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Large Set Principles and Well-founded Principles

Mamoru SHIMODA * 下田 守

Introduction

Paris and Harrington presented in 1977 a new combinatorial statement which is true but not provable in Peano arithmetic PA. After the discovery of Paris-Harrington principle PH, not a few similar works for undecidable finite combinatorial principles have been reported, most of those are model theoretic. Among few proof theoretic ones Kurata [4] presented extensions of PH and their equivalents, by studying Ketonen and Solovay [3] in detail. In addition, relating to reflection principles in fragments of PA, Ono [7] obtained refinements of some results of [4].

There are two series of extensions of PH, namely PH_n $(n = 1, 2, \ldots)$ and PH_n^* $(n = 1, 2, \ldots)$. These are confused and identified in [4], so the main result of [4] should be divided into the following two theorems.

Theorem 1 *For every positive integer n,* PH_n, $\mathrm{LSP}_{\Sigma_n}[\varepsilon_0]$, $\mathrm{WFP}_{\Sigma_n}[\varepsilon_0]$, *and* $\mathrm{TI}_{\Pi_n}[\varepsilon_0]$ *are mutually equivalent in* PA.

Proof. $\mathrm{PA} \vdash \mathrm{LSP}_{\Sigma_n}[\varepsilon_0] \leftrightarrow \mathrm{WFP}_{\Sigma_n}[\varepsilon_0] \leftrightarrow \mathrm{TI}_{\Pi_n}[\varepsilon_0]$ follows from Theorem 5.1 of [7], which is a refinement of Theorems 2.5.5 and 2.5.6 of [4]. The equivalence to PH_n in PA follows from Theorems 2.6.2 and 2.7.4 of [4].

Notice that the results of [3] are fully used in the proof of the equivalence to PH_n, i.e. for proving Theorems 2.6.2 and 2.7.4 of [4].

Theorem 2 *For each positive integer n,* PH_n^*, $\mathrm{RFN}_{\Sigma_n}(\mathrm{PA})$, $\mathrm{LSP}_{\Sigma_n}^*[\varepsilon_0]$, *and* $\mathrm{WFP}_{\Sigma_n}^*[\varepsilon_0]$ *are mutually equivalent in* PA.

Proof. $\mathrm{PA} \vdash \mathrm{PH}_n^* \leftrightarrow \mathrm{RFN}_{\Sigma_n}(\mathrm{PA})$ is shown in Theorem 1.4.6 of [4].
$\mathrm{PA} \vdash \mathrm{RFN}_{\Sigma_n}(\mathrm{PA}) \leftrightarrow \mathrm{LSP}_{\Sigma_n}^*[\varepsilon_0] \leftrightarrow \mathrm{WFP}_{\Sigma_n}^*[\varepsilon_0]$ follows from Theorem 5.7 of [7].

Now $\mathrm{RFN}_{\Sigma_n}(\mathrm{PA})$ is the uniform Σ_n-reflection principle of PA (cf. e.g. [7]). The definitions of PH_n and PH_n^* are described in [6]. Other principles in the theorems will be explained later for Γ_0-versions.

*Shimonoseki City University 下関市立大学

The distinction of the two series is essentially due to Ono [7]. We follow the notations of [7] (and [4]) with a few changes. Here the well-ordering principle of [4] and [7] is called the well-founded principle, so we write $\mathrm{WFP}_{\Sigma_n}[\varepsilon_0]$ for $\mathrm{WOP}_{\Sigma_n}[\varepsilon_0]$, and so on.[1]

Relating to Friedman-McAloon-Simpson principle FMS (cf. [1]), another undecidable combinatorial principle much stronger than PH, we obtained partial extensions of the above theorems by replacing ε_0 with Γ_0. The extensions are described as Theorem 3 and Theorem 4 in the last section.

The present paper is expanded from part of [6] with some additions and improvements. Unlike its title, [6] is a brief survey about extensions of PH and those of FMS, including some corrections of [4] and some conjectures. Theorem 4 is the proof of a conjecture in [6]. Another part of [6] is treated in [5]. This paper also includes a summary of [9], which is neccesary to prove the main theorems. The constructive properties of the system of fundamental sequences for Γ_0 are extensions of the corresponding properties of the system for ε_0 in [4], which are proved by using transfinite inductions in [2] or [3].

Much of this work was suggested by R. Kurata, to whom the author owe thanks for valuable comments. It should be mentioned that the precise work of H. Ono [7] was of much help for developing the arguments in our case.

1 Definition of the principles

In this section we give the precise definitions of the principles. We assume that our system PA contains the function symbols for primitive recursive functions and their defining equations as axioms. For the representation of the ordinals below Γ_0, we adopt the notation system of Schütte [8]. So we assume that the reader is familiar with the basic concepts in Chapter V of [8]. Accordingly, there is a primitive recursive well-ordering $\prec$ on the set of all natural numbers of order type Γ_0. Henceforth we consider only the ordinals below Γ_0 (the ordinal terms of the set OT in [8]). Throughout this paper small roman letters denote the natural numbers, while small greek letters denote the ordinals below Γ_0, which we occasionally identify with the corresponding natural numbers.

In the following the definitions of the principles are modifications of those in [7]. The *transfinite induction up to Γ_0 for Π_n-formulas* ($\mathrm{TI}_{\Pi_n}[\Gamma_0]$) is the schema

$$\forall x[\forall y(\forall z(z \prec y \to \psi(z)) \to \psi(y)) \to \psi(x)]$$

for every Π_n-formula $\psi(x)$.

Suppose that θ is a formula containing at least two free variables. Let $F(\theta)$ denote the formula $\forall x \exists! y \theta(x, y)$, which means that θ is the graph of a function. Then let $SIF(\theta)$ be the formula

$$F(\theta) \wedge \forall x \forall y \forall z(\theta(x, y) \wedge \theta(x+1, z) \to y < z),$$

which means that θ is the graph of a strictly increasing function.

[1] In [6] $\mathrm{WFP}_{\Sigma_n}[\varepsilon_0]$ is denoted as $\mathrm{WFP}_{\Delta_n}[\varepsilon_0]$ etc., but here we adopt a little more formal point of view.

The *well-founded principle of Γ_0 for Σ_n-formulas* ($\mathrm{WFP}_{\Sigma_n}[\Gamma_0]$) is the schema

$$F(\theta) \to \exists x \exists y \exists z(\theta(x,y) \wedge \theta(x+1,z) \wedge \neg(z \prec y))$$

where θ is a Σ_n-formula containing at least two free variables. It means that there is no strictly descending infinite Σ_n-sequence with respect to the well-ordering $\prec$. Here a Σ_n-sequence (resp. a Σ_n-function) is an abbreviation for a sequence (resp. a function) represented by a Σ_n-formula.

To define the large set principle the notion of a system of fundamental sequences for Γ_0 is required.

We define the system by using the notation system for ordinals below Γ_0. For $\gamma > 0$, the *normal form* of γ is $\gamma_1 + \cdots + \gamma_k$, where each γ_j is of the form $\phi\alpha_j\beta_j$, $\beta_j < \phi\alpha_j\beta_j$, and $\gamma_j \geq \gamma_{j+1}$. Every ordinal $\gamma > 0$ has a unique normal form (cf. [8, Theorem 14.8]). Define $\omega_m\alpha\beta$ by induction as:

$$\omega_0\alpha\beta = \beta, \text{ and } \omega_{m+1}\alpha\beta = \phi(\alpha, \omega_m\alpha\beta).$$

Definition 1.1 (cf. [2, §3]).

For γ and n, define $\gamma[n]$ by induction on $L\gamma$, the length of γ.

(1) If $\gamma = 0$, then $\gamma[n] = 0$.
(2) If $\gamma = \phi 00 = 1$, then $\gamma[n] = 0$.
(3) If $\gamma = \phi(\delta+1, 0)$, then $\gamma[n] = \omega_{n+1}\delta 0$.
(4) If $\gamma = \phi\alpha 0$, α is limit, then $\gamma[n] = \phi(\alpha[n], 0)$.
(5) If $\gamma = \phi(0, \eta+1)$, then $\gamma[n] = \phi 0\eta \cdot (n+1)$.
(6) If $\gamma = \phi(\delta+1, \eta+1)$, then $\gamma[n] = \omega_n(\delta, \phi(\delta+1, \eta)+1)$.
(7) If $\gamma = \phi(\alpha, \eta+1)$, α is limit, then $\gamma[n] = \phi(\alpha[n], \phi\alpha\eta)+1)$.
(8) If $\gamma = \phi\alpha\beta$, β is limit, then $\gamma[n] = \phi(\alpha, \beta[n])$.
(9) If $\gamma = \gamma_1 + \cdots + \gamma_k$ (normal form), then $\gamma[n] = \gamma_1 + \cdots + \gamma_{k-1} + \gamma_k[n]$.

We write $\alpha \underset{n}{\longrightarrow} \beta$ if $\alpha = \gamma_1, \gamma_1[n] = \gamma_2, \ldots, \gamma_k[n] = \beta$ for some $\gamma_1, \ldots, \gamma_k$.[2] We write $\alpha \underset{n}{\Longrightarrow} \beta$ if either $\alpha \underset{n}{\longrightarrow} \beta$ or $\alpha = \beta$. Obviously, $\alpha \underset{n}{\longrightarrow} \beta$ implies $\alpha[n] \underset{n}{\Longrightarrow} \beta$.

Definition 1.2 (cf. [3, 4.1] and [4, Definition 2.3.1]).

For γ and $n_1, \ldots, n_k$, define $\gamma[n_1, \ldots, n_k] = \gamma[n_1][n_2, \ldots, n_k]$ by induction on k. For a finite subset X of ω and an ordinal α, X is *α-large* if $\alpha[x_1, \ldots, x_k] = 0$, where $X = \{x_1, \ldots, x_k\}$ $(x_1 < x_2 < \cdots < x_k < \omega)$. For a function $f : \omega \longrightarrow \omega$, X is *(α, f)-large* if $f(X)$ is α-large.

In the following $(x)_i = y$ means that y is the i-th element of the sequence coded by x. Now for each ordinal α and each formula θ, let $\theta^*(s, v)$ denote the following formula:

$$\exists z[(z)_0 = \alpha \wedge (\forall w < s)\exists u \exists t\{(z)_w = u \wedge \theta(x+w, t) \wedge (z)_{w+1} = u[t]\} \wedge (z)_s = v].$$

[2] Definition 2.3 of [9] should be corrected as above.

We write $[x,y]$ *is* (α,θ)*-large* instead of $\theta^*(y \dot{-} x, 0)$. Then the *large set principle of* Γ_0 *for* Σ_n*-formulas* ($\mathrm{LSP}_{\Sigma_n}[\Gamma_0]$) is the schema

$$SIF(\theta) \to \forall\alpha\forall x\exists y([x,y] \text{ is } (\alpha,\theta)\text{-large})$$

where θ is a Σ_n-formula containing at least two free variables. $\mathrm{LSP}_{\Sigma_n}[\Gamma_0]$ means that for every strictly increasing Σ_n-function f, for any α and m, there exists a number k such that $[m,k]$ is (α,f)-large.

Next we present the definition of the two principles for definable functions. Let $Pr(x)$ be a canonical representation of the provability predicate for PA. So for a formula ψ, $Pr(\ulcorner\psi\urcorner)$ means the provability of ψ in PA, where $\ulcorner\psi\urcorner$ is the Gödel number of ψ.

The *large set principle of* Γ_0 *for* Σ_n*-definable functions* ($\mathrm{LSP}^*_{\Sigma_n}[\Gamma_0]$) is the schema

$$Pr(\ulcorner SIF(\theta)\urcorner) \to \forall\alpha\forall x\exists y([x,y] \text{ is } (\alpha,\theta)\text{-large})$$

where θ is a Σ_n-formula containing at least two free variables. It means that for all strictly increasing Σ_n-definable function f, for any α and m, there exists a number k such that $[m,k]$ is (α,f)-large. For Σ_n-definable functions, see [7, §5].

Similarly, the *well-founded principle of* Γ_0 *for* Σ_n*-definable functions* ($\mathrm{WFP}^*_{\Sigma_n}[\Gamma_0]$) is the schema

$$Pr(\ulcorner F(\theta)\urcorner) \to \exists x\exists y\exists z(\theta(x,y) \wedge \theta(x+1,z) \wedge \neg(z \prec y))$$

where θ is a Σ_n-formula containing at least two free variables. It means that there is no strictly descending infinite Σ_n-definable sequence with respect to the well-ordering $\prec$.

2 Constructive properties of the system of fundamental sequences

Here we are going to show the outline of the constructive properties of the system, details are described in [9].

Lemma 2.1 *For each limit ordinal* γ, $\{\gamma[n]\}_{n<\omega}$ *is a fundamental sequence for* γ, *i.e.,*

(1) $\gamma[n] < \gamma[n+1] < \gamma$ *for all* n,

(2) $\lim_{n<\omega} \gamma[n] = \gamma$.

Proof. By induction on length.

Hence Definition 1.1 induces a system of fundamental sequences for Γ_0. Consequently, $\gamma[n] < \gamma$ for $\gamma > 0$.

For a function $f : \omega \longrightarrow \omega$, f^n means n-times iteration of f. For each number m, define $f_k = f_k\langle m\rangle : \omega \longrightarrow \omega$ $(k < \omega)$ inductively;

$$f_0(x) = m^x,\ f_{k+1}(x) = f_k^{mx+m}(1).$$

Lemma 2.2 *Let $m \geq 2$ and $f_k = f_k\langle m\rangle$. Then for all x and k, the followings hold.*

(1) $x < f_k(x) < f_k(x+1)$.
(2) $k < f_k(x) < f_{k+1}(x)$.
(3) *If* $n < m$, *then* $f_k^n(f_k(0)) < f_{k+1}(0)$.
(4) *If* $n < m$, *then* $f_k^n(f_{k+1}(x)+1) < f_{k+1}(x+1)$.

Proof. By induction on k and/or on x.

Let n be a fixed number and $f_k = f_k\langle n+2\rangle$ $(k < \omega)$. For each γ, the number $r(\gamma) = r_n(\gamma)$ is defined by induction on $L\gamma$ as follows.

(1) If $\gamma = 0$, then $r(\gamma) = 0$.
(2) If $\gamma = \phi\alpha\beta\,(\beta < \gamma)$, then $r(\gamma) = f_{r(\alpha)}(r(\beta))$.
(3) If $\gamma = \gamma_1 + \cdots + \gamma_k$ (normal form), then $r(\gamma) = r(\gamma_1) + \cdots + r(\gamma_k)$.

Proposition 2.3 *Let $r = r_n$. If $\gamma > 0$, then $r(\gamma[n]) < r(\gamma)$.*

Proof. By induction on $L\gamma$. Use Lemma 2.2.

For α and n, let $T(\alpha, n) = \{\beta\,;\, \alpha \underset{n}{\longrightarrow} \beta\}$. The cardinality of a set A is denoted by $|A|$.

Proposition 2.4 *Let $r = r_n$. If $\gamma > 0$, then $|T(\gamma, n)| \leq r(\gamma)$. Hence for all γ and n, $T(\gamma, n)$ is finite and $\gamma \underset{n}{\longrightarrow} 0$.*

Proof. By induction on $r(\gamma)$. Use Proposition 2.3.

We call β is *α-normal* if $\beta < \phi\alpha\beta$. We write $\beta \underset{n}{\overset{\alpha}{\longrightarrow}} \gamma$ if for some $\beta_1, \ldots, \beta_k$, each β_i is α-normal $(i = 1, \ldots, k)$ and $\beta = \beta_1, \beta_1[n] = \beta_2, \ldots, \beta_k[n] = \gamma$.

Lemma 2.5 (1) $\phi\alpha 0 \underset{n}{\overset{\alpha}{\longrightarrow}} 0$.

(2) *If* $\beta \underset{n}{\overset{\alpha}{\longrightarrow}} \gamma$ *and* $\alpha \leq \tau$, *then* $\beta \underset{n}{\overset{\tau}{\longrightarrow}} \gamma$.
(3) *If* $\beta \underset{n}{\overset{\alpha}{\longrightarrow}} \gamma$ *and* $\gamma \underset{n}{\overset{\alpha}{\longrightarrow}} \delta$, *then* $\beta \underset{n}{\overset{\alpha}{\longrightarrow}} \delta$.

Lemma 2.6 *Let $n \geq 1$ and $\beta > 0$.*

(1) *If* $\beta \underset{n}{\overset{\alpha}{\longrightarrow}} \gamma$, *then* $\omega_k\alpha\beta \underset{n}{\overset{\alpha}{\longrightarrow}} \omega_k\alpha\gamma$ *for all* k.
(2) *If* $\phi\alpha\beta \underset{n}{\overset{\alpha}{\longrightarrow}} \beta$, *then* $\omega_{k+1}\alpha\beta \underset{n}{\overset{\alpha}{\longrightarrow}} \omega_k\alpha\beta \underset{n}{\overset{\alpha}{\longrightarrow}} \beta$ *for all* k.

Proof. By induction on $r(\alpha)$, $r = r_n$. Use Propositions 2.3, 2.4, and Lemma 2.5.

Lemma 2.7 *Let $n \geq 1$. Then* $\phi(\alpha, \beta+1) \xrightarrow[n]{\alpha} \phi\alpha\beta \cdot (n+1) \xrightarrow[n]{\alpha} \phi\alpha\beta + 1$.

Proof. By induction on $r(\alpha)$, $r = r_n$. Use Lemmas 2.5 and 2.6.

Lemma 2.8 *Let $n \geq 1$ and $\alpha > 0$.*
(1) *If* $\alpha \xrightarrow[n]{\alpha} \beta$, *then* $\phi\alpha 0 \xrightarrow[n]{\alpha} \phi\beta 0$.
(2) *If* $\alpha \xrightarrow[n]{\alpha} \beta$ *and* $\phi\alpha\gamma = \gamma$, *then* $\phi(\alpha, \gamma+1) \xrightarrow[n]{\alpha} \phi(\beta, \gamma+1)$.

Proof. Use Lemmas 2.5 to 2.7.

Proposition 2.9 (cf. [2, theorem 3.1], [3, Theorem 2.4], and [4, Proposition 2.2.6]).
Let $n \geq 1$. If γ is limit and α-normal, then $\gamma[i+1] \xrightarrow[n]{\alpha} \gamma[i]$ *for all* $i < \omega$. *Hence the system is (n)-built-up .*

Proof. By induction on $L\gamma$. Use Proposition 2.4 and Lemmas 2.5 to 2.8.

Proposition 2.10 (cf. [2, Proposition 1.1]).
If $\alpha \xrightarrow[m]{} \beta$ *and $m < n$, then* $\alpha \xrightarrow[n]{} \beta$.

Proof. Use the previous proposition.

Proposition 2.11 *Let $n \geq 1$ and $\alpha > 0$. If* $\alpha \xrightarrow[n]{} \mu$ *and* $\phi\alpha\beta \xrightarrow[n]{} \nu$, *then* $\phi\alpha\beta \xrightarrow[n+1]{} \phi\mu\nu$.

Proof. By induction on $r(\gamma)$, $r = r_{n+1}$. Use Lemmas 2.5 to 2.8, Propositions 2.9 and 2.10.

The *height* of γ, denoted by $h(\gamma)$, is defined by induction on $L\gamma$:
(1) If $\gamma = 0$, then $h(0) = 0$.
(2) If $\phi\alpha_1\beta_1 + \cdots + \phi\alpha_k\beta_k$ is the normal form of γ, then

$$h(\gamma) = \max\{h(\alpha_i) + 1, h(\beta_i)\,;\, i = 1, \ldots, k\}.$$

Lemma 2.12 *If $\gamma < \lambda$, then $h(\gamma) \leq h(\lambda)$.*

Proof. By induction on $L\gamma$ and on $L\lambda$.

Lemma 2.13 *$h(\gamma) \leq k$ if and only if $\gamma < \zeta_k$.*

Proof. By induction on k and on $L\gamma$.

The number $c(\gamma)$ is defined by induction on $L\gamma$:
(1) If $\gamma = 0$, then $c(0) = 0$.
(2) If $\gamma = \phi\alpha_1\beta_1 \cdot n_1 + \cdots + \phi\alpha_k\beta_k \cdot n_k + m$, $\beta_i < \phi\alpha_i\beta_i$, $\phi\alpha_1\beta_1 > \cdots > \phi\alpha_k\beta_k > 1$, then

$$c(\gamma) = \max\{c(\alpha_i) + 1,\ c(\beta_i + 1) + 1,\ n_i,\ m\,;\ i = 1, \ldots, k\}.$$

Proposition 2.14 (cf. [3, Corollary 2.2.8]).
If $h(\gamma) \leq k$ and $c(\gamma) \leq n$, then $\zeta_k \underset{n}{\longrightarrow} \gamma$.

Proof. By induction on k and on $L\gamma$. Use Lemmas 2.7, 2.12, and Proposition 2.11.

Proposition 2.15 (cf. [2, §4], [3, Proposition 2.8.1], and [4, Lemma 2.2.9]).
There exists a primitive recursive function $g(\alpha, \beta)$ such that if $\alpha > \beta$ and $n = g(\alpha, \beta)$, then $\alpha \underset{n}{\longrightarrow} \beta$.

Proof. Let $n = g(\alpha, \beta) = \max\{c(\alpha), c(\beta)\}$. It is obvious that $g(\alpha, \beta)$ is primitive recursive. (Recall that α and β are identified with the corresponding natural numbers.) Put $k = \max\{h(\alpha), h(\beta)\}$ and apply Proposition 2.14.

Notice that all the statements in this section can be proved in PA. In [2] and [3], the corresponding properties for ε_0-versions are proved by transfinite inductions, but we can prove those for Γ_0-versions only by using mathematical inductions.

3 Equivalent principles

By the constrctive properties of the system of fundamental sequences, shown in the previous section, partial extensions of Theorems 1 and 2 are to be proved.

Theorem 3 (cf. [6, Theorem 5.1]).
Let n be a positive integer. Then the following three schemata are mutually equivalent in PA:

$$(1)\ \mathrm{LSP}_{\Sigma_n}[\Gamma_0], \qquad (2)\ \mathrm{WFP}_{\Sigma_n}[\Gamma_0], \qquad (3)\ \mathrm{TI}_{\Pi_n}[\Gamma_0].$$

Proof. Since PA $\vdash$ (2) $\rightarrow$ (3) $\rightarrow$ (1) is proved in the same way as in Theorem 5.1 of [7] (cf. [3, Theorems 2.5.5 and 2.5.6]), it suffices to prove PA $\vdash$ (1) $\rightarrow$ (2).

Assume $\mathrm{WFP}_{\Sigma_n}[\Gamma_0]$ does not hold. Then there is a strictly descending Σ_n-sequence $\{\alpha_i\}_{i<\omega}$ of ordinals. Let $g(\alpha, \beta)$ be the primitive recursive function of Proposition 2.15, so $\alpha > \beta$ and $n = g(\alpha, \beta)$ implies $\alpha \underset{n}{\longrightarrow} \beta$. For each $i < \omega$, let $m_i = g(\alpha_i, \alpha_{i+1})$, then $\alpha_i \underset{m_i}{\longrightarrow} \alpha_{i+1}$. Define $\{n_i\}_{i<\omega}$ by $n_0 = m_0 + 1$ and $n_{i+1} = \max(m_{i+1}, n_i) + 1$. Since $m_i < n_i$,

$\alpha_i \xrightarrow[n_i]{} \alpha_{i+1}$ by Proposition 2.10. Define $f : \omega \longrightarrow \omega$ by $f(i) = n_i$ $(i < \omega)$. Since g is primitive recursive, f is a strictly increasing Σ_n-function, i.e. there is a Σ_n-formula ψ such that $SIF(\psi)$ holds and ψ represents the graph of f.

Define $\{\beta_i\}_{i<\omega}$ by $\beta_0 = \alpha_0$ and $\beta_{i+1} = \beta_i[n_i]$. By induction on i, $\beta_{i+1} \Longrightarrow_{n_i} \alpha_{i+1}$ holds for all $i < \omega$. Let $\alpha = \alpha_0$, then for every i,

$$\alpha[f(0), \ldots, f(i)] = \alpha[n_0, \ldots, n_i] = \beta_{i+1} \geq \alpha_{i+1} > 0.$$

So for all number i, $[0, i]$ is not (α, f)-large, in other words, $[0, i]$ is not (α, ψ)-large. Since ψ is a Σ_n-formula satisfying $SIF(\psi)$, $\mathrm{LSP}_{\Sigma_n}[\Gamma_0]$ fails.

To see that the above argument can be done in PA, we will show only the first step of the formalizing process. Recall that Propositions 2.15 and 2.10 are provable in PA. Let T be a theory obtained from PA by adding the formula $F(\theta)$ and the following formula

$$\forall x \forall y \forall z(\theta(x, y) \wedge \theta(x + 1, z) \rightarrow z \prec y)$$

as additional axioms, where θ is a Σ_n-formula. Then θ represents the graph of a strictly descending Σ_n-sequence $\{\alpha_i\}_{i<\omega}$. Define the formulas $\tau(x, s)$ and $\psi(x, t)$ by

$$\tau(x, s) \equiv \exists y \exists z(\theta(x, y) \wedge \theta(x + 1, z) \wedge g(y, z) = s)$$

and

$$\begin{aligned}\psi(x, t) \equiv \exists z[&\exists y((z)_0 = y + 1 \wedge \tau(0, y)) \wedge (\forall u < x)\exists v \exists w \exists s\{(z)_u = v \\ &\wedge w = \max(s, v) + 1 \wedge \tau(x + 1, s) \wedge (z)_{u+1} = w)\} \wedge (z)_x = t].\end{aligned}$$

Clearly τ and ψ represents the graphs of the sequences $\{m_i\}_{i<\omega}$ and $\{n_i\}_{i<\omega}$ respectively, so ψ represents the graph of the function f. Since θ is a Σ_n-formula and g is primitive recursive, τ and ψ are also Σ_n-formulas. It is easy to prove in PA that $F(\theta)$ implies $SIF(\psi)$, so $T \vdash SIF(\psi)$.

Through similar process of formalization, we can show that

$$T \vdash [0, i] \text{ is not } (\alpha, \psi)\text{-large for every } i < \omega.$$

Therefore

$$\exists \alpha \exists x \exists y([x, y] \text{ is not } (\alpha, \psi)\text{-large})$$

is provable in T. □

Next we consider the other two principles for Σ_n-definable functions.

Theorem 4 (cf. [6, Lemma 5.3]).

Let n be a positive integer. Then the following two schemata are mutually equivalent in PA:

$$(1)\ \mathrm{LSP}^*_{\Sigma_n}[\Gamma_0], \qquad (2)\ \mathrm{WFP}^*_{\Sigma_n}[\Gamma_0].$$

Proof. By a quite similar argument as in the proof of Theorem 3, we can show that (1) $\rightarrow$ (2) is provable in PA. For since $F(\theta) \rightarrow SIF(\psi)$ is provable in PA, $Pr(\ulcorner F(\theta) \urcorner)$ implies $Pr(\ulcorner SIF(\psi) \urcorner)$ in PA. Hence it suffices to prove the inverse, i.e. PA $\vdash$ (2) $\rightarrow$ (1).

Let T be the theory obtained from PA with $\mathrm{WFP}^*_{\Sigma_n}[\Gamma_0]$ and $Pr(\ulcorner SIF(\theta) \urcorner)$ as additional axioms, where θ is a Σ_n-formula. Suppose an ordinal α and a number x are given. We may assume $\alpha > 0$. Here we write ζ for θ^*, so $\zeta(s,v)$ is the following Σ_n-formula:

$$\exists z[(z)_0 = \alpha \wedge (\forall w < s)\exists u \exists t\{(z)_w = u \wedge \theta(x+w,t) \wedge (z)_{w+1} = u[t]\} \wedge (z)_s = v].$$

Since $SIF(\theta) \rightarrow F(\theta)$ and $F(\theta) \rightarrow F(\zeta)$ are provable in PA,

$$T \vdash Pr(\ulcorner F(\zeta) \urcorner).$$

Hence by $\mathrm{WFP}^*_{\Sigma_n}[\Gamma_0]$,

$$T \vdash \exists s \exists v \exists w(\zeta(s,v) \wedge \zeta(s+1,w) \wedge \neg(w \prec v)).$$

But by a property of fundamental sequences,

$$T \vdash \forall s \forall v \forall w(\zeta(s,v) \wedge \zeta(s+1,w) \wedge 0 \prec v \rightarrow w \prec v).$$

Hence $T \vdash \exists s \zeta(s,0)$. Therefore

$$\forall \alpha \forall x \exists y([x,y] \text{ is } (\alpha,\zeta)\text{-large})$$

is provable in T. $\square$

We conjecture that the equivalent principles in Theorem 3 are also equivalent to FMS_n, which is an extension of FMS [1]. We also conjecture that the equivalent ones in Theorem 4 are equivalent to FMS^*_n, which is another extension of FMS and is proved to be equivalent to $\mathrm{RFN}_{\Sigma_n}(\mathrm{ATR}_0)$ in [5]. The definitions of FMS_n and FMS^*_n can be found in [6].

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Mamoru SHIMODA
Shimonoseki City University
Shimonoseki 751
Japan