i''(z)}{g_i'(z)} - 1 \right) + 1 - \sum_{i=1}^n (\alpha_i + \gamma_i) \right] \right| > 0$, $f_i \in \Sigma^*(\delta)$, $g_i \in \Sigma_k(\delta)$, then we have

$$\lambda > c + 2 - (1 - \delta) \sum_{i=1}^n (\alpha_i + \gamma_i).$$

Then, by the hypothesis (2.11), we have $\lambda > 1$. Therefore, $I_n(f_i, g_i)(z) \in \Sigma_N(\lambda)$, $\lambda > 1$.

If we set $\gamma_i = 0$ in Theorem 2.2, then we have [8, Theorem 2.6].

Further, Putting $c = 1, \gamma_i = 0$ in Theorem 2.2, we get

Corollary 2.3. For $i = 1, 2, \dots, n$, let $f_i \in \Sigma$, $\alpha_i \geq 0$. If $f_i \in \Sigma^*(\delta)$, and

$$\sum_{i=1}^n \alpha_i > \frac{2}{1 - \delta},$$

then $\mathcal{H}_n(z) \in \Sigma_N(\lambda)$, $\lambda > 1$.

In addition, taking $c = 1, \alpha_i = 0$ in Theorem 2.2, we receive

Corollary 2.4. For $i = 1, 2, \dots, n$, let $g_i \in \Sigma$, $\gamma_i \geq 0$. If $g_i \in \Sigma_k(\delta)$, and

$$\sum_{i=1}^n \gamma_i > \frac{2}{1 - \delta},$$

then $\mathcal{H}_{\gamma_1, \dots, \gamma_n}(z) \in \Sigma_N(\lambda)$, $\lambda > 1$.

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