

Skeletons of some relatives of the n -cube

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Abstract

We study the skeleton of several polytopes related to the n -cube, the halved n -cube, and the folded n -cube. In particular, the Gale polytope of the n -cube, its dual and the duals of the halved n -cube and the complete bipartite sub-graphs polytope.

1 Introduction

The general references are [2, 6, 12] for polytopes, [4] for graphs and [5] for lattices. We first recall some basic properties of the cube and the halved cube.

The vertices of the n -cube $\gamma_n = [0, 1]^n$ are all the 2^n characteristic vectors χ^S for $S \subset N = \{1, 2, \dots, n\}$, that is, $\chi_i^S = 1$ for $i \in S$ and 0 otherwise. With $|S \Delta S'|$ denoting the size of the symmetric difference of the subsets S and S' , two vertices χ^S and $\chi^{S'}$ are adjacent if and only if $|S \Delta S'| = 1$. The skeleton of γ_n is denoted by $H(n, 2)$ and the skeleton of its dual, the cross-polytope $\beta_n = \gamma_n^*$, is $K_{2 \times n}$, which is also called the Cocktail-Party graph. The diameter of the n -cube and its dual are, respectively, n and 2.

The *halved n -cube* $h\gamma_n$ (see Section 8.6 of [6]) is obtained from the n -cube γ_n by selecting the vertex of even cardinality on each edge, that is, $h\gamma_n$ is the convex hull of all the 2^{n-1} characteristic vectors χ^S for $S \subset N = \{1, 2, \dots, n\}$ and $|S|$ even. Two vertices χ^S and $\chi^{S'}$ are adjacent if and only if $|S \Delta S'| = 2$. The skeleton of the halved n -cube is denoted by $\frac{1}{2}H(n, 2)$; its diameter is $\lfloor \frac{n}{2} \rfloor$.

2 Skeleton of the dual halved n -cube

The halved 3-cube is a regular tetrahedron α_3 . The halved 4-cube is the simplicial polytope $h\gamma_4 = \beta_4$. For $n > 4$, the facets of $h\gamma_n$ -cube are partitioned into the following two orbits of its symmetry group $2^{n-1}\text{Sym}(n)$. The orbit O_1^n consists of the $2n$ facets belonging to the facets of the n -cube and defined by the inequalities:

$$x_i \leq 1 \quad \text{for } i \in N, \quad (1)$$

$$x_i \geq 0 \quad \text{for } i \in N. \quad (2)$$

The orbit O_2^n consists of the 2^{n-1} facets cutting off the vertices of odd cardinality from the n -cube and defined by the inequalities:

$$\sum_{i=1}^n x_i(1 - 2\chi_i^A) \leq |A| - 1 \quad \text{for } A \subset N \text{ and } |A| \text{ odd.} \quad (3)$$

The facets defined by the inequalities (1), (2) and (3) are respectively denoted by F_1^i , F_0^i and F^A . Since the symmetries of a polytope preserve adjacency and linear independence, we can describe the properties of its facets by simply considering a representative facet of each orbit. The facets $F_1^i \simeq F_0^i \simeq h\gamma_{n-1}$ (here and in the following “ $\simeq$ ” denotes the affine equivalency) and each facet F^A is the simplex containing the n vertices: $\chi^{A \cup \{i\}}$ for $i \in \bar{A}$ and $\chi^{A \setminus \{i\}}$ for $i \in A$.

The skeleton of the dual halved n -cube, denoted by $h\gamma_n^*$, is the graph whose nodes are the facets of $h\gamma_n$, two facets being adjacent if and only if their intersection is a face of codimension 2. This skeleton is given below.

Lemma 2.1 *The facets of O_1^n and O_2^n form, respectively, the coclique $\bar{K}_{2n}$, and the coclique $\bar{K}_{2^{n-1}}$; each facet F^A is adjacent, either to F_1^i if $i \in A$, or to F_0^i if $i \in \bar{A}$ for each $i \in N$.*

Corollary 2.2 *For $n \geq 4$, the skeleton of the dual halved n -cube is a bipartite graph of diameter 4.*

PROOF. Since the valency of a facet belonging to O_1^n , respectively to O_2^n , is half the size of O_2^n , respectively of O_1^n , we have $\delta(h\gamma_n^*) \leq 4$. On the other hand, the facets F_1^i and F_0^i , having no common neighbour, we get $\delta(h\gamma_n^*) > 3$. $\square$

Corollary 2.3 *The halved n -cube has $n \cdot 2^{n-2}$ faces of codimension 2 which are all simplices, that is $h\gamma_n$ is quasi-simplicial. For $n \rightarrow \infty$, $h\gamma_n$ is asymptotically simplicial.*

PROOF. Since the number of faces of codimension 2 of a polytope is half of the total valency of the skeleton of its dual, the result is a straightforward calculation. All faces of codimension 2 being incident to the simplex facets of $h\gamma_n$, the halved n -cube is a quasi-simplicial. $\square$

3 Gale transform of the n -cube

Let A be a $(2^n - n - 1) \times 2^n$ matrix whose rows form a basis for the space of all the affine dependencies on the vertices of the n -cube. A Gale transform of γ_n is the collection of the 2^n points in $\mathbb{R}^{2^n - n - 1}$ which are the columns of A .

We consider the matrix A induced by the following $2^n - n - 1$ affine dependencies on the vertices of γ_n :

$$(1 - |T|)\chi^\emptyset + \sum_{i \in T} \chi^{\{i\}} - \chi^T = 0 \quad \text{for } T \subset N \text{ and } |T| \geq 2. \quad (4)$$

Since each column of A corresponds to a vertex χ^S of γ_n for $S \subset N$, we simply denote by v^S the vector formed by this column of A . For example, the first column of A corresponds to $\chi^\emptyset$ and forms the vector $v^\emptyset$ whose $2^n - n - 1$ coordinates are $v_T^\emptyset = (1 - |T|)$, where $\mathbb{R}^{2^n - n - 1}$ is naturally indexed by $T \subset N$, $|T| \geq 2$.

A *Gale polytope*, $Gale(P)$, of a polytope P is the convex hull of a Gale transform of P . In the following we consider $Gale(\gamma_n)$ associated to the affine dependencies (4). The polytope $Gale(\gamma_3)$ is a prism over a tetrahedron; see also Example 5.6 in [3] for relation with Lawrence polytopes. For $n \geq 4$, we introduce some edges and facets of $Gale(\gamma_n)$ in order to compute its diameter and the one of its dual.

Consider the following inequalities, where x_T for $T \subset N$ and $|T| \geq 2$ are the coordinates of a point x in $\mathbb{R}^{2^n - n - 1}$ indexed by $T \subset N$, $|T| \geq 2$.

$$-x_A \leq 1 \quad \text{for } |A| = 2, \quad (e_1)$$

$$x_{A \setminus \{i\}} - x_A \leq 1 \quad \text{for } |A| \geq 3 \text{ and } i \in A, \quad (e_2)$$

$$x_A \leq 1 \quad \text{for } |A| = 2, \quad (e_3)$$

$$x_{A \cup \{i\}} - x_A \leq 1 \quad \text{for } |A| \geq 2 \text{ and } i \notin A, \quad (e_4)$$

$$2 \sum_{j \in N} x_{\{j\}} - 2x_{\{i\}} + (n-1)(x_N - 1) \leq 0 \quad \text{for } i \in N, \quad (e_5)$$

$$\sum_{|T| \geq 2} x_T - 2^n(x_A + x_B) \leq 2^n - 1 \quad \text{for } |A|, |B| \geq 2 \text{ and } 2(|A| + |B|) \leq n+3. \quad (e_6)$$

One can easily check that each of those inequalities induces an edge of $Gale(\gamma_n)$. More precisely, (e_1) and (e_2) induce the edges $[v^\emptyset, v^A]$ for $|A| \geq 2$, $(e_3), (e_4)$ and (e_5) induce the edges $[v^i, v^A]$ for $|A| \geq 1$ and $i \notin A$ or $A = N$ and (e_6) induce the edges $[v^A, v^B]$ for $|A|, |B| \geq 2$ and $2(|A| + |B|) \leq n + 3$.

Property 3.1 *The diameter of $Gale(\gamma_n)$ is at most 2. Moreover, $\delta(Gale(\gamma_3)) = 2$ and $\delta(Gale(\gamma_4)) = 1$.*

PROOF. The vertices $v^\emptyset$ and v^A are respectively linked by the edges $[v^\emptyset, v^N]$ and $[v^N, v^A]$ for $|A| = 1$ and by the edge $[v^\emptyset, v^A]$ for $|A| \geq 2$. The vertices v^i and v^j always form an edge, v^i and v^A are linked by $[v^i, v^j]$ and $[v^j, v^A]$ with $j \notin A$, for $2 \leq |A| \leq n - 1$, and $[v^i, v^N]$ form an edge. Finally, the vertices v^A and v^B are linked by the edges $[v^A, v^\emptyset]$ and $[v^\emptyset, v^B]$ for $|A|, |B| \geq 2$. $\square$

We then consider the following 2^{n-1} inequalities.

$$2^{n-1}x_{\bar{A}} - \sum_{|T| \geq 2} x_T \leq 1 \quad \text{for } A \subset N \text{ and } |A| \leq 1,$$

$$2^{n-1}(x_A + x_{\bar{A}}) - \sum_{|T| \geq 2} x_T \leq 1 \quad \text{for } A \subset N \text{ and } 2 \leq |A| \leq n - 1.$$

One can easily check that each of those inequalities induces a facet G^A of $Gale(\gamma_n)$ for $A \subset N$ and $|A| \leq n - 1$. Since each facet G^A contains all vertices except the pair $\{v^S, v^{\bar{S}}\}$, we call them the *huge facets*.

Lemma 3.2 *The huge facets form the clique K_{2^n-1} in the skeleton of $Gale^*(\gamma_n)$.*

PROOF. Let us first consider $g = G^A \cap G^B$ with $A, B \subset N$ and $2 \leq |A|, |B| \leq n - 1$. The face g contains all the vertices of $Gale(\gamma_n)$ except $\{v^A, v^{\bar{A}}, v^B, v^{\bar{B}}\}$. We show that g is of codimension 2 by exhibiting a family V of $2^n - n - 2$ affinely independent vertices belonging to g , this will imply that G^A and G^B are adjacent. Namely, V is formed by the vertices v^S with $S \notin \{A, \bar{A}, B, \bar{B}\}$ and $|S| \geq 2$ and the vertices $\{v^i, v^j\}$ with $1 \leq i < j \leq n$ such that $v_A^i = v_B^j = 1$ and $v_B^i = v_A^j = 0$. In the case $0 \leq |A|, |B| \leq 1$, V is formed by the vertices v^S with $S \notin \{\bar{A}, \bar{B}\}$ and $|S| \geq 2$. Finally, in the case $0 \leq |A| \leq 1$ and $2 \leq |B| \leq n - 1$, V is formed by the vertices v^S with $S \notin \{\bar{A}, B, \bar{B}\}$ and $|S| \geq 2$ and the vertex $v^\emptyset$. $\square$

Property 3.3 *The huge facets form a dominating clique in the skeleton of $Gale^*(\gamma_n)$.*

PROOF. Since the pairs $\{v^S, v^{\bar{S}}\}$ form a partition of all the vertices of $Gale(\gamma_n)$, for any facet F , at least one huge facet G^A satisfies $|G^A \cap F| = |F| - 1$. This implies that G^A is adjacent to F ; in other words, the huge facets form a dominating clique. $\square$

Corollary 3.4 *The diameter of $\text{Gale}^*(\gamma_n)$ is at most 3. Moreover, it is 2 for $n = 3, 4$.*

Conjecture 3.5 *For $n \geq 4$, the diameters of the Gale polytope of the n -cube and of its dual are 1 and 2, respectively.*

4 Complete bipartite subgraphs polytope

We recall that the *folded n -cube* $\square_n$ is the graph whose vertices are the 2^{n-1} partitions of $N = \{1, \dots, n\}$ into two subsets, S and $\bar{S}$; two partitions being adjacent when their common refinement contains a singleton. In particular, $\square_4 = K_{4,4}$ and $\square_5 = \frac{1}{2}H(5, 2)$, also called the Clebsch graph.

The *complete bipartite subgraphs polytope* c_n , which is also called the cut polytope of the complete graph, is a relative of the folded n -cube. More precisely, the vertices of c_n are the 2^{n-1} incidence vectors $\delta(S)$ in $\mathbb{R}^{\binom{n}{2}}$ of the partitions of N , that is, $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. It is easy to check that the squared Euclidian distance between two partitions, seen as vertices of c_n , is $d(n - d)$, where d is their path distance, in the graph $\square_n$. Now, $c_3 = h\gamma_3 = \alpha_3$ and c_4 is combinatorially equivalent to the simplicial 6-dimensional cyclic polytope with 8 vertices. The symmetry group of c_n is isomorphic to the automorphism group of $\square_n$, see [10]. See [11] for a detailed treatment of c_n .

The skeleton of c_n is the clique $K_{2^{n-1}-1}$, see [1]. The determination of all the facets of c_n for large n seems to be hopeless, but a wide range of facets has been already found (including all for $n \leq 7$). It seems that the huge majority of them are simplices for large n , that is, c_n is asymptotically simplicial, as well as $h\gamma_n$. In [7] it was conjectured (and proved for $n \leq 7$) that $\delta(c_n^*) \leq 4$; moreover, $\delta(c_4^*) = \delta(c_5^*) = 2$ and $\delta(c_6^*) = 3$. Actually, the skeleton of c_4^* is the line graph of the folded 4-cube.

Remark 4.1 *Using the basis of the space of affine dependencies on c_5 given in [8], we found by computer that $\text{Gale}(c_5) \simeq h\gamma_5$; recall that $\square_5 = \frac{1}{2}H(5, 2)$. Clearly, $\text{Gale}(h\gamma_4) \simeq \alpha_3$ and $\text{Gale}(h\gamma_5) \simeq c_5$; more generally, for n odd, $\text{Gale}(h\gamma_n)$ can be obtained from the following basis of $2^{n-1} - n - 1$ affine dependencies:*

$$(n-1) \sum_{i \in X} x_{N \setminus \{i\}} - |A| \sum_{i \in N} x_{N \setminus \{i\}} + (n-1)x_A = 0 \text{ for } |A| \text{ even, } 2 \leq |A| \leq n-2.$$

Finally, we mention $cont_m$, the contact polytope of the lattice $\mathbb{Z}(V_m)$ in $\mathbb{R}^{\binom{m}{2}}$ studied in [9], where V_m denotes the set of vertices of c_m , that is, $cont_m$ is the convex hull of all vectors of this lattice having the minimal length $\mu = \min(4, m-1)$. Clearly, it comes from the construction A given in Chapters 5, 7 of [5] with V_m seen as a linear binary code with $n = \binom{m}{2}$, $M = 2^{m-1}$ and $d = m-1$. We have,

- $cont_2 = \text{conv}\{\pm e_1\} = \beta_1$ and $\mathbb{Z}(V_2) = \mathbb{Z} = A_1$,
- $cont_3 = \text{conv}\{\pm e_i \pm e_j : 1 \leq i \neq j \leq 3\}$ is the cubo-octahedron (the vertices of this Archimedean solid are the midpoints of the edges of γ_3) and $\mathbb{Z}(V_3)$ is the face-centered cube lattice $A_3 \cong D_3$,
- $cont_4 = \text{conv}\{\pm \delta(i), \pm \delta(i) - 2e_{ij} : 1 \leq i \neq j \leq 4\} \simeq h\gamma_6$,
- $cont_5$ is a 10-polytope with the following 100 vertices: $\{\pm 2e_{ij} : 1 \leq i \leq j \leq 5\} \cup \{\delta(i) - 2\sum_{\{jk\} \in X} e_{jk} : 1 \leq i \leq 5, X \subset E(K_{i,\{1,2,3,4,5\}-i})\}$. So, $cont_5$ is the union of $2\beta_{10}$ and five 4-cubes γ_4 , this polytope has 4 624 facets divided into 4 orbits of its symmetry group $2^5 Sym(5)$, moreover, the orbit formed by the 384 facets equivalent to the one induced by the inequality $\sum_{\{ij\} \in C_{1,2,3,4,5}} x_{ij} \leq 2$ forms a dominating set in the skeleton of $cont_5^*$,
- for $m \geq 6$, $cont_m = \text{conv}\{\pm 2e_{ij} : 1 \leq i \leq j \leq m\} \simeq \beta_{\binom{m}{2}}$.

So, the kissing number of the lattice, that is the number of vertices of $cont_m$, is $\tau = 2, 12, 32, 100, m(m-1)$ for $m = 2, 3, 4, 5, \geq 6$.

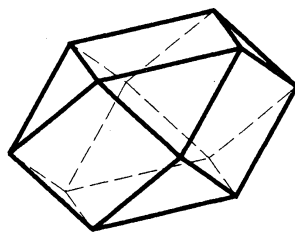


Figure 4.1: The contact polytope of $\mathbb{Z}(V_3)$ is a cubo-octahedron

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