

On some analytic properties of a function associated with the Selberg class satisfying certain special conditions

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1 Introduction

J.Kaczorowski defined the associated Euler totient function for a class of generalized L -functions including the Riemann zeta function, Dirichlet L -functions and obtained an asymptotic formula (see [4]) : By a polynomial Euler product expressions we mean a function $F(s)$ of a complex variable $s = \sigma + it$ which for $\sigma > 1$ is defined by a Euler product expressions of the form

$$F(s) = \prod_p F_p(s) = \prod_p \prod_{j=1}^d \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}, \quad (1.1)$$

where p runs over primes and $|\alpha_j(p)| \leq 1$ for all p and $1 \leq j \leq d$. We take the smallest $d \in \mathbb{Z}_{>0}$ such that there is at least one prime p_0 satisfying

$$\prod_{j=1}^d \alpha_j(p_0) \neq 0,$$

where d is called the *Euler degree* of F . Note that the L -functions from number theory including the Riemann zeta function $\zeta(s)$ and Dirichlet L -function $L(s, \chi)$ and Dedekind zeta and Hecke L -functions of algebraic number fields, as well as the (normalized) L -functions of holomorphic modular forms and, conjecturally, many other L -functions are polynomial Euler products expressions.

M.Řeško described the analytic property of some function connected with the Euler totient function (see [6]) : Let $\mathbb{H}$ be the upper half-plane. We describe basic analytic properties of the function $f(z)$ defined for $z \in \mathbb{H}$ as follows :

$$f(z) = \lim_{n \rightarrow \infty} \sum_{\substack{\rho \\ 0 < \text{Im } \rho < T_n}} \frac{e^{\rho z} \zeta(\rho - 1)}{\zeta'(\rho)}, \quad (1.2)$$

where $\{T_n\}$ denotes a real sequence yielding appropriate groupings of the zeros, and the summation is over non-trivial zeros of $\zeta(s)$ with positive imaginary part. For simplicity we assume here that all the zeros of $\zeta(s)$ have simple. M.Řeško showed the holomorphy of $f(z)$ for $\text{Im } z > 0$, meromorphic continuation to the whole z -plane with its principal part, and a functional relation containing its reflection property. The functional equation for $f(z)$ connects the values of the function f at the points z and $\bar{z}$. Define a smooth curve $\tau : [0, 1] \ni t \mapsto \tau(t) \in \mathbb{C}$ such that $\tau(0) = -1/4$, $\tau(1) = 5/2$ and $0 < \text{Im } \tau(t) < 1$ for $t \in (0, 1)$, and define it by $\ell(-1/4, 5/2)$. The analytic property of $f(z)$ is described by the following theorems:

Theorem 1.1 (Theorem 1.in [6]). *The function $f(z)$ is holomorphic on $\mathbb{H}$ and for $z \in \mathbb{H}$ we have*

$$2\pi i f(z) = f_1(z) + f_2(z) - e^{5z/2} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{5/2}(z - \log n)}, \quad (1.3)$$

where the functions

$$f_1(z) = \int_{-1/4+i\infty}^{-1/4} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds, \quad (1.4)$$

$$f_2(z) = \int_{\ell(-1/4, 5/2)} \frac{\zeta(s-1)}{\zeta(s)} e^{sz} ds \quad (1.5)$$

are holomorphic on $\mathbb{H}$ and on the whole z -plane respectively, and the last term is a meromorphic function on the whole z -plane, whose poles are at $z = \log n$ of the second order with the residues $-\varphi(n)/2\pi i$ ($n = 1, 2, \dots$). Here $\varphi(n)$ denotes Euler's totient function.

Theorem 1.2 (Theorem 2.in [6]). *The function $f(z)$ can be continued analytically to a meromorphic function on the whole z -plane, which satisfies the functional equation*

$$f(z) + \overline{f(\bar{z})} = B(z) \quad (1.6)$$

and

$$B(z) = -\frac{6}{\pi^2} e^{2z} + \frac{1}{2\pi^2} \sum_{k,n=1}^{\infty} \frac{\mu(k)}{n^2 k} \left[\frac{1}{(nke^z - 1)^2} + \frac{2}{nke^z - 1} + \frac{1}{(nke^z + 1)^2} - \frac{2}{nke^z + 1} \right], \quad (1.7)$$

where $B(z)$ is a meromorphic function on the whole z -plane with the poles of the second order at $z = -\log nk$, $n, k = 1, 2, \dots$ and $\mu(k)$ is the Möbius function.

We now provide the Selberg class $\mathcal{S}$ defined as follows : $f \in \mathcal{S}$ if

- (i) (ordinary Dirichlet series) $f(s) = \sum_{n=1}^{\infty} a_f(n)n^{-s}$, converges absolutely for $\sigma > 1$;
- (ii) (analytic continuation) there exists $m \in \mathbb{Z}_{\geq 0}$ such that $(s-1)^m f(s)$ is entire of finite order;
- (iii) (functional equation) $f(s)$ satisfies a functional equation of the form $\Phi(s) = \omega \overline{\Phi(1-\bar{s})}$, where

$$\Phi(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) f(s) = \gamma(s) f(s), \quad (1.8)$$

say, with $r \geq 0$, $Q > 0$, $\lambda_j > 0$, $\text{Re } \mu_j \geq 0$ and $|\omega| = 1$;

- (iv) (Ramanujan conjecture) $a_f(n) \ll n^\epsilon$ holds for any $\epsilon > 0$;

- (v) (Euler product) $f(s) = \prod_p \exp \left(\sum_{\ell=0}^{\infty} \frac{b_f(p^\ell)}{p^{\ell s}} \right)$ for $\sigma > 1$, where $b_f(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b_f(n) \ll n^\vartheta$ for some $\vartheta < 1/2$.

The empty products are hereafter to be equal to 1.

2 The extension of $f(z)$ to the subclass of $\mathcal{S}$

If a functions $F \in \mathcal{S}$ has a polynomial Euler product expressions (1.1), the subclass of $\mathcal{S}$ of the functions with polynomial Euler product expressions is denoted by $\mathcal{S}^{\text{poly}}$. Establishing the results which extend Theorem 1.1 and 1.2 to the function $F \in \mathcal{S}^{\text{poly}}$ are the main aim of the present paper. Let $\{\rho\}$ denote the non-trivial zeros of F with positive imaginary parts, and assume that each ρ is simple. Moreover, let $\{T_n\}$ denote a real sequence yielding appropriate groupings of the zeros, where its precise definition is to be given by (3.2) below. For $\text{Im } z > 0$ and $F \in \mathcal{S}^{\text{poly}}$, we consider the function defined by

$$f(z, F) = \lim_{n \rightarrow \infty} \sum_{\substack{\rho \\ 0 < \operatorname{Im} \rho < T_n}} \frac{e^{\rho z} \zeta(\rho - 1)}{F'(\rho)}. \quad (2.1)$$

If there are trivial zeros of $F(s)$ on the imaginary axis in $\mathbb{H}$, they are to be incorporated in the summation. The reason why $\zeta(s)$ appears in the numerator on the right hand side is that the Barnes type integral (5.3) below for the Whittaker function can then be applied under the hypothesis $(r, \lambda_j) = (1, 1)$ for all j in (1.8) (see Lemma 5.1 and Section 6).

Fact 2.1 (Fact 2.1 in [3]). *The limit in (2.1) exists for all $z \in \mathbb{H}$.*

We will prove Fact 2.1 in the next section.

3 Proof of Fact 2.1.

We prove Fact 2.1, for which the following Lemma is used.

Lemma 3.1 (Lemma 4. in [8]). *Let $F \in \mathcal{S}$ and let T be sufficiently large, and fix $H = D \log \log T$ with a large constant $D > 0$. We take any subinterval $[n, n+1]$ with n chosen such that $n \in \mathbb{Z}_{>0} \cap [T-H, T+H]$. Then, there are the lines $t = t_0$ such that*

$$|F(\sigma + it_0)|^{-1} = O(\exp(C(\log T)^2)), \quad (3.1)$$

uniformly in $\sigma \geq -2$, where C is a positive constant.

Let T be sufficiently large. We fix $H = D \log \log T$, where D is a large positive constant. We take any subinterval $[n, n+1]$, where n is a positive integer in $[T-H, T+H]$. Then, by Lemma 3.1 there are the lines $t = T_n$ such that

$$|F(\sigma + iT_n)|^{-1} = O(\exp(C_1(\log T)^2)) \quad (3.2)$$

uniformly for $\sigma \geq -2$, where C_1 is a positive constant. Since T_n is contained in the interval $[T-H, T+H]$, we can see that $T_n \sim T$ as n tends to infinity. Let $\alpha = \min\{\operatorname{Im} \rho > 0\}/2$ and $\mathcal{L}$ denote the contour consisting of the line segments

$$[b, b + iT_n], [b + iT_n, a + iT_n], [a + iT_n, a], [a, (a+b)/2 + i\alpha], [(a+b)/2 + i\alpha, b],$$

where $\max\{-3/2, \max\{\operatorname{Re} \rho < 0\}/2\} < a < 0, b > 5/2$. We assume that the real part of $s = a + it$ ($t \in \mathbb{R}$) does not coincide the poles of $\Gamma(s + \mu)\Gamma(s - \mu)$. We consider the following contour integral round $\mathcal{L}$:

$$\int_{\mathcal{L}} \frac{\zeta(s-1)}{F(s)} e^{sz} ds. \quad (3.3)$$

Since we assume the order of ρ is simple, we have by residue theorem

$$\begin{aligned} \int_{\mathcal{L}} \frac{\zeta(s-1)}{F(s)} e^{sz} ds &= \int_{a+iT_n}^a \frac{\zeta(s-1)}{F(s)} e^{zs} ds + \int_L \frac{\zeta(s-1)}{F(s)} e^{zs} ds \\ &\quad + \int_b^{b+iT_n} \frac{\zeta(s-1)}{F(s)} e^{zs} ds + \int_{b+iT_n}^{a+iT_n} \frac{\zeta(s-1)}{F(s)} e^{zs} ds \\ &= 2\pi i \sum_{\substack{\rho \\ 0 < \operatorname{Im} \rho < T_n}} \frac{e^{\rho z} \zeta(\rho - 1)}{F'(\rho)}, \end{aligned} \quad (3.4)$$

where the path L above consists of joining the two line segments $[a, (a+b)/2 + i\alpha]$ and $[(a+b)/2 + i\alpha, b]$. We now estimate the integral along the line segment $[b + iT_n, a + iT_n]$. For $a \leq \sigma \leq b$, we have by (3.2),

$$|F(\sigma + iT_n)|^{-1} = O(\exp(C(\log T)^2)),$$

which, with the vertical estimate for $\zeta(s-1)$, shows with $z = x + iy$ that

$$\left| \int_{a+iT_n}^{b+iT_n} \frac{\zeta(s-1)}{F(s)} e^{zs} ds \right| \ll (b-a) T_n^c \exp\{C(\log T)^2 - yT_n + |x|(|a| + |b|)\} \quad (3.5)$$

for $T, T_n \geq 1$ with a constant $c = c(a, b) > 0$, where the last bound tends to zero as $n \rightarrow \infty$. By Theorem 4.1 below, the convergence of the other integrals in (3.4) are ensured (see (4.5)-(4.7)). The limit in (2.1) therefore exists. $\square$

4 Main theorems

Letting $n \rightarrow \infty$, we have

$$\int_{a+i\infty}^a \frac{\zeta(s-1)}{F(s)} e^{zs} ds + \int_L \frac{\zeta(s-1)}{F(s)} e^{zs} ds + \int_b^{b+i\infty} \frac{\zeta(s-1)}{F(s)} e^{zs} ds = 2\pi i f(z, F), \quad (4.1)$$

with $f(z, F)$ in (2.1). To evaluate the integral along the vertical line with $s = b + it$ ($t \geq 0$), we prepare the Dirichlet series expansion of $\zeta(s-1)/F(s)$ for $\sigma > 2$.

Definition 4.1 (p.34 in [4]). For $\sigma > 1$ and $F \in \mathcal{S}^{\text{poly}}$, we define the Dirichlet coefficients μ_F as follows :

$$\frac{1}{F(s)} = \sum_{n=1}^{\infty} \frac{\mu_F(n)}{n^s} = \prod_p \prod_{j=1}^d \left(1 - \frac{\alpha_j(p)}{p^s}\right). \quad (4.2)$$

Remark 4.2 (p.34 in [4]). By (4.2), $|\mu_F(n)| \leq \tau_d(n)$, where $\tau_d(n)$ is the divisor function of order d , so that $\zeta^d(s) = \sum_{n=1}^{\infty} \tau_d(n)/n^s$ for $\sigma > 1$. In particular $\tau_1(n) = 1$ for all n .

Using (4.2) in $\sigma > 2$, we obtain

$$\frac{\zeta(s-1)}{F(s)} = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \quad (4.3)$$

where

$$g(n) = \sum_{d|n} \mu_F(d) \frac{n}{d}. \quad (4.4)$$

Theorem 4.1 (Theorem 4.1 in [3]). *Let $\max\{-3/2, \max\{\operatorname{Re} \rho < 0\}/2\} < a < 0, b > 5/2$. Then, the function (2.1) is holomorphic on $\mathbb{H}$, and for $z \in \mathbb{H}$ the formula*

$$2\pi i f(z, F) = f_1(z, F) + f_2(z, F) - e^{bz} \sum_{n=1}^{\infty} \frac{g(n)}{n^b(z - \log n)}, \quad (4.5)$$

is valid, where the functions

$$f_1(z, F) = \int_{a+i\infty}^a \frac{\zeta(s-1)}{F(s)} e^{sz} ds, \quad (4.6)$$

$$f_2(z, F) = \int_L \frac{\zeta(s-1)}{F(s)} e^{sz} ds \quad (4.7)$$

are holomorphic on $\mathbb{H}$ and on the whole z -plane, and the last term on the right is a meromorphic function on the whole z -plane with the poles at $z = \log n$ ($n = 1, 2, \dots$).

We need not use the condition of a which does not coincide the poles of $\Gamma(s+\mu)\Gamma(s-\mu)$ in proving of Theorem 4.1.

Theorem 4.2 (Theorem 4.2 in [3]). *For any $F \in \mathcal{S}^{\text{poly}}$ with $(r, \lambda_j) = (1, 1)$ for all j in (1.8) and $0 \leq \mu < 1$, the function (2.1) has a meromorphic continuation to $y > -\pi$.*

The complex number μ_1 when $r = 1$ in (1.8) is hereafter referred to as μ . The L -functions associated with holomorphic cusp forms and Dedekind zeta functions of the imaginary quadratic fields are the examples of F considered in Theorem 4.2. Letting

$$\mathbb{H}^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}, \quad (4.8)$$

we next study the function, for $z \in \mathbb{H}^-$,

$$f^-(z, F) = \lim_{n \rightarrow \infty} \sum_{\substack{\rho \\ -T_n < \operatorname{Im} \rho < 0}} \frac{e^{\rho z} \zeta(\rho - 1)}{F'(\rho)}. \quad (4.9)$$

If there are trivial zeros of $F(s)$ on the imaginary axis in $\mathbb{H}^-$, they are incorporated in the summation. The existence of the limit on the right hand side of (4.9) is proved similarly to Fact 2.1.

Corollary 4.3 (Corollary 4.3 in [3]). *For any $F \in \mathcal{S}^{\text{poly}}$ satisfying the same conditions as in Theorem 4.2, the function (4.9) has a meromorphic continuation to $y < \pi$.*

Theorem 4.4 (Theorem 4.4 in [3]). *For any $F \in \mathcal{S}^{\text{poly}}$ satisfying the same conditions as in Theorem 4.2, the function (2.1) can be continued analytically to the whole z -plane. In addition to the condition as in Theorem 4.2, we assume that the Dirichlet series coefficients $a_F(n)$ of $F \in \mathcal{S}^{\text{poly}}$ is real-valued for all n . Then, the function (2.1) satisfies the functional equation*

$$f(z, F) + \overline{f(\bar{z}, F)} = B(z, F), \quad (4.10)$$

where

$$B(z, F) = \frac{1}{2\pi i} (f_1(z, F) + f_1^-(z, F)) - \frac{e^{2z}}{F(2)} \quad (4.11)$$

for all $z \in \mathbb{C}$ and the function $f_1^-(z, F)$ is holomorphic on $\mathbb{H}^-$.

5 Some auxiliary results on the Whittaker function

We first introduce the Whittaker function $W_{\kappa, \mu}(z)$ (via the confluent hypergeometric function $\Psi(\alpha, \gamma; z)$ below), which is necessary to prove our main theorems, and then prepare some auxiliary results, i.e. its integral expression and asymptotic expansions.

Definition 5.1 (The confluent hypergeometric function of the second kind ([1])). Let $\Psi(\alpha, \gamma; z)$ be the *confluent hypergeometric function of the second kind* defined by

$$\Psi(\alpha, \gamma; z) = \frac{1}{\Gamma(\alpha)(e^{2\pi i \alpha} - 1)} \int_{\infty}^{(0+)} e^{-zw} w^{\alpha-1} (1+w)^{\gamma-\alpha-1} dw \quad (5.1)$$

for any $(\alpha, \gamma) \in \mathbb{C}^2$ and for $|\arg z| < \pi$. Here the path of integration is a contour in the w -plane which consists of the upper real axis from $e^{0\pi i} \infty$ to $e^{0\pi i} \delta$ with a small $\delta > 0$, the circle with the center $w = 0$ and the radius δ through which $\arg w$ varies from 0 to 2π , and the lower real axis from $e^{2\pi i} \delta$ to $e^{2\pi i} \infty$.

Definition 5.2 (The Whittaker function ([5])). The *Whittaker function* $W_{\kappa, \mu}(z)$, which has large applicability, e.g. in number theory and physics, is defined by

$$W_{\kappa, \mu}(z) = z^{\mu+1/2} e^{-z/2} \Psi\left(\frac{1}{2} - \kappa + \mu, 2\mu + 1; z\right) \quad (|\arg z| < \pi), \quad (5.2)$$

where (to avoid many-valuedness) the domain of z is to be restricted in the z -plane cut along the negative real axis with $|\arg z| < \pi$.

Lemma 5.1 (Barnes type integral for the Whittaker function ([7], [9])). *The Barnes integral for $W_{\kappa, \mu}(z)$ asserts*

$$W_{\kappa, \mu}(z) = \frac{e^{-z/2} z^{\kappa}}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{\Gamma(s) \Gamma(-s - \kappa - \mu + \frac{1}{2}) \Gamma(-s - \kappa + \mu + \frac{1}{2})}{\Gamma(-\kappa - \mu + \frac{1}{2}) \Gamma(-\kappa + \mu + \frac{1}{2})} z^s ds, \quad (5.3)$$

for $|\arg z| < 3\pi/2$, and $\kappa \pm \mu + 1/2 \neq 0, 1, 2, \dots$; the contour has loops if necessary so that the poles of $\Gamma(s)$ from those of $\Gamma(-s - \kappa - \mu + 1/2) \times \Gamma(-s - \kappa + \mu + 1/2)$ are on opposite sides of it.

In (5.3), it holds for all finite values of c provided that the contour of integration can always be deformed so as to separate the poles $\Gamma(s)$ and those of the other Γ -factors. By Stirling's formula for $\Gamma(s)$ (cf. [2]), the integral in (5.3) represents a function of z which is holomorphic at all points in the domain $|\arg z| \leq 3\pi/2 - \alpha$ with any small $\alpha > 0$. The asymptotic expansions for $\Psi(\alpha; \gamma; z)$ as $z \rightarrow 0$ readily asserts the following proposition.

Proposition 5.2 (The asymptotic expansions for $\Psi(a; b; z)$ as $z \rightarrow 0$ ([7])). *We have the asymptotic expansions, as $z \rightarrow 0$ through $|\arg z| < \pi$,*

$$\Psi(a; b; z) = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(|z|^{\operatorname{Re} b-2}) \quad (\operatorname{Re} b \geq 2, b \neq 2), \quad (5.4)$$

$$= \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(|\log z|) \quad (b = 2), \quad (5.5)$$

$$= \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(1) \quad (1 < \operatorname{Re} b < 2), \quad (5.6)$$

$$= \frac{\Gamma(1-b)}{\Gamma(1+a-b)} + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + O(|z|) \quad (\operatorname{Re} b = 1, b \neq 1), \quad (5.7)$$

$$= -\frac{1}{\Gamma(a)} \left\{ \log z + \frac{\Gamma'}{\Gamma}(a) + 2C_0 \right\} + O(|z \log z|) \quad (b = 1), \quad (5.8)$$

where C_0 is Euler's constant.

By the definition (5.2) and Proposition 5.2, we have the following asymptotic expansions as $z \rightarrow 0$ for $W_{\kappa, \mu}(z)$.

Proposition 5.3 (The asymptotic expansions in $z \rightarrow 0$ for $W_{\kappa, \mu}(z)$).

$$W_{\kappa, \mu}(z) = \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right)} z^{1/2-\mu} + O(z^{3/2-\operatorname{Re} \mu}) \quad (\operatorname{Re} \mu \geq 1/2, \mu \neq 1/2), \quad (5.9)$$

$$= \frac{1}{\Gamma(1-\kappa)} + O(|z \log z|) \quad (\mu = 1/2), \quad (5.10)$$

$$= \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \kappa\right)} z^{1/2-\mu} + O(|z|^{\operatorname{Re} \mu+1/2}) \quad (0 < \operatorname{Re} \mu < 1/2), \quad (5.11)$$

$$= \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \mu - \kappa\right)} z^{\mu+1/2} + \frac{\Gamma(2\mu)}{\Gamma\left(\mu + \frac{1}{2} - \kappa\right)} z^{-\mu+1/2} + O(|z|^{\operatorname{Re} \mu+3/2}) \quad (\operatorname{Re} \mu = 0, \mu \neq 0), \quad (5.12)$$

$$= -\frac{z^{1/2}}{\Gamma\left(\frac{1}{2} - \kappa\right)} \left(\log z + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - \kappa\right) + 2C_0 \right) + O(|z|^{3/2} |\log z|) \quad (\mu = 0). \quad (5.13)$$

6 Proof of Theorem 4.2

We prove that the function $f(z, F)$ ($z = x + iy$) has a meromorphic continuation to $y > -\pi$. By Theorem 4.1, the function

$$f_1(z, F) = \int_{a+i\infty}^a \frac{\zeta(s-1)}{F(s)} e^{zs} ds = - \int_a^{a+i\infty} \frac{\zeta(s-1)}{F(s)} e^{zs} ds$$

is convergent for $y > 0$. We recall the hypotheses that $(r, \lambda_j) = (1, 1)$ for all j in (1.8) and $0 \leq \mu < 1$. We rewrite the functional equation (1.8) under these hypotheses as follows :

$$\begin{aligned} Q^s \Gamma(s + \mu) F(s) &= \overline{\omega Q^{1-\bar{s}} \Gamma(1 - \bar{s} + \mu) F(1 - \bar{s})} \\ &= \omega Q^{1-s} \Gamma(1 - s + \mu) \overline{F(1 - \bar{s})}, \end{aligned}$$

where the conditions of Q and ω are the same as noted in (1.8). Hence

$$\frac{1}{F(s)} = \overline{\omega} Q^{2s-1} \frac{\Gamma(s + \mu)}{\Gamma(1 - s + \mu)} \frac{1}{\overline{F(1 - \bar{s})}}, \quad (6.1)$$

which yields, from the reciprocal relation $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$, that

$$\frac{1}{F(s)} = \frac{\overline{\omega}}{\pi} Q^{2s-1} \sin \pi(s-\mu) \Gamma(s+\mu) \Gamma(s-\mu) \frac{1}{F(1-\overline{s})}. \quad (6.2)$$

By (6.2) and the functional equation for $\zeta(s)$, on setting

$$\begin{aligned} H(s, w) &= (2\pi Q^2)^s \zeta(2-s) \Gamma(s-\mu) \Gamma(s+\mu) \Gamma(2-s) e^{ws}, \\ K(s, w) &= \frac{H(s, w)}{F(1-\overline{s})}, \end{aligned} \quad (6.3)$$

we have

$$\begin{aligned} f_1(z, F) &= J_- \left(z + \frac{3\pi i}{2} \right) + J_+ \left(z - \frac{\pi i}{2} \right) + J_- \left(z + \frac{\pi i}{2} \right) + J_+ \left(z - \frac{3\pi i}{2} \right) \\ &= \sum_{j=1}^4 f_{1j}(z, F), \end{aligned} \quad (6.4)$$

say, where

$$J_{\pm}(w) = \frac{\pm i e^{\pm \pi i \mu} \overline{\omega}}{(2\pi)^3 Q} \int_a^{\alpha+i\infty} K(s, w) ds. \quad (6.5)$$

It follows from, by Stirling's formula,

$$\Gamma(s+\mu) \Gamma(s-\mu) \Gamma(2-s) \asymp e^{-3\pi|t|/2} |t|^{a+1/2} \quad (\text{as } t \rightarrow \pm\infty), \quad (6.6)$$

$\max\{-3/2, \max\{\operatorname{Re} \rho < 0\}/2\} < a < 0$, and the vertical estimates for $\zeta(2-s)$ and $\overline{F(1-\overline{s})}$ that $f_{1j}(z, F)$ ($j = 1, \dots, 4$) are holomorphic for $y > -3\pi$, for $y > -\pi$, for $y > -2\pi$ and for $y > 0$ respectively. We next show that $f_{14}(z, F)$ can be continued to $0 < y < 3\pi$. Splitting the integral in $f_{14}(z, F)$, we have

$$\begin{aligned} f_{14}(z, F) &= J_+ \left(z - \frac{3\pi i}{2} \right) \\ &= \frac{\overline{\omega} e^{\mu \pi i}}{(2\pi)^3 Q} \left\{ \int_{a-i\infty}^{\alpha+i\infty} - \int_{a-i\infty}^a \right\} K \left(s, z - \frac{3\pi i}{2} \right) ds \\ &= I_1(z, F) + I_2(z, F), \end{aligned}$$

say, where

$$I_1(z, F) = \frac{\overline{\omega} e^{\mu \pi i}}{(2\pi)^3 Q} \int_{a-i\infty}^{\alpha+i\infty} K \left(s, z - \frac{3\pi i}{2} \right) ds, \quad (6.7)$$

$$I_2(z, F) = \frac{\overline{\omega} e^{\mu \pi i}}{(2\pi)^3 Q i} \int_{a-i\infty}^a K \left(s, z - \frac{3\pi i}{2} \right) ds. \quad (6.8)$$

Here, we recall the hypothesis that the real part of $s = a + it$ ($t \in \mathbb{R}$) does not coincide with the poles of $\Gamma(s+\mu)\Gamma(s-\mu)$. From (6.6), we can see that the integral $I_2(z, F)$ is convergent for $y < 3\pi$. From (6.6) again, $I_1(z, F)$ converges absolutely for $0 < y < 3\pi$. Substituting the Dirichlet series expansions in (4.2) and $\zeta(2-s)$, and then integrating term-by-term, we have

$$I_1(z, F) = \frac{\overline{\omega} e^{\mu \pi i}}{(2\pi)^3 Q} \sum_{k,n=1}^{\infty} \frac{\overline{\mu_F(k)}}{kn^2} \int_{a-i\infty}^{\alpha+i\infty} e^{(\log(2\pi nk Q^2) + z - 3\pi i/2)s} \Gamma(s+\mu) \Gamma(s-\mu) \Gamma(2-s) ds, \quad (6.9)$$

where the interchange of the order of integration and summation can be justified by absolute convergence as follows :

For $0 < y < 3\pi$, each s -integral in (6.9) is bounded as

$$\ll (nk)^a e^{ax} \int_{-\infty}^{\infty} e^{-(y-3\pi/2)t - 3\pi|t|/2} (|t|+1)^{a+1/2} dt \ll (nk)^a,$$

and hence the series in (6.9) is bounded above as

$$\ll \sum_{k,n=1}^{\infty} \frac{|\overline{\mu_F(k)}|}{k^{1-a}n^{2-a}} < +\infty.$$

by (4.2) and $a < 0$. We next prepare the following notations which is used to describe the residues of the integrand in (6.9) and also used later in similar situations : for $\mu \notin \{0, 1/2\}$,

$$S_{k,n,m}(w, \mu) = \frac{(-1)^m}{m!} \Gamma(2\mu - m) \Gamma(2 - \mu + m) (2\pi n k Q^2)^{\mu-m} e^{w(\mu-m)} \quad (6.10)$$

and for $\mu = 0$,

$$S_{k,n,m}(w, 0) = \frac{m+1}{m!} \frac{e^{-mw}}{(2\pi n k Q^2)^m} \left\{ \log(2\pi n k Q^2) + w + \sum_{h=1}^m \frac{1}{h} - C_0 - \frac{1}{m+1} \right\} \quad (6.11)$$

with Euler's constant C_0 , and also for $\mu = 1/2$,

$$\begin{aligned} S_{k,n,m}\left(w, \frac{1}{2}\right) &= \frac{\Gamma\left(\frac{3}{2} + m\right)}{(m!)^2} \frac{e^{(1/2-m)w}}{(2\pi n k Q^2)^{m-1/2}} \left\{ m \left(\psi\left(\frac{3}{2} + m\right) - 2\psi(m+1) \right) \right. \\ &\quad \left. - m \left(\log(2\pi n k Q^2) + w \right) + 1 \right\} \end{aligned} \quad (6.12)$$

with Euler's psi function $\psi(s) = (\Gamma'/\Gamma)(s)$. Then we have the following lemma.

Lemma 6.1 (Lemma 7.1 in [3]). *The residues of the integrand in (6.9) are given as follows :*

(i) for $\mu \notin \{0, 1/2\}$, those at $s = \pm\mu - m$ ($m = 0, 1, \dots$) equal

$$R_{k,n,m}^{(1)}(z, \pm\mu) = S_{k,n,m}\left(z - \frac{3\pi i}{2}, \pm\mu\right);$$

(ii) for $\mu = 0$, those at $s = -m$ ($m = 0, 1, \dots$) equal

$$R_{k,n,m}^{(1)}(z, 0) = S_{k,n,m}\left(z - \frac{3\pi i}{2}, 0\right);$$

(iii) for $\mu = 1/2$, those at $s = 1/2 - m$ ($m = 0, 1, \dots$) equal

$$R_{k,n,m}^{(1)}(z, 1/2) = S_{k,n,m}\left(z - \frac{3\pi i}{2}, \frac{1}{2}\right).$$

Proof. The reciprocal relation $\Gamma(s)$ shows that $\Gamma(-m + \epsilon) = (-1)^m \pi / \Gamma(1 + m - \epsilon) \sin(\pi\epsilon)$ ($m = 0, 1, \dots$), and this implies the Laurent series expansion, as $\epsilon \rightarrow 0$,

$$\Gamma(-m + \epsilon) = \frac{(-1)^m}{m!} \epsilon^{-1} \{1 + \psi(1 + m)\epsilon + O(\epsilon^2)\}, \quad (6.13)$$

which with the Taylor series expansion

$$\Gamma(2 + m - \epsilon) = (m+1)! \{1 - \psi(2 + m)\epsilon + O(\epsilon^2)\}. \quad (6.14)$$

Readily asserts the cases (i) and (ii), by substituting (6.13) and (6.14) into the integrand in (6.9), and by noting $\psi(1 + m) = \sum_{h=1}^m (1/h) + \psi(1)$ with $\psi(1) = -C_0$, while (iii) by noting $\Gamma(s - 1/2)\Gamma(s + 1/2) = (s - 1/2)\Gamma(s - 1/2)^2$ and by substituting (6.13), instead of (6.14), the Taylor series expansion, as $\epsilon \rightarrow 0$,

$$\Gamma\left(\frac{3}{2} + m - \epsilon\right) = \Gamma\left(\frac{3}{2} + m\right) \left\{ 1 - \psi\left(\frac{3}{2} + m\right) \epsilon + O(\epsilon^2) \right\}.$$

□

Since $\lim_{T \rightarrow +\infty} \int_{a \pm iT}^{a \pm i\infty} = 0$ for $0 < y < 3\pi$ in (6.9), we obtain if $\mu \notin \{0, 1/2\}$, by the residue theorem,

$$\int_{a-i\infty}^{a+i\infty} K\left(s, z - \frac{3\pi i}{2}\right) ds = - \int_C K\left(s, z - \frac{3\pi i}{2}\right) ds - 2\pi i \left\{ \sum_{m_1=0}^{M_1} R_{k,n,m_1}^{(1)}(z, \mu) + \sum_{m_2=0}^{M_2} R_{k,n,m_2}^{(1)}(z, -\mu) \right\}, \quad (6.15)$$

where C is a contour separating the poles of $\Gamma(2-s)$ from those of $\Gamma(s+\mu)\Gamma(s-\mu)$ in opposite sides, and M_1 and M_2 are the integers chosen correspondingly. Here the split residual terms in (6.15) when $\mu \in \{0, 1/2\}$ are to be understood as

$$\sum_{m=0}^{M'} R_{k,n,m}^{(1)}(z, 0) \quad (\text{if } \mu = 0); \quad \sum_{m=0}^{M''} R_{k,n,m}^{(1)}(z, 1/2) \quad (\text{if } \mu = 1/2), \quad (6.16)$$

where the same convention is used hereafter. The integral over C in (6.15) can be evaluated, on replacing s with $2-s$, by the formula (5.3), and this shows that it equals

$$T_{k,n}(z - 3\pi i/2),$$

where and in the sequel

$$T_{k,n}(w) = 2\pi i (2\pi n k Q^2)^{1/2} \exp\left(\frac{e^{-w}}{4\pi n k Q^2} + \frac{w}{2}\right) \Gamma(2-\mu)\Gamma(2+\mu) W_{-3/2,\mu}\left(\frac{e^{-w}}{2\pi n k Q^2}\right). \quad (6.17)$$

Therefore, we have

$$I_1(z, F) = \frac{\bar{\omega} e^{\mu\pi i} i}{(2\pi)^3 Q} \sum_{k,n=1}^{\infty} \frac{\overline{\mu_F(k)}}{kn^2} \left[T_{k,n}\left(z - \frac{3}{2}\pi i\right) - 2\pi i \left\{ \sum_{m_1=0}^{M_1} R_{k,n,m_1}^{(1)}(z, \mu) + \sum_{m_2=0}^{M_2} R_{k,n,m_2}^{(1)}(z, -\mu) \right\} \right]. \quad (6.18)$$

The following lemma ensures the convergence for the series on the right hand side in (6.18).

Lemma 6.2 (Lemma 7.2 in [3]). *For $F \in \mathcal{S}^{poly}$ with $(r, \lambda_j) = (1, 1)$ for all j and $0 \leq \mu < 1$ in (1.8), the series on the right hand side in (6.18) is absolutely and uniformly convergent on every compact subset on the whole z -plane.*

By Lemma 6.2, for any $F \in \mathcal{S}^{poly}$ with $(r, \lambda_j) = (1, 1)$ for all j in (1.8) and $0 \leq \mu < 1$, we have the following analytic continuation of $f_1(z, F)$ for $y > -\pi$:

$$f_1(z, F) = J_-\left(z + \frac{3\pi i}{2}\right) + J_+\left(z - \frac{\pi i}{2}\right) + J_-\left(z + \frac{\pi i}{2}\right) + \frac{\bar{\omega} e^{\mu\pi i} i}{(2\pi)^3 Q} \times \sum_{k,n=1}^{\infty} \frac{\overline{\mu_F(k)}}{kn^2} \left[T_{k,n}\left(z - \frac{3}{2}\pi i\right) - 2\pi i \left\{ \sum_{m_1=0}^{M_1} R_{k,n,m_1}^{(1)}(z, \mu) + \sum_{m_2=0}^{M_2} R_{k,n,m_2}^{(1)}(z, -\mu) \right\} \right] + \frac{\bar{\omega} e^{\mu\pi i}}{(2\pi)^3 Q i} \int_{a-i\infty}^a K\left(s, z - \frac{3\pi i}{2}\right) ds, \quad (6.19)$$

where the first integral is holomorphic for $y > -3\pi$, the second for $y > -\pi$, the third for $y > -2\pi$, the fourth double sum for $y > -\infty$ by Lemma 6.2, and the last for $y < 3\pi$. Therefore, (6.19) completes the proof of the continuation of $f(z, F)$ to the region $y > -\pi$. $\square$

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References

- [1] A. Erdélyi (ed.), W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, Vol. I, McGraw-Hill, New York, 1953.
- [2] A. Ivić, The Riemann-Zeta function, Dover Publ. Inc., Mineola, New York, 1985.
- [3] H. Iwata, On some analytic properties of a function associated with the Selberg class satisfying certain special conditions, Tsukuba J. Math. (to appear).
- [4] J. Kaczorowski, On a generalization of the Euler totient function, Monatsh. Math 170 (2013), 27-48.
- [5] N. N. Lebedev, special Functions & Their Applications, Dover Publ. Inc., Mineola, New York, 1972.
- [6] M. Rękoś, On some complex explicit formulae connected with the Euler's φ function. I, Funct. Approx. Comment. Math 29 (2001), 113-124.
- [7] L. J. Slater, Confluent Hypergeometric Functions, Cambridge University Press, 1960.
- [8] K. Srinivas, Distinct zeros of functions in the Selberg class, Acta Arith. 103.3 (2002), 201-207.
- [9] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, 4th ed., Cambridge Univ. Press, 1927.