

Selected results on functions of uniformly bounded characteristic

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It is a great honor for me to be able to speak in the opening session of this assembly.*

Functions of uniformly bounded characteristic are functions meromorphic on a Riemann surface with "uniformly" bounded Shimizu-Ahlfors characteristic functions, so that we must begin with the definition of the characteristic function.

1. Shimizu-Ahlfors' characteristic function.

Let R be a Riemann surface, each point of which will be identified with its local-parametric image in the complex plane $\mathbb{C} = \{ |z| < \infty \}$ if there is no risk of misunderstanding. By a pair, w, D , we always mean a point $w \in R$ and a domain (open and connected set) $D, w \in D \subset R$, such that the boundary ∂D consists of a finite number of mutually disjoint, analytic, simple, and closed curves. The radius $r = r(w, D)$ of D with respect to w is defined by

$$r = \exp\{\lim(g_D(z, w) + \log|z - w|)\},$$

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where $z \rightarrow w$ within the parametric disk of center w , and g_D is the Green function of D . Set

$$D_t = \{z \in D; g_D(z, w) \geq \log(r/t)\}, \quad 0 < t < r.$$

Let $M(R)$ be the family of meromorphic functions on R , and set $f^\# = |f'|/(1 + |f|^2)$ for $f \in M(R)$, the spherical derivative of f (not a function in general). The Shimizu-Ahlfors characteristic function is then defined by

$$T(D, w, f) = \pi^{-1} \int_0^r t^{-1} \left[\iint_{D_t} f^\#(z)^2 dx dy \right] dt.$$

The terminology is justified because we obtain the familiar one in the specified case $R = \{z \in \mathbb{C}; |z| < \rho\}$, $0 < \rho \leq +\infty$, $w = 0$.

Since $R \in O_G$ cannot carry nonconstant, nonnegative, and superharmonic function, we shall hereafter assume that $R \notin O_G$, or, there exist the Green functions $g(z, w) = g_R(z, w)$. Set

$$T(w, f) = T(R, w, f) = \lim T(D, w, f) \leq \infty,$$

where $D \uparrow R$, the directed limit. Then $T(w, f)$ is the function on R . By definition, $f \in UBC(R)$ (of uniformly bounded characteristic on R) if the supremum of $T(w, f)$ for $w \in R$ is finite.

By elementary considerations we obtain

Theorem 1.1. For $f \in M(R)$ and for $w \in R$, we have

$$T(w, f) = \pi^{-1} \iint_R f^\#(z)^2 g(z, w) dx dy \quad (z = x + iy).$$

Thus, $f \in UBC(R)$ if and only if the Green potential $T(w, f)$ of the measure $\pi^{-1} f^\#(z)^2 dx dy$ is bounded on R .

2. Harmonic majoration and F. Riesz' decomposition.

It follows from the celebrated Florack-Behnke-Stein theorem that for $f \in M(R)$ there exist holomorphic functions f_1 and f_2 with no common zero on R such that $f = f_1/f_2$. Then $\phi = (1/2)\log(|f_1|^2 + |f_2|^2) > -\infty$ is subharmonic because $\Delta\phi(z) = 2f^\#(z)^2 \geq 0$. With the aid of the Green formula we have

$$T(D, w, f) = \phi_D^\wedge(w) - \phi(w)$$

for each pair w, D , where $\phi_D^\wedge$ is the least harmonic majorant of ϕ in D , namely,

$$\phi_D^\wedge(w) = -\frac{1}{2\pi} \int_{\partial D} \phi(z) dg_D^*(z, w),$$

the Poisson integral of ϕ on ∂D being positively oriented.

Let $BC(R)$ be the family of $f \in M(R)$ such that there exists $w = w(f) \in R$ with $T(w, f) < \infty$.

Theorem 2.1. (The F. Riesz decomposition of ϕ on R .)

For each $f \in BC(R)$ there exists the least harmonic majorant $\phi_R^\wedge$ of ϕ on R , the smallest among all the harmonic functions not less than ϕ on R , such that

$$\phi(w) = \phi_R^\wedge(w) - T(w, f), \quad w \in R.$$

Remark. The function $T(w, f)$ is of C^∞ with respect to the real variables u and v with $w = u + iv$.

Corollary 2.1.1. If $f \in BC(R)$, then $T(w, f) < \infty$ for each $w \in R$.

Corollary 2.1.2. If $f \in BC(R)$, and if f has two ex-
pressions $f = f_1/f_2 = F_1/F_2$, described at the beginning of this
section, then we set

$$\phi = (1/2) \log(|f_1|^2 + |f_2|^2) \quad \text{and}$$

$$\Phi = (1/2) \log(|F_1|^2 + |F_2|^2).$$

Then the difference $\phi - \Phi$ is harmonic in R .

Here we consider the Nevanlinna-Parreau-Sario characteristic function $T_S(D, w, f)$ of $f \in M(R)$. Set

$$m_S(D, w, f) = -\frac{1}{2\pi} \int_{\partial D} \log^+ |f(z)| dg_D^*(z, w).$$

Let $n(t, f)$ be the number of the roots of the equation $f = \infty$ in D_t and let $n(0, f)$ be the limit of $n(t, f)$ as $t \rightarrow 0$. Set

$$N_S(D, w, f) = \int_0^r t^{-1} [n(t, f) - n(0, f)] dt + n(0, f) \log r,$$

$$T_S(D, w, f) = m_S(D, w, f) + N_S(D, w, f),$$

$$T(w, f) = \lim_{w \in D \uparrow R} T_S(D, w, f).$$

We compare T with T_S in

Theorem 2.2. For $f \in M(R)$,

$$|T(w, f) - T_S(w, f)| \leq k(w, f), \quad w \in R,$$

where k is a constant; read $T = \infty$ if and only if $T_S = \infty$.

Corollary 2.2.1. Let $f \in M(R)$. Then $f \in BC(R)$ if and

only if there exists $w \in R$ such that $T_S(w, f) < \infty$.

3. Removable singularity; classification of Riemann surfaces.

A closed set E on R is said to be of capacity zero if the intersection of E with each parametric disk, considered to be a subset of $\mathbb{C}$, is of logarithmic capacity zero. We claim that a compact set on R of capacity zero is always UBC-removable, namely,

Theorem 3.1. Let E be a compact set of capacity zero on R . Then, for each $f \in \text{UBC}(R \setminus E)$ there exists $F \in \text{UBC}(R)$ such that the restriction of F to $R \setminus E$ coincides with f .

Let $\text{BMOA}(R)$ be the family of functions f holomorphic in R with

$$\sup_{w \in R} T^*(w, f) < \infty,$$

where

$$T^*(w, f) = T^*(R, w, f) = \lim_{w \in D \uparrow R} T^*(D, w, f)$$

with

$$T^*(D, w, f) = \pi^{-1} \int_0^r t^{-1} \left[\iint_{D_t} |f'(z)|^2 dx dy \right] dt.$$

An easy calculation yields the Green potential expression:

$$T^*(w, f) = \pi^{-1} \iint_R |f'(z)|^2 f(z, w) dx dy, \quad w \in R.$$

If $|f|^2$ admits a harmonic majorant on R , then

$$|f(w)|^2 = (|f|^2)_R^\wedge(w) - 2T^*(w, f), \quad w \in R,$$

where $(|f|^2)_R^\wedge$ is the least harmonic majorant of $|f|^2$. This is the F. Riesz decomposition of the subharmonic function $|f|^2$. Let $UBCA(R)$ be the family of all the pole-free members of $UBC(R)$.

Theorem 3.2. $BMOA(R) \subset UBCA(R)$ and the inclusion relation is proper in case R is the open unit disk $\Delta = \{|z| < 1\}$.

Let O_X be the family of Riemann surfaces R such that $R \in O_G$ or $R \notin O_G$ with $X(R) \equiv \mathbb{C}$.

Theorem 3.3. $O_{UBCA} \subsetneq O_{BMOA}$.

4. Counting function.

For $f \in M(R)$ and for w, D we set

$$N(D, w, f) = \sum_{\substack{f(b)=\infty \\ b \in D}} g_D(w, b).$$

Then $N(D, w, f) = N_S(D, w, f)$ if $f(w) \neq \infty$. Actually,

$$N(D, w, f) = \int_0^r t^{-1} n(t, f) dt.$$

For $z \in \mathbb{C}^* = \mathbb{C} \cup \{\infty\}$, we set

$$N(D, w, z, f) = N(D, w, \frac{1}{f-z}) = \sum_{\substack{f(\zeta)=z \\ \zeta \in D}} g_D(w, \zeta),$$

with $N(D, w, \infty, f) = N(D, w, f)$. We further set

$$N(w, z, f) = \lim_{w \in D \uparrow R} N(D, w, z, f).$$

We call $f \in M(R)$ Lindelöfian if $N(w, z, f) < \infty$ for each pair of points $w \in R, z \in \mathbb{C}^* \setminus \{f(w)\}$. It is known that $f \in BC(R)$ if and only if f is Lindelöfian. Thus, $N(w, z, f)$ serves for deciding whether $f \in BC(R)$ or not.

The Riemann sphere $\mathbb{C}^*$ has the chordal distance $\chi(a, b)$, the Euclidean distance for the subspace $\mathbb{C}^*$ of $\mathbb{R}^3$. Let

$$\Gamma(a, \rho) = \{z \in \mathbb{C}^*; \chi(z, a) = \rho\}, \quad a \in \mathbb{C}^*, 0 < \rho < 1.$$

For $w \in R, 0 < \rho < 1$, and $f \in M(R)$ we set

$$C(w, \rho, f) = \sup_{z \in \Gamma(f(w), \rho)} N(w, z, f).$$

Theorem 4.1. The following are mutually equivalent for
 $f \in M(R)$.

- (1) $f \in UBC(R)$.
- (2) There exists $\rho, 0 < \rho < 1$, such that $\sup_{w \in R} C(w, \rho, f) < \infty$.
- (3) For each $\rho, 0 < \rho < 1$, $\sup_{w \in R} C(w, \rho, f) < \infty$.

5. The case $R = \Delta$.

By the uniformization theory there exists an analytic projection map π from Δ onto $R \not\subset O_G$. In many cases we can reduce the problems on R to Δ via π . The following result is fundamental.

Theorem 5.1. For $f \in M(R)$ and for $\delta \in \Delta$, we have

$$T(R, \pi(\delta), f) = T(\Delta, \delta, f \circ \pi).$$

Corollary 5.1.1. For $f \in M(R)$ we have

$$f \in UBC(R) \Leftrightarrow f \circ \pi \in UBC(\Delta).$$

Set

$$N(\Delta) = \{f \in M(\Delta); \sup_{z \in \Delta} (1 - |z|^2) f^\#(z) < \infty\}$$

and

$$N(R) = \{f \in M(R); f \circ \pi \in N(\Delta)\};$$

each member of $N(R)$ is called a normal meromorphic function on R .

It immediately follows from $UBC(\Delta) \subset N(\Delta)$ that

Theorem 5.2. $UBC(R) \subset N(R)$.

In case $R = \Delta$, the inclusion is sharp; there exists a holomorphic function f in Δ which is

- (i) normal in Δ ;
- (ii) of Hardy class $H^p(\Delta)$ for each $0 < p < \infty$;
- (iii) not a member of $UBC(\Delta)$.

Each $f \in UBC(\Delta)$ has, as a member of $BC(\Delta)$, the decomposition $f = (b_1/b_2)F$, where b_1 and b_2 are Blaschke products without common zeros, and $F \in BC(\Delta)$ is pole- and zero-free.

Theorem 5.3. $f \in UBC(\Delta) \Rightarrow F \in UBC(\Delta)$.

The converse is false. There exists a Blaschke quotient

$b_1/b_2 \notin N(\Delta)$ so that we have only to let $F = 1$.

Algebraically $UBC(\Delta)$ is not good:

Theorem 5.4. $UBC(\Delta)$ is closed neither for summation nor for multiplication.

For $f \in M(\Delta)$ and $w \in \mathbb{C}^*$ we let $n(w, f)$ be the number of the roots of the equation $f = w$ in Δ . Our next result is concerned with exceptional sets.

Theorem 5.5. Let $f \in M(\Delta)$ and let $k \geq 0$ be an integer.

Then,

$$\text{cap}\{w \in \mathbb{C}^*; n(w, f) \leq k\} > 0 \Rightarrow f \in UBC(\Delta),$$

where cap means the logarithmic capacity.

Another theorem on the value distribution is

Theorem 5.6. Suppose for $f \in M(\Delta)$ that

$$\iint_{\Delta} f^{\#}(z)^2 dx dy < \infty.$$

Then

$$\lim_{|w| \rightarrow 1} T(\Delta, w, f) = 0.$$

A sequence $\{z_n\}_{n=1}^{\infty}$ of points in Δ is called interpolating if

$$\inf_{n \geq 1} \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| > 0.$$

If $\{z_n\}$ is interpolating, then $\sum (1 - |z_n|) < \infty$.

Theorem 5.7. Let $\{a_n^{(k)}\}_{n=1}^{\infty}$ ($k = 1, 2$) be disjoint interpolating sequence of points in Δ . Set for $k = 1, 2$,

$$B_k(z) = \prod_{n=1}^{\infty} \frac{|a_n^{(k)}|}{a_n^{(k)}} (a_n^{(k)} - z) / (1 - \overline{a_n^{(k)}} z)$$

($|a_n^{(k)}|/a_n^{(k)} = 1$ if $a_n^{(k)} = 0$). Then the following are mutually equivalent.

(I) $B_1/B_2 \in N(\Delta)$.

(II) $B_1/B_2 \in UBC(\Delta)$.

(III) $\{a_n^{(1)}\} \cup \{a_n^{(2)}\}$ is interpolating.

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