

**Construction of the fundamental solution
for a degenerate equation and
a local version of Riemann-Roch theorem**

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§1. Introduction. Let M be a compact Kaehler manifold whose complex dimension is n . The following Riemann-Roch theorem holds for M .

$$\sum_{q=0}^n (-1)^q \dim H_q(M) = \int_M (2\pi i)^{-n} [Td(T(M))]_{2n},$$

where

$$Td(T(M)) = \det\left(\frac{\Omega}{e^{\Omega} - 1}\right)$$

with a curvature form Ω .

Analytical proofs of the above theorem are based on the following formula :

$$(1.1) \quad \sum_{q=0}^n (-1)^q \dim H_q(M) = \int_M \sum_{q=0}^n (-1)^q \operatorname{tr} e_q(t, x, x) dv,$$

where $e_q(t, x, y)$ denotes the kernel of the fundamental solution $E_q(t)$ for the heat equation for $\Delta_q = d''\vartheta'' + \vartheta''d''$ acting on differential $(0, q)$ -forms $A^{(0,q)}(M) = \Gamma(\wedge^q T^{*(0,1)}(M))$. The author have shown the local version of Gauss-Bonnet-Chern theorem for a compact manifold with boundary, by constructing the fundamental solution according to the technique of symbolic calculus of pseudodifferential operators [10]. In this note, we show that the following formula, which is called a local version of Riemann-Roch theorem holds.

local version of Riemann-Roch Theorem

$$(1.2) \quad \sum_{q=0}^n (-1)^q \operatorname{tr}_q(t, x, x) dv = (2\pi i)^{-n} [Td(T(M))]_{2n} + 0(t^{\frac{1}{2}})$$

We shall give a rough sketch of a proof of the above formula, constructing the fundamental solution according to the method of symbolic calculus for a degenerate parabolic operator as [8] instead of that of a parabolic operator. (See §4). Our point is that we can prove the above formula by only calculating the main term of the fundamental solution, introducing a new weight of symbols of pseudodifferential operators.

There are several papers for this problem. T.Kotake[7] proved this formula for manifolds of dimension 1. Then V.K.Patodi[12] has proved for Kaehler manifolds of any dimension. P.B.Gilkey[6] also has shown, using invariant theory. E.Getzler[5] treated this problem by different approach.

§2. The representation of Δ . Let M be a smooth Kaehler manifold with a hermitian metric g . Set $Z_1, Z_2, \dots, Z_n$ be a local orthonormal frame of $T^{1,0}(M)$ in a local patch of chart U . And let $\omega^1, \omega^2, \dots, \omega^n$ be its dual.

The differential d'' and its dual ϑ'' acting on $A^{0,q}(M)$ are given as follow, using the Levi-Civita connection ∇ .

$$d'' = \sum_{j=1}^n e(\bar{\omega}^j) \nabla_{\bar{Z}_j}, \quad \vartheta'' = - \sum_{j=1}^n i(\bar{Z}_j) \nabla_{Z_j}$$

where we use the following notations.

Notations.

$$Z_{\bar{j}} = \bar{Z}_j, \quad \omega^{\bar{j}} = \bar{\omega}^j \quad j = 1, \dots, n,$$

$$e(\omega^\alpha) \omega = \omega^\alpha \wedge \omega, \quad i(Z_\alpha) \omega(Y_1, \dots, Y_{p-1}) = \omega(Z_\alpha, Y_1, \dots, Y_{p-1}).$$

Let $c_{\alpha,\beta}^\gamma, \alpha, \beta, \gamma \in \{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ be the following functions and let $R(Z_\alpha, \bar{Z}_\beta)$ be the curvature transformation.

$$\nabla_{Z_\alpha} Z_\beta = \sum_{\gamma} c_{\alpha,\beta}^\gamma Z_\gamma,$$

$$R(Z_\alpha, Z_\beta) = [\nabla_{Z_\alpha}, \nabla_{Z_\beta}] - \nabla_{[Z_\alpha, Z_\beta]},$$

$$R(Z_i, Z_{\bar{j}})Z_{\beta} = \sum_{\gamma} R^{\gamma}_{\beta i \bar{j}} Z_{\gamma}.$$

Remark 1. Owing to that $\{Z_j, \bar{Z}_j\}_{j=1, \dots, n}$ is an orthonormal frame, we have $R^i_{\bar{j} k \bar{\ell}} = R_{i \bar{j} k \bar{\ell}} = g(R(Z_k, \bar{Z}_{\ell})\bar{Z}_j, Z_i) = R(Z_i, \bar{Z}_j, Z_k, \bar{Z}_{\ell})$.

From the fact that our connection is the Riemannian connection and that M is a Kaehler manifold we have

Proposition 1. *The coefficients $c^{\gamma}_{\alpha, \beta}$ of connection form satisfy*

$$c^{\bar{i}}_{\alpha j} = c^i_{\alpha, \bar{j}} = 0, \quad c^i_{\alpha, j} = -c^{\bar{j}}_{\alpha, \bar{i}},$$

$$\alpha \in \{1, \dots, n, \bar{1}, \dots, \bar{n}\}, i, j \in \{1, \dots, n, \bar{1}, \dots, \bar{n}\}$$

$$R(Z_i, Z_j) = 0, \quad R(\bar{Z}_i, \bar{Z}_j) = 0,$$

$$R^{\bar{k}}_{\ell i \bar{j}} = R^k_{\ell i \bar{j}} = 0, \quad \bar{R}^k_{\ell i \bar{j}} = R^{\bar{k}}_{\ell i \bar{j}},$$

$$\bar{R}_{\ell \bar{k} i \bar{j}} = R_{\ell j i \bar{k}},$$

$$[Z_{\alpha}, Z_{\beta}] = \sum_{\gamma} (c^{\gamma}_{\alpha, \beta} - c^{\gamma}_{\beta, \alpha}) Z_{\gamma}.$$

We have the following representation for $\Delta = d'' \vartheta'' + \vartheta'' d''$ which is known as Bocher-Kodaira formula.

Lemma 1.

On $A^{0,*}(M) = \sum_{q=0}^n A^{0,q}(M)$ we have

$$\Delta = -\frac{1}{2} \left\{ \sum_{j=1}^n (\nabla_{Z_j} \nabla_{\bar{Z}_j} + \nabla_{\bar{Z}_j} \nabla_{Z_j}) - \sum_{i,j=1}^n (c^{\bar{j}}_{i, \bar{i}} \nabla_{\bar{Z}_j} + c^j_{\bar{i}, i} \nabla_{Z_j}) - \sum_{j=1}^n R(Z_j, \bar{Z}_j) \right\}$$

We use the following notations in the rest of this paper.

$$e(\bar{\omega}^j) = a_j^*, \quad \iota(\bar{Z}_m) = a_m,$$

$$a_I = a_{i_1} a_{i_2} \cdots a_{i_p}, \quad a_I^* = a_{i_p}^* \cdots a_{i_1}^* \quad \text{for } I = \{i_1 < i_2 < \cdots < i_p\},$$

$$\omega^{\bar{I}} = \bar{\omega}^{i_1} \wedge \bar{\omega}^{i_2} \wedge \cdots \wedge \bar{\omega}^{i_p} \quad \text{for } I = \{i_1 < i_2 < \cdots < i_p\},$$

By the above notations we have

$$\nabla_{Z_\alpha}(\varphi_{\bar{J}}\omega^{\bar{J}}) = (Z_\alpha\varphi_{\bar{J}})\omega^{\bar{J}} - \sum_{k,\ell=1}^n \varphi_{\bar{J}}c_{\alpha,k}^\ell a_k^*a_\ell(\omega^{\bar{J}})$$

for $\varphi_{\bar{J}}\omega^{\bar{J}} \in A^{0,*}(M)$, $\omega^{\bar{J}} = \bar{\omega}^{i_1}\bar{\omega}^{i_2}\dots\bar{\omega}^{i_p}$ ($J = (i_1, i_2, \dots, i_p)$), using

$$\nabla_{Z_\alpha}(\bar{\omega}^\ell) = - \sum_{k=1}^n c_{\alpha,k}^\ell \bar{\omega}^k.$$

So we have the local representation of Δ .

$$\begin{aligned} \Delta = & -\frac{1}{2} \sum_{j=1}^n \{ (Z_j I - G_j)(\bar{Z}_j I - G_{\bar{j}}) + (\bar{Z}_j I - G_{\bar{j}})(Z_j I - G_j) \} \\ & + \frac{1}{2} \sum_{i,j=1}^n \{ c_{i,\bar{i}}^{\bar{j}}(\bar{Z}_j I - G_{\bar{j}}) + c_{\bar{i},i}^j(Z_j I - G_j) \} - \frac{1}{2} \sum_{j,k,\ell=1}^n R_{\ell\bar{k}j\bar{j}} a_k^* a_\ell \end{aligned}$$

on $A^{0,*}(M)$. Here

$$G_\alpha = \sum_{\ell,m=1}^n c_{\alpha,\ell}^m a_\ell^* a_m$$

and I is an identity operator on $\wedge^*(T^*(M))$.

The following proposition is fundamental for a_i, a_j^* .

Proposition 2.

$$a_i a_j + a_j a_i = 0,$$

$$a_i^* a_j^* + a_j^* a_i^* = 0,$$

$$a_i a_j^* + a_j^* a_i = \delta_{ij},$$

$$[a_i^* a_j, a_k^* a_\ell] = \delta_{jk} a_i^* a_\ell - \delta_{i\ell} a_k^* a_j.$$

§3. Berezin-Patodi formula. Let V be a vector space of dimension n with inner product and let $\wedge^p(V)$ be its anti-symmetric p tensors. Set $\wedge^*(V) = \sum_{p=0}^n \wedge^p(V)$. Let $\{v_1, \dots, v_n\}$ be an orthonormal basis for V . Set a_i^* be a linear transformation on $\wedge^*(V)$ defined by $a_i^* v = v_i \wedge v$ and set a_i be an adjoint operator of a_i^* on $\wedge^*(V)$. Then

$\{a_i^*, a_j\}$ satisfy Proposition 2. The following Theorem 1 and Corollary are shown in [3] under the above assumptions.

Theorem 1(Berezin-Patodi[3]). For any linear operator A on $\wedge^*(V)$, we can write uniquely in the form $A = \sum_{I,J} \alpha_{I,J} a_I^* a_J$ and

$$\sum_{p=0}^n \text{tr}[(-1)^p A_p] = (-1)^n \alpha_{\{1,2,\dots,n\}\{1,2,\dots,n\}},$$

where $A_p = A|_{\wedge^p(V)}$.

Corollary . (1) If multi index I and J satisfy $\sharp(I) < n$ or $\sharp(J) < n$, we have

$$\sum_{p=0}^n \text{tr}[(-1)^p a_I^* a_J] = 0.$$

(2) Let π and σ be elements of permutation of n . Then

$$\sum_{p=0}^n \text{tr}[(-1)^p a_{\pi(1)}^* a_{\sigma(1)} a_{\pi(2)}^* a_{\sigma(2)} \cdots a_{\pi(n)}^* a_{\sigma(n)}] = (-1)^n \text{sign}(\pi) \text{sign}(\sigma).$$

§4. Fundamental solution for a degenerate operator. In this paper we use the pseudo-differential operators of Weyl symbols, that is, a symbol $p(x, \xi) \in S_{\rho, \delta}^m(\mathbf{R}^n)$ defines an operator as

$$Pu(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} e^{i(x-y) \cdot \xi} p\left(\frac{x+y}{2}, \xi\right) u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbf{R}^n).$$

Definition 1. We denote the symbol multi-product $p_1(x, D)p_2(x, D) \cdots p_\ell(x, D)$ of pseudo-differential operators $p_j(x, D)$ with symbol $p_j(x, \xi)$ by $(p_1 \circ \cdots \circ p_\ell)(x, \xi)$. We use the notation $\sigma(P)$ to denote a symbol of a pseudo-differential operator P .

Theorem 2. Let $p_j \in S_{\rho, \delta}^{m_j}(\mathbf{R}^n)$, $(j = 1, 2)$, $\delta < 1$, $\rho \geq \delta$. Then for any $N \in \mathbf{N}$ we have the expansion

$$(5.1) \quad p_1 \circ p_2 = \sum_{k=0}^{N-1} \frac{(2i)^{-k}}{k!} \sigma_k(p_1, p_2) + q_N,$$

where

$$(5.2) \quad \sigma_k(p, q) = \sum_{|\alpha|+|\beta|=k} (-1)^{|\beta|} \frac{k!}{\alpha! \beta!} p_{(\beta)}^{(\alpha)} q_{(\alpha)}^{(\beta)}$$

with $q_N \in S_{\rho, \delta}^{m_1+m_2-(\rho-\delta)N}(\mathbf{R}^n)$.

Definition 2. (1) ∇p means a vector

$$\begin{pmatrix} \nabla_x p \\ \nabla_\xi p \end{pmatrix} = {}^t \left(\frac{\partial}{\partial x_1} p, \dots, \frac{\partial}{\partial x_n} p, \frac{\partial}{\partial \xi_1} p, \dots, \frac{\partial}{\partial x_n} p \right)$$

for a linear transformation $p(x, \xi)$ with parameter (x, ξ) .

(2) J is a transformation on $\mathbf{C}^n \times \mathbf{C}^n$ defined by

$$J \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -u \end{pmatrix}$$

for $u, v \in \mathbf{C}^n$. We also use the same notation J in case $u = {}^t(u_1, \dots, u_n)$, u_j is a linear transformation on some vector space.

(3) H_p is the Hessian matrix of p .

(4) $\langle t, s \rangle = \sum_{j=1}^k t_j s_j$ for a pair of vectors $t = {}^t(t_1, \dots, t_k)$, $s = {}^t(s_1, \dots, s_k)$.

We consider the construction of the fundamental solution $U(t)$ for a degenerate parabolic system

$$\begin{cases} LU = \left(\frac{d}{dt} + P \right) U(t) = 0 & \text{in } (0, T) \times \mathbf{R}^m, \\ U(0) = I & \text{on } \mathbf{R}^m, \end{cases}$$

for the Cauchy problem on $\mathbf{R}^m$ (See I-Iwasaki [8]). Here P is a differential operator of a symbol $p(x, \xi) = p_2(x, \xi) + p_1(x, \xi) + p_0(x, \xi)$, where $p_j(x, \xi)$ are homogeneous of order j with respect to ξ .

Condition (A).

$$(A) - (1) \quad p_2(x, \xi) = \sum_{j=1}^d b_j(x, \xi) c_j(x, \xi) \quad (c_j = \bar{b}_j),$$

where $b_j \in S_{1,0}^1$ are scalar symbols.

$$(A) - (2) \quad p_1 + \text{tr}_+ \left(\frac{\mathcal{M}}{2} \right) \geq c|\xi|$$

for some positive constant c on the characteristic set $\Sigma = \{(x, \xi) \in \mathbf{R}^m \times \mathbf{R}^m;$
 $b_j(x, \xi) = 0 \text{ for any } j\}$, where $\text{tr}_+ \mathcal{M}$ is the sum of all positive eigenvalues of $\mathcal{M}$:

$$\mathcal{M} = i\Xi^* J \Xi.$$

Here $\Xi = (\nabla c_1, \dots, \nabla c_d, \nabla b_1, \dots, \nabla b_d)$.

Set $\mathbf{b} = {}^t(b_1, \dots, b_d)$, $\mathbf{c} = {}^t(c_1, \dots, c_d)$. Then we have

Theorem 3. *Let $p(x, \xi)$ satisfy Condition (A). Then the fundamental solution $U(t)$ is constructed as a pseudo-differential operator of a symbol $u(t)$ belonging to $S_{\frac{1}{2}, \frac{1}{2}}^0$ with parameter t . Moreover $u(t)$ has the following expansion for any N :*

$$u(t) - \sum_{j=0}^{N-1} u_j(t) \text{ belongs to } S_{\frac{1}{2}, \frac{1}{2}}^{-\frac{N}{2}},$$

$$u_0(t) = \exp \varphi, \quad u_j(t) = f_j(t) u_0(t) \in S_{\frac{1}{2}, \frac{1}{2}}^{-\frac{j}{2}},$$

where

$$\varphi = -\frac{t}{2} \left(\begin{pmatrix} \mathbf{c} \\ \mathbf{b} \end{pmatrix}, F\left(\frac{\mathcal{M}t}{2}\right) \begin{pmatrix} \mathbf{b} \\ \mathbf{c} \end{pmatrix} \right) - \frac{1}{2} \text{tr}[\log\{\cosh(\frac{\mathcal{M}t}{2})\}] - p_1 t,$$

$$F(s) = s^{-1} \tanh s.$$

§5. Sketch of the proof. We shall apply the above discussion for our operator Δ . Fix a point $\hat{z} \in M$ and choose a local chart U of a neighborhood $\hat{z}$. Choose a local coordinate $z_1, z_2, \dots, z_n$ of U such that

$$\hat{z} = 0, \quad Z_j = \frac{\partial}{\partial z_j} + \sum_{k=1}^n \kappa_{jk}(z, \bar{z}) \frac{\partial}{\partial z_k}, \quad \kappa_{jk}|_{z=0} = 0.$$

We use the following notation in the rest of paper.

$$\bar{z}_j = z_j, \quad \hat{G}_j = G_j|_{z=0}, \quad \tilde{Z}_j = \sum_{k=1}^n \kappa_{jk}(z, \bar{z}) \frac{\partial}{\partial z_k}.$$

Then we have

$$\kappa_{j\bar{k}} = \bar{\kappa}_{jk}, \quad \kappa_{j\bar{k}} = \kappa_{\bar{j}k} = 0 \quad (1 \leq j, k \leq n).$$

Proposition 3. For the derivatives of $\kappa_{\alpha\beta}(z, \bar{z})$ we have

$$\{Z_\alpha(\kappa_{\beta\gamma}) - Z_\beta(\kappa_{\alpha\gamma})\}|_{z=0} = (c_{\alpha,\beta}^\gamma - c_{\beta,\alpha}^\gamma)|_{z=0}, \quad \alpha, \beta, \gamma \in \Gamma.$$

$$\sum_\gamma \{Z_\alpha(\kappa_{\beta\gamma}) - Z_\beta(\kappa_{\alpha\gamma})\}|_{z=0} \hat{G}_\gamma = (-\nabla_{[Z_\alpha, Z_\beta]} + [Z_\alpha, Z_\beta])|_{z=0}.$$

Set

$$W(z, \bar{z}) = \sum_{j=1}^n (z_j \hat{G}_j + \bar{z}_j \hat{G}_{\bar{j}}) = \sum_{\alpha \in \Lambda} z_\alpha \hat{G}_\alpha$$

$$\Lambda = \{1, \dots, n, \bar{1}, \dots, \bar{n}\}.$$

Lemma 2. We have

$$(Z_\alpha - G_\alpha)e^W = e^W(Z_\alpha - F_\alpha), \quad \alpha \in \Lambda$$

where

$$(5.1) \quad F_\alpha = G_\alpha - \hat{G}_\alpha - \sum_{\beta \in \Lambda} \kappa_{\alpha\beta} \hat{G}_\beta + \frac{1}{2} \sum_{\beta \in \Lambda} [\hat{G}_\alpha, \hat{G}_\beta] z_\beta + \tilde{F}_\alpha$$

with

$$(5.2) \quad \tilde{F}_\alpha = \sum_{\beta \in \Lambda} z_\beta [G_\alpha - \hat{G}_\alpha, \hat{G}_\beta] - \frac{1}{2} \sum_{\beta \in \Lambda} [\kappa_{\alpha\beta} \hat{G}_\beta, W]$$

$$-I_3(1, C_2(Z_\alpha W : W) : W) + I_2(1, C_2(G_\alpha : W) : W),$$

where

$$I_j(t, B : A) = \int_0^t \frac{(t-s)^{j-1}}{(j-1)!} e^{-sA} B e^{sA} ds$$

and

$$C_2(B : A) = [[B, A], A].$$

Corollary . For the same W we have

$$\Delta e^W = e^W \tilde{\Delta},$$

where

$$(5.3) \quad \begin{aligned} \tilde{\Delta} = & -\frac{1}{2} \sum_{j=1}^n \{ (Z_j I - F_j)(\bar{Z}_j I - F_{\bar{j}}) + (\bar{Z}_j I - F_{\bar{j}})(Z_j I - F_j) \} \\ & + \frac{1}{2} \sum_{i,j=1}^n \{ c_{i,\bar{i}}^{\bar{j}}(\bar{Z}_j I - F_{\bar{j}}) + c_{i,i}^j(Z_j I - F_j) - \frac{1}{2} R \end{aligned}$$

with

$$(5.4) \quad R = e^{-W} \left(\sum_{j,k,\ell=1}^n R_{\ell\bar{k}j\bar{j}} a_k^* a_\ell \right) e^W.$$

So the symbol of $\tilde{\Delta}$ is given by

$$\sigma(\tilde{\Delta}) = -\frac{1}{2} \sum_{j=1}^n \{ (p_j I - F_j)(q_j I - F_{\bar{j}}) + (q_j I - F_{\bar{j}})(p_j I - F_j) \} - \frac{1}{2} R,$$

where

$$r_1 = -\frac{1}{4} \sum_{k,j=1}^n \{ Z_k(c_{j,j}^k) + \bar{Z}_k(c_{j,\bar{j}}^{\bar{k}}) \} I + \frac{1}{2} \sum_{i,j=1}^n \{ c_{i,\bar{i}}^{\bar{j}}(q_j I - F_{\bar{j}}) + c_{i,i}^j(p_j I - F_j) \},$$

F_α and R are given by (5.1) and (5.4) respectively. Here

$$\sigma(Z_j) = p_j I \quad \sigma(\bar{Z}_j) = q_j I.$$

By (5.1), (5.2) and Proposition 3 we have

Proposition 4. For the symbols $(p_k I - F_k), (q_k I - F_{\bar{k}})$, we have

$$\begin{aligned} & \langle J\nabla(p_k I - F_k), \nabla(p_j I - F_j) \rangle|_{z=0} = i\sigma([Z_k, Z_j]), \\ & \langle J\nabla(p_k I - F_k), \nabla(q_j I - F_{\bar{j}}) \rangle|_{z=0} = i\{R(Z_k, \bar{Z}_j) + \sigma([Z_k, \bar{Z}_j])\}, \\ & \langle J\nabla(q_k I - F_{\bar{k}}), \nabla(q_j I - F_{\bar{j}}) \rangle|_{z=0} = i\sigma([\bar{Z}_k, \bar{Z}_j]). \end{aligned}$$

If we apply Theorem 3 to construction of the fundamental solution for $\frac{\partial}{\partial t} + \tilde{\Delta}$ considering $a_i a_j^*$ as $S_{1,0}^2$, we obtain, in this case, the fundamental solution with $\mathcal{M}$ whose main part is given by

$$\mathcal{M} = - \begin{pmatrix} \mathcal{M}_0 & 0 \\ 0 & \bar{\mathcal{M}}_0 \end{pmatrix},$$

where $(\mathcal{M}_0)_{ij} = R(Z_i, \bar{Z}_j)$. So the kernel $\tilde{u}_0(t, x, x)$ of pseudodifferential operator with symbol $u_0(t, x, \xi)$ is obtained as

$$\tilde{u}_0(t, x, x) = (2\pi t)^{-n} \det\left(\frac{t\mathcal{M}_0}{\exp(t\mathcal{M}_0) - 1}\right)$$

Then we have

$$\text{str } \tilde{u}_0(t, x, x) = (2\pi t)^{-n} \text{str} \left[\det\left(\frac{t\mathcal{M}_0}{\exp(t\mathcal{M}_0) - 1}\right) \right]$$

Applying Theorem 1, we have

$$\text{str } \tilde{u}_0(t, x, x) dv = (2\pi i)^{-n} [\text{Td}(\text{TM})]_{2n},$$

where

$$\text{Td}(T(M)) = \det\left(\frac{\Omega}{e^\Omega - 1}\right)$$

with curvature form Ω , that is,

$$\Omega = (\Omega_\ell^k), \quad \Omega_\ell^k = \sum_{i,j=1}^n R_{\bar{k},\ell,i,\bar{j}} \omega^i \wedge \bar{\omega}^j.$$

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