

(Restricted) Quantized Enveloping Algebras
of Simple Lie superalgebras
and Universal R-Matrices

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In this note, we define a (Jimbo type) quantized enveloping superalgebras $U_q(G)$ of complex simple Lie superalgebras G of types A, B, C, D (all types) and types F_4 and G_3 (distinguished types). We can get a defining relations of $U_q(G)$, which are consist of q -Serre relations and additional relations. They were unknown even if $q=1$. Moreover we define a restricted quantum groups $u_\zeta(G)$ at a root of unity ζ .

Finally, we consider a Hopf algebrization of the Hopf superalgebra $u_\zeta(G)$, and construct the universal R-matrix of $u_\zeta(G)^\sigma$. Our construction is due to Drinfeld's quantum double construction. By using quantum double construction, we can also show a Poincaré-Birkhoff-Witt type theorem for $U_q(G)$ and $u_\zeta(G)$.

In [Y1-2], we introduced the (Drinfeld type) quantized enveloping superalgebras $U_h(G)$, showed $\underbrace{U_h(G)}_{\text{that}}$ is an h -adic topologically free $C[[h]]$ -Hopf algebra, and gave an explicit formula of universal R-matrix of $U_h(G)^\sigma$. The arguments used in this note are the essentially same arguments as we used in [Y2].

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§1. Quantum double construction.

Let K be a field. Suppose $\text{char}(K) = 0$. Let $(A, \Delta, S, \varepsilon)$ is K -Hopf algebras with coproduct $\Delta : A \rightarrow A \otimes A$, antipode, $S : A \rightarrow A$ and counit $\varepsilon : A \rightarrow K$.

Moreover we assume that there is a symmetric Hopf-pairing $\langle, \rangle : A \otimes A \rightarrow K$, namely $\langle, \rangle$ is a symmetric K -bilinear form such that

- (1) $\langle \Delta(x), y \otimes z \rangle = \langle x, y \otimes z \rangle$,
- (2) $\langle S(x), y \rangle = \langle x, S(y) \rangle$,
- (3) $\langle 1, x \rangle = \varepsilon(x)$

where $x, y, z \in A$.

We call a Hopf-algebra $A^{\text{op}} = (A, \Delta^{\text{op}}, S, \varepsilon)$ the opposite Hopf-algebra of A where $\Delta^{\text{op}} = \tau \circ \Delta$ and $\tau(x \otimes y) = y \otimes x$.

Proposition 1.1. (Quantum double) There is a unique K -Hopf algebra $(D = D(A), \Delta_D, S_D, \varepsilon_D)$ satisfying:

- (1) As K -vector spaces, $D \cong A \otimes A$.
- (2) The K -linear maps $A \rightarrow A \otimes A$ ($x \rightarrow x \otimes 1$) and $A^{\text{op}} \rightarrow A \otimes A$ ($x \rightarrow 1 \otimes x$) are homomorphisms of Hopf-algebras.
- (3) The product of D is defined as follows; if $x, y \in A$ and $\Delta^{(2)}(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)} \otimes x_i^{(3)}$ and $\Delta^{(2)}(y) = \sum_i y_i^{(1)} \otimes y_i^{(2)} \otimes y_i^{(3)}$, then

$$(v \otimes x) \cdot (y \otimes w) = \sum_{i,j} \langle x_i^{(1)}, y_j^{(3)} \rangle \langle x_i^{(3)}, S(y_j^{(1)}) \rangle (v y_j^{(2)} \otimes x_i^{(2)} w).$$

Proposition 1.2. (Universal R -matrix of $D(A)$) Assume that $\dim A < \infty$ and $\langle, \rangle$ is non-degenerate. Let $\{e_i\}$ and $\{e^i\}$ are two bases of A such that $\langle e_i, e^j \rangle = \delta_{ij}$. Then $R = \sum_i (e_i \otimes 1) \otimes (1 \otimes e^i) \in D \otimes D$ satisfies:

- (0) $R^{-1} = (1 \otimes S^{-1})(R)$.
- (1) $R \Delta_D(a) R^{-1} = \Delta_D^{\text{op}}(a)$ ($a \in D$).
- (2) $(1 \otimes \Delta_D)(R) = R_{13} R_{12}$, $(\Delta_D \otimes 1)(R) = R_{23} R_{13}$.

Remark. From (1) and (2), we can easily see that R satisfies the Yang-Baxter equation :

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.$$

Therefore R is called the *universal R -matrix of D* .

§2. Quantized enveloping (super)algebras.

Here we give an abstract definition of Quantized enveloping (super)algebras by using the Quantum double construction.

Let $\mathbb{E}$ be an N -dimensional K -vector space. Assume that there is a non-degenerate bi-linear form $(,) : \mathbb{E} \times \mathbb{E} \rightarrow K$ with a basis $\{\underline{e}_i \mid 1 \leq i \leq N\}$ such that $(\underline{e}_i, \underline{e}_j) = 0$ ($i \neq j$), $(\underline{e}_i, \underline{e}_i) \in \mathbb{Z} - \{0\}$. Let $\Pi = \{\alpha_i \in \mathbb{E} \mid 1 \leq i \leq n\}$ be the set of linearly independent elements. Suppose that $(\alpha_i, \alpha_j) \in (1/4)\mathbb{Z}$. Let $p : \Pi \rightarrow \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ be the function. Write $p(i)$ for $p(\alpha_i)$. We call p the parity function. Put $P_+ = \mathbb{Z}\underline{e}_1 \oplus \dots \oplus \mathbb{Z}\underline{e}_N$.

Let $q \in K^\times$. Let $U_q^{\sim} b_+^\sigma$ be a K -algebra with generators $\{E_i \ (1 \leq i \leq n), K_\lambda \ (\lambda \in P_+), \sigma\}$ and defining relations:

$$(U^{\sim}.1) \ \sigma^2 = 1, \sigma E_i \sigma = (-1)^{p(i)} E_i, \sigma K_\lambda \sigma = K_\lambda,$$

$$(U^{\sim}.2) \ K_0 = 1, K_\lambda K_\mu = K_{\lambda+\mu} \ (\lambda, \mu \in P_+),$$

$$(U^{\sim}.3) \ K_\lambda E_i K_\lambda^{-1} = q^{(\alpha_i, \lambda)} E_i.$$

Moreover $U_q^{\sim} b_+^\sigma$ has a K -Hopf algebra such that

$$(U^{\sim}.4) \ \Delta(\sigma) = \sigma \otimes \sigma, S(\sigma) = \sigma, \varepsilon(\sigma) = 1,$$

$$(U^{\sim}.5) \ \Delta(K_\lambda) = K_\lambda \otimes K_\lambda, S(K_\lambda) = K_\lambda^{-1}, \varepsilon(K_\lambda) = 1,$$

$$(U^{\sim}.6) \ \Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \sigma^{p(i)} \otimes E_i, S(E_i) = -K_{\alpha_i}^{-1} \sigma^{p(i)} E_i, \varepsilon(E_i) = 0.$$

Let $U_q^{\sim} b_+$ (resp. $U_q^{\sim} n_+, T$) be an unital subalgebra generated by the elements $\{E_i \ (1 \leq i \leq n), K_\lambda \ (\lambda \in P_+)\}$ (resp. $\{E_i \ (1 \leq i \leq n)\}, \{K_\lambda \ (\lambda \in P_+)\}$).

Let $\mathbb{I}$ be the set of finite sequences of $\{1, \dots, n\}$. Put $E_I = E_{i_1} E_{i_2} \dots E_{i_p}$ for $I = (i_1, i_2, \dots, i_p) \in \mathbb{I}$ and put $E_\emptyset = 1$.

Lemma 2.1. As a K -vector space, $U_q^{\sim} b_+^\sigma$ has a basis elements such that

$E_I K_\lambda \sigma^c$ ($I \in \mathbb{I}, \lambda \in P_+, c \in \{0, 1\}$). In particular, we have

$$U_q^{\sim} b_+^\sigma \cong U_q^{\sim} n_+ \otimes T \otimes K\langle \sigma \rangle \text{ as } K\text{-vector spaces.}$$

Proposition 2.2. There is a symmetric Hopf-pairing

$\langle , \rangle : U_{q^+}^{\sigma} \otimes U_{q^+}^{\sigma} \rightarrow K$ such that

$$(P.1) \quad \langle \sigma, E_I K_{\lambda} \sigma^c \rangle = \delta_{I\phi} (-1)^c,$$

$$(P.2) \quad \langle K_{\mu}, E_I K_{\lambda} \sigma^c \rangle = \delta_{I\phi} q^{(\mu, \lambda)},$$

$$(P.3) \quad \langle E_i, E_I K_{\lambda} \sigma^c \rangle = \delta_I(i).$$

We put $I_{b_+}^{\sigma} = \text{Ker} \langle , \rangle$ and put $u_{q^+}^{\sigma} = U_{q^+}^{\sigma} / I_{b_+}^{\sigma}$.

Let $D(u_{q^+}^{\sigma})$ be the quantum double of $u_{q^+}^{\sigma}$ with respect to $\langle , \rangle$. For $X \in u_{q^+}^{\sigma}$, we write X, X^{op} for $X \otimes 1, 1 \otimes X \in D(u_{q^+}^{\sigma})$ respectively.

Lemma 2.3. In $D(u_{q^+}^{\sigma})$, the following equations hold:

$$(D \sim .1) \quad \sigma \cdot \sigma^{op} = \sigma^{op} \cdot \sigma, \quad \sigma K_{\lambda}^{op} \sigma = K_{\lambda}^{op}, \quad \sigma E_i^{op} \sigma = (-1)^{p(i)} E_i^{op},$$

$$\sigma^{op} K_{\lambda} \sigma^{op} = K_{\lambda}, \quad \sigma^{op} E_i \sigma^{op} = (-1)^{p(i)} E_i,$$

$$(D \sim .2) \quad K_{\lambda} \cdot K_{\mu}^{op} = K_{\mu}^{op} \cdot K_{\lambda},$$

$$K_{\lambda} E_i^{op} K_{\lambda}^{-1} = q^{-(\alpha_i, \lambda)} E_i^{op}, \quad K_{\lambda}^{op} E_i K_{\lambda}^{op-1} = q^{-(\alpha_i, \lambda)} E_i,$$

$$(D \sim .3) \quad E_i \cdot E_j^{op} - E_j^{op} \cdot E_i = \delta_{ij} (K_{\alpha_i}^{op} \sigma^{op p(i)} - K_{\alpha_i} \sigma^{p(i)}).$$

Let L be an ideal of K -algebra $D(u_{q^+}^{\sigma})$ generated by $\sigma \cdot \sigma^{op} - \sigma^{op} \cdot \sigma$ and $K_{\lambda} \cdot K_{\lambda}^{op} - K_{\lambda}^{op} \cdot K_{\lambda}$ ($\lambda \in P_+$). It is clear that L is a Hopf-ideal. Put

$$u_q^{\sigma} = u_{q^+}^{\sigma} (E, \Pi, p) = D(u_{q^+}^{\sigma}) / L.$$

Put $u_{q^{n+}} = U_{q^{n+}}^{\sigma} / (I_{b_+}^{\sigma} \cap U_{q^{n+}}^{\sigma})$, $t = T / (I_{b_+}^{\sigma} \cap T)$.

Lemma 2.4. (1) As K -vector spaces,

$$u_q^\sigma \cong u_{q^{n_+}} \otimes t \otimes K\langle \sigma \rangle \otimes u_{q^{n_+}} (Xt\sigma^c Y^{op} \leftarrow X \otimes t \otimes \sigma^c \otimes Y). \\ (c=0,1)$$

(2) For $1 \leq i \leq N$, let $\gamma_i = \min\{\gamma \mid K_{\underline{\varepsilon}_i}^\gamma = 1\} \in \mathbb{Z}_+ \cup \{+\infty\}$. Then the elements $K_{\underline{\varepsilon}_1}^{\delta_1} \dots K_{\underline{\varepsilon}_N}^{\delta_N}$ ($0 \leq \delta_i < \gamma_i$) form a K -basis of t .

(3) Let u_q be an unital subalgebra of u_q^σ generated by the elements $\{E_i, \dot{F}_i = E_i^{op} \sigma^{p(i)} (1 \leq i \leq n), K_\lambda (\lambda \in P_+)\}$. Then there is a Hopf-superalgebra structure on u_q with coproduct $\dot{\Delta}$ defined by

$$\dot{\Delta}(K_\lambda) = K_\lambda \otimes K_\lambda, \dot{\Delta}(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i, \dot{\Delta}(\dot{F}_i) = \dot{F}_i \otimes K_{\alpha_i}^{-1} + 1 \otimes \dot{F}_i.$$

Theorem 2.5. Assume that q is an indeterminate and $K = C(q)$. Suppose that $(\alpha_i, \alpha_i) > 0$, $(\alpha_i, \alpha_i) \leq 0$ and $2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \in \mathbb{Z}$. Let $\underline{G}$ be the Kac-Moody Lie algebra defined for $(,) : \mathbb{E} \times \mathbb{E} \rightarrow K$ and Π . Then u_q is isomorphic to the Drinfeld-Jimbo quantized enveloping algebra $U_q(\underline{G})$.
(Jimbo type)

Theorem 2.6. Let $\underline{G}$ be the simple C-Lie algebra. Suppose that Π is the set of the simple roots of $\underline{G}$. Assume that $K = C$. Let ζ be an m -th root of unity such that $m \gg 1$. Then u_ζ is isomorphic to the Lusztig's primitive quantum group at root of unity $u_\zeta(\underline{G})$.

Theorem 2.5 can be immediately proved by Proposition 2.4.1 in [T]. Theorem 2.6 also seems to be well-known. For example, see [R].

§3. Root Systems of Simple Lie Superalgebras.

Let $\mathbb{G}$ be simple Lie superalgebras of types $A_{N-1}, B_N, C_N, D_N, F_4, G_3$.

Let $(\mathbb{E}, \Pi, p)$ be a triple related to a root system of $\mathbb{G}$. From now on, we only

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treat triples $(\mathbb{E}, \Pi, p)$ following Dynkin diagrams.

In the following diagrams, the element under i -th dot denotes the i -th simple root $\alpha_i \in \Pi$. The i -th dot $\times$ stands for $\circ$ (resp. $\otimes$) if $(\alpha_i, \alpha_i) \neq 0$ (resp. $= 0$). If

i -th dot is $\circ$, $\otimes$ or $\bullet$, then we define $p(\alpha_i) = 0, 0, 1$ respectively. We also

define a diagonal matrix $\mathbb{D} = (d_1, \dots, d_n)$ such that $A = \mathbb{D}^{-1}((\alpha_i, \alpha_j))$ is a Cartan matrix of $\mathbb{G}$.

$$(A_{N-1}) \quad \begin{array}{ccccccc} & 1 & & 2 & & & N-1 \\ \times & \text{---} & \times & \text{---} & \dots & \text{---} & \times \\ \underline{\epsilon}_1 - \underline{\epsilon}_2 & & \underline{\epsilon}_2 - \underline{\epsilon}_3 & & & & \underline{\epsilon}_{N-1} - \underline{\epsilon}_N \end{array}, \quad (\underline{\epsilon}_i, \underline{\epsilon}_i) = \pm 1, \quad \mathbb{D} = \text{diag}(1, \dots, 1),$$

$$(B_N) \quad \begin{array}{ccccccc} & 1 & & 2 & & & N-1 & N \\ \times & \text{---} & \times & \text{---} & \dots & \text{---} & \times \text{---} & \bullet \\ \underline{\epsilon}_1 - \underline{\epsilon}_2 & & \underline{\epsilon}_2 - \underline{\epsilon}_3 & & & & \underline{\epsilon}_{N-1} - \underline{\epsilon}_N & \underline{\epsilon}_N \end{array}, \quad \begin{array}{ccccccc} & 1 & & 2 & & & N-1 & N \\ \times & \text{---} & \times & \text{---} & \dots & \text{---} & \times \text{---} & \bullet \\ \underline{\epsilon}_1 - \underline{\epsilon}_2 & & \underline{\epsilon}_2 - \underline{\epsilon}_3 & & & & \underline{\epsilon}_{N-1} - \underline{\epsilon}_N & \underline{\epsilon}_N \end{array}, \quad (\underline{\epsilon}_i, \underline{\epsilon}_i) = \pm 1, \quad \mathbb{D} = \text{diag}(1, \dots, 1, 1/2),$$

$$(C_N) \quad \begin{array}{ccccccc} & 1 & & 2 & & & N-1 & N \\ \times & \text{---} & \times & \text{---} & \dots & \text{---} & \times \text{---} & \circ \\ \underline{\epsilon}_1 - \underline{\epsilon}_2 & & \underline{\epsilon}_2 - \underline{\epsilon}_3 & & & & \underline{\epsilon}_{N-1} - \underline{\epsilon}_N & 2\underline{\epsilon}_N \end{array}, \quad (\underline{\epsilon}_i, \underline{\epsilon}_i) = \pm 1, \quad \mathbb{D} = \text{diag}(1, \dots, 1, 2),$$

$$(D_N) \quad \begin{array}{ccccccc} & & & & N-1 & & \\ & 1 & & 2 & & & N-2 & \circ & \underline{\epsilon}_{N-1} - \underline{\epsilon}_N \\ \times & \text{---} & \times & \text{---} & \dots & \text{---} & \times & \text{---} & \circ & \underline{\epsilon}_N \\ \underline{\epsilon}_1 - \underline{\epsilon}_2 & & \underline{\epsilon}_2 - \underline{\epsilon}_3 & & & & \underline{\epsilon}_{N-1} - \underline{\epsilon}_N & \underline{\epsilon}_{N-1} + \underline{\epsilon}_N & & \end{array}, \quad \begin{array}{ccccccc} & & & & N-1 & & \\ & 1 & & 2 & & & N-2 & \otimes & \underline{\epsilon}_{N-1} - \underline{\epsilon}_N \\ \times & \text{---} & \times & \text{---} & \dots & \text{---} & \times & \text{---} & \otimes & \underline{\epsilon}_N \\ \underline{\epsilon}_1 - \underline{\epsilon}_2 & & \underline{\epsilon}_2 - \underline{\epsilon}_3 & & & & \underline{\epsilon}_{N-1} - \underline{\epsilon}_N & \underline{\epsilon}_{N-1} + \underline{\epsilon}_N & & \end{array}, \quad (\underline{\epsilon}_i, \underline{\epsilon}_i) = \pm 1, \quad \mathbb{D} = \text{diag}(1, \dots, 1, 1),$$

$$(F_4) \quad \begin{array}{ccccccc} & 1 & & 4 & & 3 & & 2 \\ \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \otimes \\ \underline{\epsilon}_2 - \underline{\epsilon}_3 & & \underline{\epsilon}_3 - \underline{\epsilon}_4 & & \underline{\epsilon}_1 & & (\underline{\epsilon}_1 - \underline{\epsilon}_2 - \underline{\epsilon}_3 - \underline{\epsilon}_4)/2 \end{array}$$

$$(\underline{\epsilon}_1, \underline{\epsilon}_1) = 6, (\underline{\epsilon}_2, \underline{\epsilon}_2) = (\underline{\epsilon}_3, \underline{\epsilon}_3) = (\underline{\epsilon}_4, \underline{\epsilon}_4) = -2, \quad \mathbb{D} = \text{diag}(2, 1, 1, 2),$$

$$(G_3) \quad \begin{array}{ccccc} & 1 & & 3 & & 2 \\ & \otimes & \text{---} & \text{---} & \text{O} < \equiv \equiv \equiv & \text{O} \end{array},$$

$$(\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_1) = -2, (\underline{\mathbf{e}}_2, \underline{\mathbf{e}}_2) = 2, (\underline{\mathbf{e}}_3, \underline{\mathbf{e}}_3) = -6, \mathbb{D} = \text{diag}(1, 3, 1).$$

§4. Defining relations of $u_q^{\sigma}(\mathbb{E}. \Pi. p)$ of Simple Lie Superalgebras $\mathbb{G}$.

Here we give defining relations of u_{q,n_+} of $u_q^{\sigma}(\mathbb{E}.II.p)$ (see Lemma 2.4) when q is not a root of unity.

Put $P_+ = \mathbf{Z}\alpha_1 \oplus \dots \oplus \mathbf{Z}\alpha_N$. We extend p to $p : P_+ \rightarrow \mathbf{Z}/2\mathbf{Z}$ additively.

For $\delta = m_1\alpha_1 + \dots + m_N\alpha_N \in P_+$, let $(u_{q^+n_+})_\delta$ be a K -subspace of $u_{q^+n_+}$ spanned by elements $E_{i_1}E_{i_2}\dots E_{i_p}$ ($\#\{i_a = i\} = m_i$). Then we have $u_{q^+n_+} = \bigoplus_{\delta \in P_+} (u_{q^+n_+})_\delta$.

For $\delta, v \in P_+$ and $X_\delta \in (u_{q^+})_\delta$, $X_v \in (u_{q^+})_v$, put

$$\text{ad}_{[1,1]} X_\delta(X_\nu) = [X_\delta, X_\nu] = X_\delta X_\nu - (-1)^{p(\delta)p(\nu)} q^{-(\delta, \nu)} X_\nu X_\delta.$$

Theorem 4.1. Let (E, Π, p) be a triple introduced in §3. Assume that q is not a root of unity. Let $u_{q^{n+}}$ be of $u_q^\sigma(E, \Pi, p)$ (see Lemma 2.4). Then, as K -algebra, $u_{q^{n+}}$ is defined with the generators E_i ($1 \leq i \leq n$) and the relations:

$$(r1) \quad [E_i, E_j] = 0 \text{ if } (\alpha_i, \alpha_j) = 0,$$

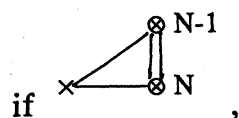
$$(r2) \quad (\text{ad}_{t_i, j} E_i)^{m_{ij}}(E_j) = 0 \text{ if } (\alpha_i, \alpha_i) \neq 0 \text{ and } m_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \in \mathbb{Z},$$

$$(r3) \quad (\text{ad}_{E_{N-1}} E_N)^3(E_{N-1}) = 0 \text{ if } \begin{matrix} N-1 & N \\ \times & \longrightarrow \bullet \end{matrix},$$

$$(r4) \quad [[[E_i, E_j], E_k], E_l] = 0$$

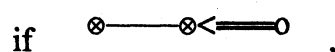
$$\begin{array}{ccccc} i & j & k & i & j & k & i & j & k \\ \text{if } \times \text{---} \otimes \text{---} \times, & \times \text{---} \otimes \text{---} \text{---} > 0 & \text{or} & \times \text{---} \otimes \text{---} \text{---} > \bullet \end{array}$$

$$(r5) \quad [[E_{N-2}, E_{N-1}], E_N] = [[E_{N-2}, E_N], E_{N-1}]$$



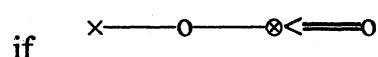
$$(r6) \quad [[[E_{N-2}, E_{N-1}], [E_{N-2}, E_{N-1}], E_N], E_{N-1}] = 0$$

N-2 N-1 N



$$(r7) \quad [[[[[[E_{N-3}, E_{N-2}], E_{N-1}], E_N], E_{N-1}], E_{N-2}], E_{N-1}] = 0$$

N-3 N-2 N-1 N



§5. Root vectors of $u_q^{\sigma}(\mathbb{E}, \Pi, p)$ of Simple Lie Superalgebras $\mathbb{G}$.

Here we assume that there is $m \gg 1$ satisfying $q^{\underline{m}} \neq 1$ for $1 \leq \underline{m} \leq m$. Assume that $(\mathbb{E}, \Pi, p)$ is the triple in §3. Let Φ be the set of roots of $\mathbb{G}$ and Φ_+ the set of positive roots with respect to Π . Let Φ_+^{red} be the set of positive roots defined by

$\Phi_+^{\text{red}} = \{\beta \in \Phi_+ \mid \beta/2 \notin \Phi_+\}$. For $\beta = c_1\alpha_1 + \dots + c_N\alpha_N \in P_+$, put $\text{ht}(\beta) = c_1 + \dots + c_N$, $g(\beta) = \min\{i \mid i \neq 0\}$ and $c_\beta = c_{g(\beta)}$.

Define a half integer $\underline{\text{ht}}(\beta)$ by $\underline{\text{ht}}(\beta) = \text{ht}(\beta)/c_\beta$. For $\alpha, \beta \in P_+$, we say that $\alpha < \beta$ if they satisfy one of the following $\in \frac{1}{2}\mathbb{Z}$

- (1) $g(\alpha) < g(\beta)$,
- (2) $g(\alpha) = g(\beta)$ and $\underline{\text{ht}}(\alpha) < \underline{\text{ht}}(\beta)$,
- (3) Π is of type D_N , $p(\underline{\epsilon}_i - \underline{\epsilon}_N) = 0$ and $\alpha = \underline{\epsilon}_i - \underline{\epsilon}_N$, $\beta = 2\underline{\epsilon}_i$ or

$$\alpha = 2\varepsilon_i, \beta = \varepsilon_i + \varepsilon_N \text{ or } \alpha = \varepsilon_i - \varepsilon_N, \beta = \varepsilon_i + \varepsilon_N.$$

We define q -root vectors E_β ($\beta \in \Phi_+^{\text{red}}$) of u_{q,n_+} of $u_q^\sigma(\mathbb{E}, \Pi, p)$ as follows.

Definition 5.1. For $\beta \in \Phi_+^{\text{red}}$, we define the element $E_\beta \in u_{q,n_+}$ as follows. (For type F_4 , (resp. G_3), we write E_{abcd} and $\dot{E}_{abcd}$ (resp. E_{abc} and $\dot{E}_{abc}$) for $E_{\alpha_1 + b\alpha_4 + c\alpha_3 + d\alpha_2}$ and $\dot{E}_{\alpha_1 + b\alpha_4 + c\alpha_3 + d\alpha_2}$ (resp. $E_{\alpha_1 + b\alpha_3 + c\alpha_2}$ and $\dot{E}_{\alpha_1 + b\alpha_3 + c\alpha_2}$).

(1) We put $E_{\alpha_i} = E_i$ ($1 \leq i \leq n$).

(2) Let $\alpha \in \Phi_+^{\text{red}}$ and $1 \leq i \leq n$ be such that $g(\beta) < i$ and $\alpha + \alpha_i \in \Phi$. Put $\dot{E}_{\alpha+\alpha_i} = [\dot{E}_\alpha, E_i]$. If Π is of type B_N , $i = N$ and $\alpha = \varepsilon_j$ ($1 \leq j \leq N-1$), let $E_{\alpha+\alpha_N} = (q^{1/2} + q^{-1/2})^{-1} \dot{E}_{\alpha+\alpha_N}$. If Π is of type D_N , $i = N$ and $\alpha = \alpha_{N-1}$, let $E_{\alpha+\alpha_N} = (q + q^{-1})^{-1} \dot{E}_{\alpha+\alpha_N}$. If Π is of type F_4 , let $E_{1120} = (q + q^{-1})^{-1} \dot{E}_{1120}$ and $E_{1232} = (q^2 + 1 + q^{-2})^{-1} \dot{E}_{1232}$. If Π is of type G_3 , let $E_{121} = (q + q^{-1})^{-1} \dot{E}_{121}$, $E_{021} = (q + q^{-1})^{-1} \dot{E}_{021}$ and $E_{031} = (q^2 + 1 + q^{-2})^{-1} \dot{E}_{031}$. Otherwise, put $E_{\alpha+\alpha_i} = \dot{E}_{\alpha+\alpha_i}$.

(3) Let $\alpha, \beta \in \Phi_+^{\text{red}}$ such that $g(\alpha) = g(\beta)$, $\alpha < \beta$, $\text{ht}(\beta) - \text{ht}(\alpha) \leq 1$ and $\alpha + \beta \in \Phi_+^{\text{red}}$. Put $\dot{E}_{\alpha+\beta} = [\dot{E}_\alpha, \dot{E}_\beta]$. If Π is of type C_N (resp. D_N, F_4 or G_3), then $E_{\alpha+\beta}$ is defined by $(q + q^{-1})^{-1} \dot{E}_{\alpha+\beta}$ (resp. $(q + q^{-1})^{-1} \dot{E}_{\alpha+\beta}$, $(q^2 + q^{-2})^{-1} \dot{E}_{\alpha+\beta}$ or $(q^2 + 1 + q^{-2})^{-1} \dot{E}_{\alpha+\beta}$).

By using similar computations in [Y2], we have

Proposition 5.2. (1) As a K -vector space, $u_q n_+$ is spanned by the elements

$$\prod_{\alpha \in \Phi_+^{\text{red}}} E_\alpha^{n_\alpha} \quad (n_\alpha \in \mathbb{Z}_+ \text{ if } (\alpha, \alpha) \neq 0, n_\alpha = 0, 1 \text{ if } (\alpha, \alpha) = 0).$$

Here $\prod_{\alpha \in \Phi_+^{\text{red}}} E_\alpha^{n_\alpha}$ denote a product taken with a total order on Φ_+^{red}

compatible with the partial order $<$.

(2)

$$\begin{aligned} & \prod_{\alpha \in \Phi_+^{\text{red}}} E_\alpha^{n_\alpha} \prod_{\alpha \in \Phi_+^{\text{red}}} E_\alpha^{m_\alpha} \\ &= \prod_{\alpha \in \Phi_+^{\text{red}}} \delta_{n_\alpha m_\alpha} \psi(n_\alpha; (-1)^{p(\alpha)} q^{(\alpha, \alpha)}) \langle E_\alpha, E_\alpha^{n_\alpha} \rangle. \end{aligned}$$

Here $\psi(n; t) = \prod_{1 \leq i \leq n} \{(t^i - 1)/(t - 1)\}$.

§6. Poincaré-Birkhoff-Witt type Theorem $\overset{\text{of}}{\vee} u_q^{\sigma(\mathbb{E}, \Pi, p)}$ of Simple Lie Superalgebras $\mathbb{G}$.

Define $d_\alpha \in (1/2)\mathbb{Z}_+$ by $d_\alpha = |(\alpha, \alpha)|/2$ if $(\alpha, \alpha) \neq 0$, $d_\alpha = 2$ if Π is of type G_3 and $\alpha = \alpha_1 + 2\alpha_3 + c\alpha_2$, $d_\alpha = 1$ otherwise.
For $\alpha = c_1\alpha_1 + \dots + c_N\alpha_N \in P_+$, put

$$b(\alpha) = (q^{d\alpha} - q^{-d\alpha}) \langle E_\alpha, E_\alpha \rangle / \prod_{1 \leq i \leq n} (q^{d_i} - q^{-d_i})^{c_i} \quad \text{and}$$

$$\gamma_\alpha = \min\{\gamma \mid \psi(\gamma; (-1)^{p(\alpha)} q^{(\alpha, \alpha)}) = 0\} \in \mathbb{Z}_+ \cup \{+\infty\}.$$

Lemma 6.1. $b(\alpha)$ can be written as $(-1)^a q^b$ for some $a, b \in \mathbb{Z}_+$. (For the precise value of $b(\alpha)$, see [Y2; Lemma 10.3.1]).

By Proposition 5.2 and Lemma 6.1, we have:

Theorem 6.2. (*PBW-type theorem*) The elements

$$\prod_{\alpha \in \Phi_+^{\text{red}}} E_\alpha^{\delta_\alpha} \quad (0 \leq \delta_\alpha < \gamma_\alpha)$$

form a $\mathbb{K}$ -basis of $u_{\mathbb{q}} n_+$.

Proposition 6.3. Let $m > 10$ and ζ a primitive m -th root of unity. Then, as $\mathbb{K}$ -algebra, $u_\zeta n_+$ is defined with the generators E_i ($1 \leq i \leq n$) and the relations (r1-7) in Theorem 4.1 and relations

$$(rr1) \quad E_\alpha^{\gamma_\alpha} = 0 \quad (\alpha \in \Phi_+^{\text{red}}).$$

§7. Universal R -matrix $\overset{\text{of}}{\underset{\vee}{u}}_\zeta^\sigma(\mathbb{E}, \Pi, p)$ of Simple Lie Superalgebra $\mathbb{G}$.

Keep notation in §3-6. For $\alpha = c_1 \alpha_1 + \dots + c_N \alpha_N \in P_+$, put

$$F_\alpha = (\prod_{1 \leq i \leq n} (q^{-d_i} - q^{d_i})^{c_i})^{-1} (E_\alpha)^{\text{op}_{\sigma^p(\alpha)}} \quad (\text{see Lemma 4.2}) \quad \text{and}$$

$$u(\alpha) = (-1)^{\text{ht}(\alpha)} / b(\alpha).$$

Theorem 7.1. (*Universal R -matrix of* u_ζ^σ) Keep notation in Proposition 6.3.

$\epsilon \cdot u_{\zeta}^{\sigma} \otimes u_{\zeta}^{\sigma}$
 The Universal R-matrix R of $u_{\zeta}^{\sigma} = u_{\zeta}^{\sigma}(\mathbb{E}, \Pi, p)$ is given by

$$R = \left\{ \prod_{\alpha \in \Phi_+^{\text{red}}} \left(\sum_{0 \leq \delta_{\alpha} < \gamma_{\alpha}} \frac{((q^{d\alpha} - q^{-d\alpha})u(\alpha)E_{\alpha} \otimes F_{\alpha} \sigma^{p(\alpha)})^{\delta_{\alpha}}}{\psi(n_{\alpha}; (-1)^{p(\alpha)} q^{(\alpha, \alpha)})} \right) \right\}$$

$$\cdot \left\{ \frac{1}{2} \sum_{0 \leq c, d \leq 1} (-1)^{cd} \sigma^c \otimes \sigma^d \right\} \cdot \prod_{1 \leq i \leq N} \left\{ \left(\frac{1}{\gamma_i} \sum_{0 \leq \delta_i, \phi_i < \gamma_i} \zeta^{-(\underline{\epsilon}_i, \underline{\epsilon}_i) \delta_i \phi_i} K_{\underline{\epsilon}_i}^{\delta_i} \otimes K_{\underline{\epsilon}_i}^{\phi_i} \right) \right\}$$

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