

A note on the length of starlike functions

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Abstract

Let $\mathcal{S}$ be the class of analytic functions $f(z)$ normalized with $f(0) = 0$ and $f'(0) = 1$ which are univalent in the open unit disk $\mathbb{U}$. Also, let $\mathcal{S}^*$ denote the subclass of $\mathcal{S}$ consisting of functions $f(z)$ which are starlike with respect to the origin in $\mathbb{U}$. For $f(z) \in \mathcal{S}^*$, Ch. Pommerenke [J. London Math. Soc. 37(1962), 209 - 224] has shown the estimates for the length of the image curve of the circle $|z| = r < 1$. The object of the present paper is to derive the generalized theorem of the result due to Ch. Pommerenke.

1 Introduction

Let $\mathcal{S}$ denote the set of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic and univalent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} \mid |z| < 1\}$. A function $f(z) \in \mathcal{S}$ is called starlike with respect to the origin if it satisfies

$$\operatorname{Re} \left(\frac{z f'(z)}{f(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*$ the subclass of $\mathcal{S}$ consisting of all starlike functions with respect to the origin in $\mathbb{U}$. In 1962, Pommerenke [6] has shown

Theorem A *Let $f(z) \in \mathcal{S}^*$ and suppose that*

$$M(r) = \max_{|z|=r<1} |f(z)| \leq \frac{1}{(1-r)^\alpha} \quad (0 < \alpha \leq 2).$$

Then

$$L(r) = \int_0^{2\pi} r |f'(re^{i\theta})| d\theta \leq \frac{A(\alpha)}{(1-r)^\alpha},$$

where $A(\alpha)$ depends only on α and $L(r)$ denotes the length of $C(r)$ which is the image of the circle $|z| = r < 1$ under the mapping $w = f(z)$.

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Pommerenke [6, Remarks in p.214] has given the comments that Theorem A can not be improved any more, except for the factor $A(\alpha)$. This is true, but it is not absolutely perfect, because the order of infinity for $M(r)$ depends not only $(1-r)^{-\alpha}$ but $(\log(1-r)^{-1})^\beta$.

In 1958, Hayman [1] has proved that if $f(z) \in \mathcal{S}$ and $1/2 < \alpha \leq 2$, then

$$M(r) = O\left(\left(\frac{1}{1-r}\right)^\alpha\right)$$

implies that

$$L(r) = O\left(\left(\frac{1}{1-r}\right)^\alpha\right).$$

Littlewood [4] has shown that this implication breaks down for small α . On the other hand, Thomas [7] has obtained

Theorem B *Let $f(z) \in \mathcal{S}^*$. Then*

$$L(r) = O\left(\sqrt{B(r)} \log\left(\frac{1}{1-r}\right)\right) \quad (as \quad r \rightarrow 1),$$

where $B(r)$ is the area enclosed by the curve $C(r)$ which is the image curve of the circle $|z| = r < 1$ under the mapping $w = f(z)$.

It is the purpose of this paper to generalize Theorem A by Pommerenke [6].

2 Main theorem

To discuss our main theorem, we need the following lemma due to Pommerenke [6] (or also due to Hayman [1]).

Lemma *If $f(z) \in \mathcal{S}$, then, for $\lambda > 1$,*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\lambda dt \leq \lambda^2 \int_0^r \frac{M(\rho)^\lambda}{\rho} d\rho \quad (0 < r < 1).$$

Now, we give

Theorem *Let $f(z) \in \mathcal{S}^*$ and suppose that*

$$M(r) = \text{Max}_{|z|=r<1} |f(z)| = O\left(\left(\frac{1}{1-r}\right)^\alpha \left(\log \frac{1}{1-r}\right)^\beta\right),$$

where $0 < \alpha < k \leq 2$, $k > 1$, and $\beta > 0$. Then we have

$$L(r) = O\left(\left(\frac{1}{1-r}\right)^\alpha \left(\log \frac{1}{1-r}\right)^{\beta+1-\frac{\alpha}{k}}\right)$$

for $0 < \alpha < k - 1$, and

$$L(r) = O\left(\left(\frac{1}{1-r}\right)^{2\alpha(1-\frac{1}{k})+2-k} \left(\log \frac{1}{1-r}\right)^\beta\right)$$

for $0 < k - 1 \leq \alpha < k$.

Proof Application of the Hölder's inequality gives us that

$$\begin{aligned} L(r) &= \int_0^{2\pi} r |f'(re^{i\theta})| d\theta = \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right| |f(z)| d\theta \\ &\leq \left(\int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^{\frac{k}{k-\alpha}} d\theta \right)^{\frac{k-\alpha}{k}} \left(\int_0^{2\pi} |f(z)|^{\frac{k}{\alpha}} d\theta \right)^{\frac{\alpha}{k}} \\ &= I^{\frac{k-\alpha}{k}} J^{\frac{\alpha}{k}}, \end{aligned}$$

where $k > \alpha$,

$$I = \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^{\frac{k}{k-\alpha}} d\theta$$

and

$$J = \int_0^{2\pi} |f(z)|^{\frac{k}{\alpha}} d\theta.$$

By Keogh [2, Theorem 1], it is well known that

$$\int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right| d\theta = O\left(\log \frac{1}{1-r}\right) \quad (\text{as } r \rightarrow 1).$$

On the other hand, we see that

$$\left| \frac{zf'(z)}{f(z)} \right| = O\left(\frac{1}{1-r}\right) \quad (\text{as } r \rightarrow 1)$$

by Nehari [5]. Thus, we have the following estimates for $0 < \alpha < k - 1$ that

$$\begin{aligned} I &= \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^{\frac{k}{k-\alpha}} d\theta \\ &= \left(O\left(\frac{1}{1-r}\right)^{\frac{\alpha}{k-\alpha}} \right) \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right| d\theta \\ &= O\left(\left(\frac{1}{1-r}\right)^{\frac{\alpha}{k-\alpha}} \left(\log \frac{1}{1-r}\right)\right) \quad (\text{as } r \rightarrow 1), \end{aligned}$$

which shows that

$$I^{\frac{k-\alpha}{k}} = O\left(\left(\frac{1}{1-r}\right)^{\frac{\alpha}{k}} \left(\log \frac{1}{1-r}\right)^{\frac{k-\alpha}{k}}\right).$$

In order to consider for the case $0 < k-1 \leq \alpha < k$, we have to recall here the following result by Littlewood [3, p.484] that if $f(z)$ is subordinate to $F(z)$ in $\mathbb{U}$, then for each r ($0 \leq r < 1$) and each k ($k \geq 0$),

$$\int_0^{2\pi} |f(re^{i\theta})|^k d\theta \leq \int_0^{2\pi} |F(re^{i\theta})|^k d\theta.$$

Since $f(z) \in \mathcal{S}^*$, we see that

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \quad (z \in \mathbb{U}),$$

where the symbol $\prec$ means the subordination. Applying the result by Littlewood [3], we have for $1 < k \leq 2$ that

$$\begin{aligned} \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^k d\theta &\leq \int_0^{2\pi} \left| \frac{1+z}{1-z} \right|^k d\theta \leq \int_0^{2\pi} \left| \frac{1+z}{1-z} \right|^2 d\theta \\ &= O\left(\frac{1}{1-r}\right) \quad (\text{as } r \rightarrow 1). \end{aligned}$$

Therefore, for the case of $0 < k-1 \leq \alpha < k$,

$$\begin{aligned} I &= \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^k \left| \frac{zf'(z)}{f(z)} \right|^{\frac{k(\alpha-k+1)}{k-\alpha}} d\theta \\ &= \left(O\left(\frac{1}{1-r}\right)^{\frac{k(\alpha-k+1)}{k-\alpha}} \right) \int_0^{2\pi} \left| \frac{zf'(z)}{f(z)} \right|^k d\theta \\ &= O\left(\frac{1}{1-r}\right)^{\frac{(k-1)(\alpha-k+1)+1}{k-\alpha}} \quad (\text{as } r \rightarrow 1), \end{aligned}$$

which implies that

$$I^{\frac{k-\alpha}{k}} = O\left(\frac{1}{1-r}\right)^{(1-\frac{1}{k})\alpha-(k-2)}.$$

Next, we have to consider J by using the lemma due to Pommerenke [6]. By using the lemma and Schwarz lemma, we have, for $0 < \alpha < k$, that

$$J = \int_0^{2\pi} |f(z)|^{\frac{k}{\alpha}} d\theta \leq \frac{2k^2\pi}{\alpha^2} \int_0^r \frac{1}{\rho} M(\rho)^{\frac{k}{\alpha}} d\rho$$

$$\begin{aligned}
&\leq \frac{2k^2\pi}{\alpha^2} \int_0^r \frac{1}{\rho} \left\{ \frac{\rho}{(1-\rho)^\alpha} \left(\log \frac{1}{1-\rho} \right)^\beta \right\}^{\frac{k}{\alpha}} d\rho \\
&= \frac{2k^2\pi}{\alpha^2} \int_0^r \frac{\rho^{\frac{k}{\alpha}-1}}{(1-\rho)^k} \left(\log \frac{1}{1-\rho} \right)^{\frac{k\beta}{\alpha}} d\rho \\
&\leq \frac{2k^2\pi}{\alpha^2} \int_0^r \left(\frac{1}{1-\rho} \right)^k \left(\log \frac{1}{1-\rho} \right)^{\frac{k\beta}{\alpha}} d\rho \\
&\leq \frac{2k^2\pi}{\alpha^2} \left(\log \frac{1}{1-r} \right)^{\frac{k\beta}{\alpha}} \int_0^r \left(\frac{1}{1-\rho} \right)^k d\rho \\
&= O \left(\left(\frac{1}{1-r} \right)^{k-1} \left(\log \frac{1}{1-r} \right)^{\frac{k\beta}{\alpha}} \right) \quad (\text{as } r \rightarrow 1),
\end{aligned}$$

which gives us that

$$J_{\frac{\alpha}{k}} = O \left(\left(\frac{1}{1-r} \right)^{\frac{\alpha(k-1)}{k}} \left(\log \frac{1}{1-r} \right)^\beta \right).$$

Consequently, we conclude that, for $0 < \alpha < k-1$,

$$L(r) = O \left(\left(\frac{1}{1-r} \right)^\alpha \left(\log \frac{1}{1-r} \right)^{\beta+1-\frac{\alpha}{k}} \right),$$

and, for $0 < k-1 \leq \alpha < k$,

$$L(r) = O \left(\left(\frac{1}{1-r} \right)^{2\alpha(1-\frac{1}{k})+(2-k)} \left(\log \frac{1}{1-r} \right)^\beta \right).$$

This completes the proof of our main theorem.

Taking $k = 2$ in Theorem, we have

Corollary *Let $f(z) \in \mathcal{S}^*$ and suppose that*

$$M(r) = \text{Max}_{|z|=r < 1} |f(z)| = O \left(\left(\frac{1}{1-r} \right)^\alpha \left(\log \frac{1}{1-r} \right)^\beta \right),$$

where $0 < \alpha < 2$ and $\beta > 0$. Then we have

$$L(r) = O \left(\left(\frac{1}{1-r} \right)^\alpha \left(\log \frac{1}{1-r} \right)^{\beta+1-\frac{\alpha}{2}} \right) \quad (\text{for } 0 < \alpha < 1)$$

and

$$L(r) = O \left(\left(\frac{1}{1-r} \right)^\alpha \left(\log \frac{1}{1-r} \right)^\beta \right) \quad (\text{for } 1 \leq \alpha < 2).$$

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