

G-VECTOR BUNDLES OVER G-MANIFOLDS

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At first I want to talk about some of my works, which I have stated in other seminar, but it is my starting point, so let me do that.

Definition. Let G be a compact Lie group. A real or complex vector bundle $E \rightarrow X$ is a G -vector bundle if and only if

- (1) E and X are G -spaces,
- (2) the projection p is an equivariant map,
- (3) for each $x \in X$ and $g \in G$, the action $g: E_x \rightarrow E_{gx}$ is linear.

Let M be a compact G -manifold with orbit type $((H), (K))$, where H, K are closed subgroups of G with $H \subset K$.

Definition (K.Jänich). A G -manifold M is special if and only if, denoting by V_x the normal space to the orbit through x for each $x \in M$, the slice representation $G_x \rightarrow \text{Aut}(V_x)$ admits a decomposition $V_x = F_x \oplus W_x$ such that $G_x|_{F_x} = \text{the identity of } F_x$ and $G_x|_{S(W_x)}$ (the unit sphere in W_x) is transitive.

We use some notations. $M_{(K)}$: the union of all singular orbits, which is a closed submanifold of M . $M_1 = \overline{M - N(M_{(K)})}$: the closure of an invariant tubular neighborhood of $M_{(K)}$. $\widehat{\text{Vect}}_K$: the family of all K -vector bundles.

The projection $p : \partial N(M_{(K)}) \rightarrow M_{(K)}$ induces a diffeomorphism $p' : \partial \pi(M_1) \rightarrow \pi(M_{(K)})$, where π is the projection $M \rightarrow M/G$. For a pair $(F, E) \in \widehat{\text{Vect}}_K(\pi(M_{(K)})) \times \widehat{\text{Vect}}_H(\pi(M_1))$, let $\alpha_H : p'^* r^* F \rightarrow E|_{\partial \pi(M_1)}$ be an isomorphism of H -vector bundles, where we denote by r^* the restriction $\widehat{\text{Vect}}_K \rightarrow \widehat{\text{Vect}}_H$.

Definition 1. Two triples (F, E, α_H) and $(\bar{F}, \bar{E}, \bar{\alpha}_H)$ is equivalent if and only if there exists a commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{p'^* r^*} & p'^* r^* F & \xrightarrow{\alpha_H} & \partial E \subset E \\ \downarrow \rho_K & & \downarrow \rho_{H,K} & & \downarrow \partial \mathcal{F}_H \downarrow \mathcal{F}_H \\ \bar{F} & \xrightarrow{p'^* r^*} & p'^* r^* \bar{F} & \xrightarrow{\bar{\alpha}_H} & \partial \bar{E} \subset \bar{E}, \end{array}$$

where $\rho_K(\mathcal{F}_H)$ is an isomorphism of $K(H)$ -vector bundles, and $\rho_{H,K}, \partial \mathcal{F}_H$ are restrictions of them.

Theorem. Under the condition

(C) $N(H) = H \rtimes \Gamma(H)$, $\Gamma(H) = N(H)/H$, $N(K) = K \rtimes \Gamma(K)$, $\Gamma(K) = N(K)/K$, $\Gamma(K) \subset \Gamma(H) \subset G$, we have an isomorphism of semi-groups

$$\text{Vect}_G(M) \approx \{(E, F, \alpha_H)\} / (\sim),$$

where we mean by $/(\sim)$ the classification due to Definition 1.

By the theorem and an analogy of the Atiyah-Bott's proof of Bott periodicity, and using H. Minami's result about $R(O(n))$, I have obtained $K_{O(n)}(W^{2n-1}(d)) \cong R(O(n-1))$. These results are to be appeared in Osaka Journal of Mathematics. ([1])

In the determination of the K_G -group, I have essentially used the splitting of the normalizer $N(I_r \times O(n-r)) = O(r) \times O(n-r)$, $r = 1, 2$. The condition (C) is too restrictive for applications, so I want to improve it.

By a technical reason I take right actions. Let M be a compact right G -manifold with just one orbit type (H) . Denote by Γ the factor group $H \backslash N(H)$, then we have a differentiable principal bundle

$$(1) \quad \Gamma \longrightarrow M_H \longrightarrow M_H/\Gamma,$$

where $M_H = \{x \in M; G_x = H\}$ and the G -manifold M is the total space of the associated fiber bundle, that is $M \cong M_H \times_{\Gamma} (H \backslash G)$ as a G -manifold. By G.Segal

$$\text{Vect}_G(M) \approx \text{Vect}_{N(H)}(M_H).$$

(Under the condition (C), $\text{Vect}_{N(H)}(M_H) \approx \text{Vect}_H(M_H/\Gamma)$.) Since M_H is a compact differentiable manifold, there exists an open covering $\{U_i\}$ of M_H/Γ such that U_i is deformable to a point x_i of U_i and a Γ -equivalence $\varphi_i : U_i \times \Gamma \longrightarrow M_H|_{U_i}$ for each i . For any $N(H)$ -vector bundle $E \longrightarrow M_H$, there are N -vector bundles $E_i \longrightarrow U_i$ with $\varphi_i^* E_i \cong E|_{\varphi_i(U_i \times \Gamma)}$, ($N = N(H)$). Further we have an isomorphism of N -vector bundles $E_i \xrightarrow{N} U_i \times (E|x_i \times \Gamma)$, because $U_i \times \Gamma$ is N -deformable to $x_i \times \Gamma$, where N -action over Γ is given by the projection $q : N \longrightarrow \Gamma = H \backslash N(H)$. On the other hand, by G.Segal $E_i|_{x_i \times \Gamma} \xrightarrow{N} V_i \times_{\Gamma} N$, where $V_i = E_i|_{x_i \times (e)}$,

which is an H -module. Thus we have an isomorphism of N -vector bundles,

$$\begin{array}{ccc} U_i \times V_i \times_{H^N} & \xrightarrow[\approx]{\Psi_i} & E|_{\varphi_i(U_i \times \Gamma)} \\ \downarrow p_i & & \downarrow p \\ U_i \times \Gamma & \xrightarrow[\approx]{\varphi_i} & \varphi_i(U_i \times \Gamma) \end{array}$$

Denote the N -equivalence $\Psi_j^{-1} \Psi_i : (U_i \cap U_j) \times V_i \times_{H^N} \rightarrow (U_i \cap U_j) \times V_j \times_{H^N}$ by $\Psi_j^{-1} \Psi_i(x, [v, n]) = (x, G_{ji}(x)[v, n])$, then with respect to the GO -topology of $\text{Iso}_N(V_i \times_{H^N}, V_j \times_{H^N})$, G_{ji} is a continuous map for each i, j . By the usual verification (Part 1. [3]), we get the next propositions.

Proposition 1. For any N -vector bundle $E \rightarrow M_H$,

$$E \cong^N [\bigcup_i U_i \times V_i \times_{H^N}] / (G_{ji}).$$

Proposition 2. Two N -vector bundles (E, G_{ji}) and (E', G'_{lk}) are equivalent if and only if there exist continuous maps $\bar{G}_{ki} : U'_k \cap U_i \rightarrow \text{Iso}_N(V_i \times_{H^N}, V'_k \times_{H^N})$ with the property

$$\bar{G}_{kj} G_{ji} = \bar{G}_{ki} \text{ on } U'_k \cap U_j \cap U_i, \quad G'_{lk} \bar{G}_{kj} = \bar{G}_{lj} \text{ on } U'_l \cap U'_k \cap U_j$$

Now to proceed much more, we consider the case which satisfies the condition

$$(\mathcal{J}) \quad N(H) = H \cdot \Gamma : \text{semi-direct product.}$$

For example, take $SO(n)$ as G and $I_r \times SO(n-r)$ as H , then $N(I_r \times SO(n-r)) = SO(n-r) \cdot O(r)$. In fact the section $O(r) \rightarrow N(I_r \times SO(n-r))$ is given by

$$O(r) \ni A \rightarrow \begin{pmatrix} A & 0 & \dots & 0 \\ 0 & \det A & & \\ 0 & 0 & 1 & \dots \\ \vdots & & & \ddots \\ 0 & 0 & & & 1 \end{pmatrix} \in N.$$

The same property can be verified for $G = SU(n)$ and $H = I_r \times SU(n-r)$, these cases happen actually (Chap. 4, [2]).

Define an H -action over $V_i \times \Gamma$ by $(v, \gamma)h = (v \cdot h^\gamma, \gamma)$, where $h^\gamma = \gamma h \gamma^{-1}$, then under the condition (8), we have

$$V_i \times \Gamma \stackrel{N}{\sim} V_i \times_H N.$$

Then we can represent G_{ji} as follows :

$$\psi_j^{-1} \psi_i(x, v, \gamma) = (x, g_{ji}(x)(v), \gamma_{ji}(x) \gamma),$$

where γ_{ji} is the transition function of the principal bundle (1). We can prove that $g_{ji} : U_i \cap U_j \rightarrow \text{Iso}(V_i, V_j)$ is continuous and

$$(*) \quad g_{ji}(x)(vh) = [g_{ji}(x)(v)] h \gamma_{ji}(x).$$

Thus in the semi-direct product case, Proposition 2 can be stated as

Proposition 2.8. (E, g_{ji}, γ_{ji}) is N -equivalent to $(E', g'_{lk}, \gamma'_{lk})$ if and only if there exist continuous maps $\bar{g}_{ki} : U_i \cap U'_k \rightarrow \text{Iso}(V_i, V'_k)$ with the property

$$\bar{g}_{ki}(x)(vh) = [\bar{g}_{ki}(x)(v)] h \bar{\gamma}_{ki}(x),$$

$$\bar{g}_{kj}(x)g_{ji}(x) = \bar{g}_{ki}(x), \quad g'_{lk}(x)g_{kj}(x) = \bar{g}_{lj}(x),$$

where $\bar{\gamma}_{ki}$ is the equivalence between γ_{ji} and γ'_{lk} .

By Proposition 2.8, we define an equivalence relation of coordinate vector bundles $[\cup U_i \times V_i] / (g_{ji}, \gamma_{ji})$ over M_H / Γ with the property (*). We will call these bundles local H-vector bundles. Denote by $\text{Vect}_{H^*}(M_H / \Gamma)$ the semi-group of equivalence classes. Then we have

Theorem. Let M be a G -manifold with just one orbit type (H) , then under the condition (8),

$$\text{Vect}_G(M) \approx \text{Vect}_{N(H)}(M_H) \approx \text{Vect}_H(M_H / \Gamma).$$

Examples.

Consider the standard m -sphere $S^m = D_1^m \cup D_2^m$ (the union of the upper and lower hemi-spheres. Let $\gamma_{ji} : D_1^m \cap D_2^m \rightarrow \Gamma$ be transition functions of a principal bundle $\Gamma \rightarrow P \rightarrow S^m$. For any local H -vector bundle E , $D_i^m \times V_i \approx E|D_i^m$, where V_i is an H -module for $i = 1, 2$. We fix a point x_0 in $S^{m-1} = D_1^m \cap D_2^m$. By an appropriate choice of $(\bar{\gamma}_{ki}, \bar{g}_{ki})$, we can get a local H -vector bundle $(\gamma'_{ji}, g'_{ji}, D_i^m \times V_i)$ which is equivalent to $E = (\gamma_{ji}, g_{ji}, D_i^m \times V_i)$, and $g'_{12}(x_0) =$ the identity of $V_1 = V_2$ as a vector space and $\gamma'_{12}(x_0) =$ the unite of Γ . This is a usual normal form of a vector bundle over S^m (Part II, [3]). For each $v \in V_1$,

$$v \cdot_1 h = g'_{12}(x_0)(v \cdot_1 h) = [g'_{12}(x_0)(v)] \cdot_2 h \gamma'_{12}(x_0) = v \cdot_2 h,$$

where we denote by $\cdot_i h$ the action in V_i . Thus we have $V_1 = V_2$ as an H -module. Now we investigate the case $G = SO(n)$, $H = I_r \times SO(n-r)$.

(I) Case $m \geq 2$.

Since S^{m-1} is connected, then $g'_{12}(S^{m-1}) \subset SO(r) \subset O(r)$ and $SO(n-r) \cdot SO(r) = SO(n-r) \times SO(r)$ direct product, further $SO(r) \times I_{n-r}$ -action on $I_r \times SO(n-r)$ by the conjugacy is trivial. Thus we have

$$\text{Vect}_H \gamma(S^m) \cong \text{Vect}_H(S^m).$$

(II) Case $m = 1$.

We can prove the next lemma by the normal form technique.

Lemma $\text{Vect}_H^{\mathbb{C}} \gamma(S^1) \approx \hat{H}^{\gamma}$, the semi-group of isomorphism classes of γ -invariant complex H -modules.

Suppose to be $\gamma_{12}(\pm 1) = \begin{pmatrix} \pm 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix}$. Set $n-r = 2s$ or $2s+1$.

We need a well known formula for the complex representation rings. Let T be the standard maximal torus of $SO(n-r)$, then $R(T) = \mathbb{Z}[\alpha_1, \alpha_1^{-1}, \dots, \alpha_s, \alpha_s^{-1}]$, where α_k is the representation

$$\text{Diag} \left\{ \begin{pmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{pmatrix} \dots \begin{pmatrix} \cos \theta_s & -\sin \theta_s \\ \sin \theta_s & \cos \theta_s \end{pmatrix} \right\} \rightarrow e^{2\pi i \theta_k},$$

for $k = 1, 2, \dots, s$. Then we have $R(SO(2s)) = \mathbb{Z}[\lambda^1, \dots, \lambda_+^s, \lambda_-^s]$

with a relation $(\lambda_+^s + \lambda_+^{s-2} + \dots)(\lambda_-^s + \lambda_-^{s-2} + \dots) = (\lambda_+^{s-1} + \lambda_+^{s-3} + \dots)^2$, where $\lambda^k = \sigma^k[\alpha_1, \alpha_1^{-1}, \dots, \alpha_s, \alpha_s^{-1}]$ k -th elementary

symmetric function and $\lambda_{\pm}^s = \sum_{\epsilon_1 \dots \epsilon_s = \pm 1} i_1 \leq \dots \leq i_s \leq s \alpha_{i_1}^{\epsilon_1} \dots \alpha_{i_s}^{\epsilon_s}$.

By the relation

$$\begin{pmatrix} -1 & 1 & & & & & 0 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ 0 & & & & & & -1 & 1 & & & \\ & & & & & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & & & & & 0 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ 0 & & & & & & (a_{ij}) \end{pmatrix} \begin{pmatrix} -1 & 1 & & & & & 0 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ 0 & & & & & & 1 & -1 & 1 & & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & & & & & 0 \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ 0 & & & & & & a_{11} & -a_{12} & \dots & -a_{1n} \\ & & & & & & -a_{21} & & & \\ & & & & & & \vdots & & & \\ & & & & & & \vdots & & & \\ & & & & & & -a_{n1} & & & \end{pmatrix}$$

we have

$$(\alpha_1) \gamma_{12}^{(-1)} = \alpha_1^{-1} ; \quad (\alpha_k) \gamma_{12}^{(-1)} = \alpha_k, \quad k = 2, 3, \dots, s$$

$$; \quad (\lambda^k) \gamma_{12}^{(-1)} = \lambda^k, \quad k = 1, 2, \dots, s ; \quad (\lambda_{\pm}^s) \gamma_{12}^{(-1)} \neq \lambda_{\pm}^s,$$

and so

$$K(\text{Vect}_{\mathbb{H}}^{\mathbb{C}} \gamma(S^1)) = \mathbb{Z} [\lambda^1, \dots, \lambda^s] \subsetneq R(\text{SO}(n-r)).$$

When we treat the manifold $W^{2n-1}(d)$ as an $\text{SO}(n)$ -manifold,

$$W^{2n-1}(d)_{\text{SO}(n-1)} = S^1 = \{ (z_0, z_1) ; z_0^d + z_1^2 = 0, \quad |z_0| = |z_1| = 1 \}.$$

If d is odd, then the double covering $S^1 \rightarrow S^1/O(1)$ is non trivial,

and so $\gamma_{12} : S^0 \rightarrow O(1)$ is surjective. Hence if n, d odd together, then

$$K_{\text{SO}(n)}(W^{2n-1}(d)_{\text{SO}(n-1)}) \simeq \mathbb{Z} [\lambda^1, \dots, \lambda^s] \subsetneq R(\text{SO}(n-1)).$$

References

- [1] H. Matsunaga : $\mathbb{K}_G$ -groups and invariant vector fields on special G -manifolds, to appear in Osaka Journal of Mathematics.

- [2] Wu-chung Hsiang and Wu-yi Hsiang : Differentiable actions of compact connected classical groups I , Amer. J. of Math. 139 (3), 1967.
- [3] N.E.Steenrod : The topology of fiber bundles, 1951, Princeton.