

Cournot Platform Competition with Mixed-Homing*

Mark J. Tremblay

University of Nevada, Las Vegas
CESifo Research Network
tremblmj@miamioh.edu

Takanori Adachi

Graduate School of Management and
Graduate School of Economics, Kyoto University
adachi.takanori.8m@kyoto-u.ac.jp

Susumu Sato

Institute of Economic Research
Hitotsubashi University
susumusato.econ@gmail.com

July 22, 2023

Abstract

Competition in quantity dates back to Cournot (1838) for traditional markets and Katz and Shapiro (1985) for markets with direct network effects. In this paper, we consider Cournot platform competition in two-sided markets with indirect network effects while allowing for single-, multi-, and mixed-homing allocations. We find that the markup and markdown terms that are typically found in monopoly two-sided pricing are distorted toward zero when platform competition intensifies. We also generalize the monopoly platform Lerner indices from Armstrong (2006) and Weyl (2010) to include competition and mixed-homing allocations. Lastly, we show that welfare decreases in the number of platforms for the most commonly considered homing allocations, highlighting how the welfare loss from breaking up a network across smaller platforms can outweigh the welfare benefits from lower aggregate prices in a setting where platforms are homogenous.

Keywords: Two-sided markets, network externality, Lerner index, single-homing, multi-homing.

JEL Classifications: D40, L10, L20, L40.

*We are grateful to Julian Wright and three anonymous referees for helpful comments. We also thank Dongyu Guo, Makoto Hanazono, and Yusuke Zenryo for insightful suggestions on an earlier version of the paper. This is a revised version of the paper that was circulated under the title of “Cournot, Conduct, and Homing in Two-Sided Markets.” Adachi acknowledges a Fund for the Promotion of Joint International Research (16KK0054) and a Grant-in-Aid for Scientific Research (C) (18K01567; 21K01440) from the Japan Society for the Promotion of Science. Sato acknowledges a Grant-in-Aid for Research Activity Startup (20K22117) from the Japan Society for the Promotion of Science. Any remaining errors are ours.

1 Introduction

The study of competition in output was first pioneered by Cournot (1838) for traditional markets and then by Katz and Shapiro (1985) for markets with direct network effects. This paper aims to contribute to the literature by considering output competition in two-sided markets with *indirect* network effects. In practice, there are a number of platform industries where capacity constraints exist on both sides of the market. For example, both video game and smartphone industries may face capacity constraints on hardware (the consumer side) and platforms in both industries have imposed quality standards on the developer side. Similarly, newspapers are constrained in the number of papers they print and the number of advertisements within an issue. Thus, the consideration for how the choice of output impacts platform competition is worth further investigation.

To do so, we propose a model of Cournot platform competition with mixed-homing allocations where single- and multi-homing agents are simultaneously present on each side of the market. This approach allows us to consider a variety of homing allocations that are found in the two-sided market literature and observed in industry. To allow for these general homing structures, we develop market clearing conditions under symmetric competition amongst N platforms (much like the previous literature on Cournot competition in traditional markets and for markets with direct network effects).

In terms of platform pricing, we find that platform competition reduces the markup to consumers as well as the markdown to sellers under the competitive bottleneck allocation (the commonly analyzed allocation where all consumers single-home and all sellers multi-home). In the limit, this implies that greater platform competition results in the classic cross-subsidization pricing strategy where consumers (sellers) are subsidized (face markups) since the markup (markdown) term goes to zero.¹ This logic, that price distortions depend on homing decisions and platform competition, generalizes to any feasible homing allocation.

¹This pricing result coincides with optimal pricing in competitive bottleneck models where the single-homing side of the market faces a more competitive price than the multi-homing side (see Armstrong (2006) and Belleflamme and Peitz (2019) for details).

With mixed-homing on both sides of the market, greater platform competition results in both platform prices converging to their respective marginal costs—a result that is consistent with the traditional Cournot model. Comparing the mixed-homing and competitive bottleneck results we see that the assumption of an entire side multi-homing generates a discontinuity in the optimal prices which suggests that caution should be taken when considering such an assumption.

In terms of welfare, the tradeoff from platform entry is clear: As shown by Tan and Zhou (2021), more platforms reduce prices but also break up network effects and this can reduce welfare. We find that welfare under Cournot platform competition is decreasing in the number of platforms under the competitive bottleneck. While this result is consistent across a variety of homing specifications (unlike Tan and Zhou (2021) who assume all agents single-home), it is important to note that we assume homogenous platform competition. With platform differentiation, welfare can increase with platform entry as in Teh et al. (2023).

While two-sided markets have received considerable attention (Rochet and Tirole (2003), Caillaud and Jullien (2003), Armstrong (2006), Hagiu (2006) and Weyl (2010)), only a few have elements from Cournot competition or allow for flexible homing allocations. First, considering the platform literature on homing, note that several recent papers have tried to generalize away from the assumption that all agents on the same side of the market either single-home or multi-home (a common assumption in models of platform competition).

Earlier studies of platform competition with more general homing allocations include Armstrong and Wright (2007). They endogenously determine when the competitive bottleneck occurs in equilibrium where consumers single-home and pay a fee below marginal cost while sellers multi-home and pay a markup. We generalize this result by considering the effects that homing have on the markup and markdown terms found in two-sided market pricing. In contrast, a more recent study by Teh et al. (2023) considers a model where platforms compete in transaction fees rather than membership fees. Our study differs from theirs in the pricing (membership versus transaction), degree of platform differentiation, and

the homing structures. Importantly, they allow for mixed-homing by consumers who view platforms as differentiated, but all sellers multi-home in equilibrium.

Others have considered mixed-homing explicitly. For example, Jeitschko and Tremblay (2020) show how straddle pricing, which avoids marginal cost pricing by homogenous platforms, can support mixed-homing allocations. Another paper that allows for mixed-homing on both sides is Bakos and Halaburda (2020) who study platform competition *à la* Hotelling (1929) (see, e.g., Doganoglu and Wright (2006), Bryan and Gans (2019), and Belleflamme and Peitz (2019) for additional studies on mixed-homing by one side of the market). They show that the effects of cross-subsidization diminish when multi-homing takes place on both sides of the market. Lastly, Ivaldi and Zhang (2021) highlight how the existence of multi-homing impacts platform mergers by simulating hypothetical media platform mergers when multi-homing by advertising is or is not allowed. These papers highlight the importance of a more general homing structure when considering platform competition.

Turning to the literature on two-sided markets with elements from Cournot competition, the earliest work is by Schiff (2003) who shows that a Cournot duopoly with compatibility (multi-homing agents) generates more welfare than a monopoly platform which generates more welfare than a Cournot duopoly without compatibility (single-homing by all). This is one of the earliest works to demonstrate how a monopoly platform can generate greater welfare than a more competitive environment. Since then, only a few other studies, Gabszewicz and Wauthy (2014) and Correia-da Silva et al. (2019), consider Cournot platform competition and those papers are the closest to ours in several respects.²

We overlap with Gabszewicz and Wauthy (2014) in that both models have heterogeneous network effects (some consumers benefit from video games more than others while some video game developers are more profitable than others), platforms are ex-ante homogeneous, and

²Reisinger et al. (2009), Alexandrov et al. (2011), Anderson et al. (2012) and Anderson et al. (2019) consider N platforms using a Salop circle approach. However, these models consider media or search platforms where indirect network externalities do not exist between the two sides of the market. In contrast, Baranes et al. (2019) consider how four platforms on the Salop (1979) circle compete, or merge, when network externalities are present on both sides; however, they assume that all agents single-home on both sides of the market.

platforms compete *à la* Cournot. However, they assume that agents single-home and that only one of the two platforms can have the most participation on both sides of the market.³ This results in asymmetric platforms ex-post with the larger platform charging higher prices. Instead, our approach specifies an alternative criteria that allows for mixed-homing on each side of the market and generates symmetric platforms ex-post.

Taking a different approach, Correia-da Silva et al. (2019) assume that network benefits are homogeneous on each side of the market and that the Cournot competing platforms have different marginal costs (we assume that marginal costs are the same across platforms). The authors take advantage of their ability to consider asymmetric marginal costs and construct an average-marginal-cost-preserving merger analysis. They find that if the pre-merger externality-adjusted prices are above average marginal costs on both sides of the market, then the merger harms agents on both sides.⁴ Our equilibrium criteria is similar to Correia-da Silva et al. (2019) in that we both have a marginal agent on each side of the market that is indifferent between all platforms present in the market; however, the homogeneous nature of agents on each side of the market in Correia-da Silva et al. (2019) prevents the consideration of multi- or mixed-homing.

Combined, Gabszewicz and Wauthy (2014), Correia-da Silva et al. (2019), and our work highlight the difficulty in modeling Cournot platform competition with all the elements one would desire (endogenous homing, asymmetric costs for merger analysis, and (a)symmetric equilibria). Our contribution is to allow for mixed-homing allocations in Cournot platform competition and this allows us to generalize the monopoly platform Lerner index and make direct comparisons to the previous work by Cournot (1838) and Katz and Shapiro (1985).

³Belleflamme et al. (2020) also consider competition between two asymmetric platforms with all agents single-homing.

⁴In a different vein, Sato (2021) uses an aggregative-games approach to study a merger of multi-product firms in the presence of direct and indirect network externalities but he also assumes that all agents single-home.

2 A Refresher on Traditional Markets

Consider Cournot competition in a traditional market. Suppose that there are N competing firms, denoted by $X = 1, 2, \dots, N$, that sell homogeneous goods. Each firm chooses their output q^X for $X = 1, 2, \dots, N$. Inverse demand for the product is given by: $p = p(Q)$, where p is the price, $Q = \sum q^X$ is the total quantity, $p(Q)$ is the downward sloping inverse demand function, and $\epsilon = \frac{\partial Q}{\partial p} \cdot \frac{p}{Q} < 0$ is the price elasticity of demand. Each firm has profit given by $\pi^X = [p(Q) - c] \cdot q^X$, where c denotes the marginal cost. Maximizing profit with respect to firm X 's output yields a first-order condition given by $\frac{\partial \pi^X}{\partial q^X} = p'(Q) \cdot q^X + [p(Q) - c] = 0$. Symmetry implies that $q^X = q = \frac{Q}{N}$ for all $X = 1, 2, \dots, N$, and so we have

$$L := \frac{p - c}{p} = \frac{1}{N} \cdot \frac{1}{-\epsilon} \text{ or } p = c + \frac{1}{N} \cdot \frac{p}{-\epsilon}, \quad (1)$$

where L denotes the Lerner index.

3 A Two-Sided Market

In a two-sided market, two groups of agents (consumers and sellers)⁵ benefit from indirect network externalities that exist across the two groups. Let the consumer side be denoted as side 1 and the seller side as side 2, and suppose that there are N (symmetric) competing platforms.⁶

3.1 Consumers and Sellers

First consider the consumer side of the market. Consumers benefit from interaction with the seller side of the market, and some consumers benefit more from sellers than others.

⁵Depending on the industry, a seller can be a content provider, game developer, book publisher, etc.

⁶Unlike the traditional Cournot model which remains tractable under cost asymmetries, allowing for cost asymmetries in the Cournot two-sided model that we consider requires restrictive assumptions on agent homing. Adachi et al. (2023) consider a different approach that focuses on necessary cost synergies for platform mergers to improve consumer surplus and they find that all sellers multi-home and consumers single- or multi-home depending on their preferences.

For example, some consumers benefit more from apps (teens) than others (parents).⁷ Let consumer types be denoted by $\tau_1 \in [0, \bar{\tau}_1]$ and be distributed uniformly, and suppose that consumers with lower τ_1 types have greater network benefits than consumers with higher τ_1 types. Specifically, the utility from joining platform X , $X = 1, 2, \dots, N$, exclusively for a consumer of type τ_1 is given by:

$$u_1^X(\tau_1) = \alpha_1(\tau_1) \cdot q_2^X - p_1^X, \quad (2)$$

where q_2^X denotes the number of sellers on platform X , p_1^X denotes the consumer price of platform X , and $\alpha_1(\cdot)$ is a decreasing function that denotes the network externality received by consumers.⁸

Now consider the seller side of the market. Sellers benefit from greater consumer participation on a platform. Like consumers, sellers are heterogeneous in their network gains. That is, we allow some sellers to be more successful than others (e.g., game developers differ in terms of sales and profitability). Let seller types be denoted by $\tau_2 \in [0, \bar{\tau}_2]$ and let sellers be distributed uniformly. The utility from joining platform X , $X = 1, 2, \dots, N$, exclusively for a seller of type τ_2 is given by:

$$u_2^X(\tau_2) = \alpha_2(\tau_2) \cdot q_1^X - p_2^X, \quad (3)$$

where q_1^X denotes the number of consumers on platform X , p_2^X denotes the seller price of platform X , and $\alpha_2(\cdot)$ is a decreasing function that denotes the network externality received by sellers.⁹

Consumers and sellers might have the option to multi-home (join more than one plat-

⁷Similar arguments can be made for consumer heterogeneity in game content for the gaming industry, in video content for the video streaming industry, etc.

⁸Furthermore, suppose that $\alpha_1(\cdot)$ is twice continuously differentiable. Capturing heterogenous network effects through the $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ functions is isomorphic to having each consumer (seller) draw their α_1 (α_2) from some distribution. This is done by Gabszewicz and Wauthy (2014) who use a uniform distribution which corresponds to the case of $\alpha_i(\tau_i) = 1 - \tau_i$ in our framework.

⁹Furthermore, suppose that $\alpha_2(\cdot)$ is twice continuously differentiable.

form). There are several issues that arise when modeling multi-homing. The most important, and most difficult, is how to deal with duplicated interactions between the same buyer and the same seller across multiple platforms.¹⁰ For example, if a video game is available on the Xbox, then a multi-homing gamer might not benefit if that game is also available on the Playstation but the gamer may benefit if the same game is on the Nintendo Switch because the Switch is portable. Similarly, some advertisers might benefit from the same consumer seeing their ad on Facebook and Instagram while other advertisers may not. To allow for the coexistence of single-homing and multi-homing agents, we assume that multi-homing agents do not benefit from duplication so that an agent on side i that multi-homes on a set of platforms, $\mathcal{X} \subset \{1, 2, \dots, N\}$, earns utility

$$u_i^{\mathcal{X}}(\tau_i) = \alpha_i(\tau_i) \cdot \tau_j^{\mathcal{X}} - \sum_{X \in \mathcal{X}} p_i^X,$$

where $\tau_j^{\mathcal{X}}$ denotes the unique number of agents on side j that participate across the $\mathcal{X}$ platforms.¹¹

In addition to this multi-homing utility specification, we detail further structure on agent homing and equilibrium selection in Subsections 3.3 and 3.4. Notably, we assume that homing on each side of the market is exogenously given by two parameters that capture the amount of single-homing on each side of the market. This allows us to consider all mixed homing allocations in isolation, but it also implies that homing allocations are determined independent of other primitives such as N .¹² We also assume that, for simplicity, a multi-

¹⁰Bakos and Halaburda (2020) arguably do the best job of investigating this issue.

¹¹This is known as “no double counting” in Bakos and Halaburda (2020) and is the most common assumption in the limited literature on this issue.

¹²In many ways, the limitations of our approach stem from this issue of how to deal with duplicated interactions between consumers and sellers that multi-home across platforms. From a micro-founded perspective, the most desirable approach would be to develop a model with duplication parameters on each side of the market that endogenously determine the single-homing parameters that we exogenously define. With such an approach one would expect that larger (smaller) benefits from duplication would make multi-homing more (less) attractive. In this case a duplication parameter on side i that is close to one would correspond to a ϕ_i close to one. Unfortunately existing studies by Bakos and Halaburda (2020) and Jeitschko and Tremblay (2020) highlight how such an approach will produce multiple equilibrium allocations since the homing decisions on one side of the market still impact the desirability of multi-homing for the other side.

homing agent on side i joins all platforms whenever possible (i.e., as long as the lowest q_i^X is not filled up): this simplifies the rationing rule provided in Subsection 3.3.¹³

3.2 The Platforms

Each platform X , $X = 1, 2, \dots, N$, faces a marginal cost for an additional side i agent that is given by $c_i \geq 0$, and so platform profits are expressed as:

$$\Pi^X = [p_1^X - c_1] \cdot q_1^X + [p_2^X - c_2] \cdot q_2^X. \quad (4)$$

3.3 Platform Strategies and Market Clearing Conditions

Much like in traditional Cournot competition, we assume that each platform commits to output choices for both sides of the market, taking the output choices of their competitors as given. More specifically, the game proceeds as follows: First, each platform X chooses q_1^X and q_2^X to maximize profits, taking the q_1^Y and q_2^Y for $Y \neq X$ as given. These output choices act as binding capacity constraints that are observed by all agents on both sides. Second, each agent on side i forms expectations on the prices, p_i^X , and chooses their homing pattern in a way that is described below. Lastly, the market clearing prices p_i^X are realized in a manner that is consistent with agents' expectations (as well as their homing decisions) and the output decisions by each platform.

For this to occur, we impose the following conditions:

1. Given the quantity selections by all platforms, the side 1 and side 2 prices for each platform are such that the marginal participating agent on each side of the market earns zero utility from every platform.¹⁴

For this reason, we move forward under these assumptions and with the development of agent homing and equilibrium selection in Subsections 3.3 and 3.4.

¹³Alternatively, we can replace this assumption with the following rationing rule: the agents who chose to multi-homing on a strict subset of platforms are never allocated to platforms.

¹⁴This condition largely drives ex-post symmetry across platforms. An alternative approach, as considered in Gabszewicz and Wauthy (2014), is to allow beliefs to drive an asymmetric outcome where the larger

2. Given the quantity selections and market clearing prices, each unit of output on each side of the market is distributed to single-homing and multi-homing agents based on the following rationing rule: (i) given that the lowest output on side i is denoted by $q_i^L = \min\{q_i^1, \dots, q_i^N\}$, an amount of $(1 - \phi_i)q_i^L$ is allocated to multi-homing agents for all the platforms, where $\phi_i \in [0, 1]$ denotes the extent of single-homing activity, and (ii) the remaining output is allocated randomly to single-homing agents.¹⁵

It is helpful to breakdown each of these conditions. For Condition 1, note that the marginal participating agent will always be an agent that single-homes. That is, from equations (2) and (3) and since $u_i^{\mathcal{X}} \leq \sum_{X \in \mathcal{X}} u_i^X$ for $\mathcal{X} \subset \{1, 2, \dots, N\}$ and for $i = 1, 2$, we see that for every τ'_i such that $u_i^{\mathcal{X}}(\tau'_i) \geq 0$ we have that $u_i^X(\tau'_i) \geq 0$ for some $X \in \mathcal{X}$. This implies the marginal participating agents will be given by setting equations (2) and (3) equal to zero.¹⁶ More specifically, let τ_i^* be the marginal participating agent on side i , where τ_i^* is given by

$$u_i^X(\tau_i^*) = 0 \text{ or } p_i^X = \alpha_i(\tau_i^*) \cdot q_j^X, \quad (5)$$

for all $X = 1, \dots, N$. In this case, the $\tau_i \in [0, \tau_i^*]$ are interested in participating in the market and the τ_i^* agent earns zero utility from every platform available. Lastly, Condition 1 requires that the entire market output on side i , denoted by $Q_i = \sum_{X=1}^N q_i^X$, be sold to the $\tau_i \in [0, \tau_i^*]$ participating agents. Naturally, homing allocations will play an important role in establishing the relationship between τ_i^* and Q_i (e.g., if all agents on side i single-home,

platform has higher prices. For example, if there are two platforms and all agents single-home, then $q_2^A > q_2^B$ would generate side 1 prices so that $\alpha_1(\tau_1^A) \cdot q_2^A = p_1^A > p_1^B = \alpha_1(\tau_1^B) \cdot q_2^B$ and the consumers with $\tau_1 \in [0, \tau_1^A]$ ($\tau_1 \in (\tau_1^A, \tau_1^B]$) would join platform A (platform B); from here, the critical τ_1^A and τ_1^B capture participation on side 1 and generate the corresponding q_1^A and q_1^B that are consistent with the original $q_2^A > q_2^B$. However, the q_1^A and q_1^B corresponding with the $q_2^A > q_2^B$ may not align with the q_1^A and q_1^B chosen by the platform and so such a market clearing approach does not allow for arbitrary output choices. Gabszewicz and Wauthy (2014) avoid this issue by assuming that only one of the platforms can be larger than the other, instead of that the marginal agent is indifferent between all platforms, and this results in an asymmetric platform equilibrium.

¹⁵ Maintaining Condition 1, but instead using a Pareto allocation where the low τ_i s get the platform with the largest q_j (which maximizes agent surplus) is another output distribution specification that clears the market. In addition, such a Pareto allocation of outputs would eliminate a black market that might be brought on by randomization.

¹⁶ We assume that agents outside option from not joining a platform is valued at zero.

then $\tau_i^* = Q_i$), and we derive this relationship explicitly in the next subsection.

Given the market clearing prices given by equation (5), Condition 2 implies that outputs are first distributed to some fraction of multi-homing agents and then distributed randomly to single-homing agents. An assumption on output allocation, not uniform randomization specifically, is necessary since some platforms might be more desirable than others. For example, if there are two platforms (A and B) and Platform A has lots of games relative to Platform B ($q_2^A > q_2^B$), then market clearing prices will imply that $p_1^A = \alpha_1(\tau_1^*)q_2^A > \alpha_1(\tau_1^*)q_2^B = p_2^B$; in this case, the marginal consumer earns zero utility from either platform but all other participating consumers, the $\tau_1 \in [0, \tau_1^*)$, prefer Platform A over Platform B . Thus, Condition 2 offers one of many possible output distribution mechanisms that ensures that the output clears.¹⁷ Our formulation is motivated by the fact that many desirable platforms face output distributional problems in practice. For example, the Nintendo Switch, Playstation 5, and Xbox Series X have had output shortages due to high demand (corresponding to Platform A in the discussion above), and this has resulted in an inefficient allocation of products that is often solved through a black market.¹⁸

Conditions 1 and 2 highlight the many ways in which the homogeneous Cournot model for platform markets is much more complicated than for traditional ones. In particular, the two-sided nature of platform competition can create differentiation between platforms (e.g., platforms that differ in size also differ in attractiveness to agents). As a result, we employ these market clearing conditions to consider platform competition that captures the spirit of traditional homogeneous Cournot competition.

Finally note that we assume that the homing distribution (ϕ_1, ϕ_2) is known to the platforms prior to choosing their output. While this assumption may appear strong, there are many industries where platforms might know the distribution of agent homing prior to production. For example, when gaming platforms like Microsoft, Nintendo, and Sony produce a new video game console, they likely have an idea of the homing distribution based on the

¹⁷For example, a Pareto allocation as discussed in footnote 15.

¹⁸Similarly, iPhone shortages have left some consumers with inferior cell phones in the past.

homing distribution for the previous console generation.¹⁹

3.4 Agent Homing

A common feature in the literature on platform competition is to exogenously assume particular homing allocations between the two sides of the market.²⁰ To better understand how different homing allocations impact our main results, we consider the scenario where the amount of single-homing on each side of the market is defined by a parameter, ϕ_i , for $i = 1, 2$. This allows for extreme allocations such as the competitive bottleneck allocation, where $\phi_1 = 1$ and $\phi_2 = 0$, as well as the more general allocations of mixed-homing that remain feasible. Not surprisingly, not all allocations are feasible; for example, if nearly all sellers multi-home (ϕ_2 close to zero), then very few consumers will multi-home (ϕ_1 close to 1). In fact, we expect the amount of multi-homing to be inversely related across the two sides and this is indeed what we find when we investigate which allocations are feasible.

Specifically, suppose that there exist critical values $\hat{\tau}_i \in [0, \tau_i^*)$ for side $i = 1, 2$ such that only the agents with type $\tau_i \leq \hat{\tau}_i$ are willing to multi-home. As motivation for the direction of the $\tau_i \leq \hat{\tau}_i$ inequality, note that if a multi-homing agent on side i joins all the platforms, then they obtain the utility

$$U_i^M(\tau_i) = \alpha_i(\tau_i)\tau_j^* - \sum_{X=1}^N p_i^X = \alpha_i(\tau_i)\tau_j^* - \alpha_i(\tau_i^*)Q_j. \quad (6)$$

Similarly, if the agent single-homes, then the expected utility is given by

$$U_i^S(\tau_i) = \alpha_i(\tau_i)q_j^e - p_i^e = \alpha_i(\tau_i)q_j^e - \alpha_i(\tau_i^*)q_j^e, \quad (7)$$

where p_i^e is the expected price on side i , and $q_j^e \in [\min_{X=1,\dots,N} q_j^X, \max_{X=1,\dots,N} q_j^X]$ is the

¹⁹Similarly, smartphone platforms and video streaming platforms likely know the amount of consumer and content multi-homing.

²⁰See Belleflamme and Peitz (2019), Bakos and Halaburda (2020) and Jeitschko and Tremblay (2020) for exceptions.

expected number of side j agents a single-homing side i agent interacts with, which depends on the rationing rule draw. Altogether, this implies that there exists a unique value $\hat{\tau}_i$ such that side i agents with $\tau_i \leq \hat{\tau}_i$ are willing to multi-home.

To ensure that our homing allocations follow those described by Condition 2, we assume that the rationing rule that assigns multi-homing agents to platforms is Pareto efficient so that the agents with the lowest type τ_i are assigned as the multi-homing agents (this is the Pareto efficient rationing rule since agents with the lowest type earn the largest network benefits). In this case, the $\tau_i \in [0, \tau_i^M]$ multi-home, where $\tau_i^M = (1 - \phi_i)q_i^L$ with $q_i^L = \min\{q_i^1, \dots, q_i^N\}$, and this approach generates an equilibrium so long as the allocation $(\phi_1$ and $\phi_2)$ results in $\hat{\tau}_i > (1 - \phi_i)q_i^L$; otherwise, the outcome is inconsistent and we say that the homing allocation is infeasible. While this approach is far from ideal, it still generalizes from the common approach of specifying a particular homing allocation for the entirety of an analysis (e.g., the competitive bottleneck allocation) and it allows for some comparative statics on homing allocations. As we discuss in the next section (see Eq. (11) below), not all homing allocations will be feasible in equilibrium under Cournot platform competition which is not surprising; as mentioned above, why would there be large amounts of consumer multi-homing when there is sizable seller multi-homing?²¹

At this point, even though allocations with sizable multi-homing on both sides are infeasible, we allow for general allocations in what follows so that we can make direct comparisons to various findings in the literature.²² Thus, $\phi_i \in [0, 1]$ continues to denote the fraction of output that are sold to single-homing agents on side i , as defined by the platform with the lowest output on side i ($q_i^L = \min\{q_i^1, \dots, q_i^N\}$), so that a $(1 - \phi_i)q_i^L$ amount of output for each platform is allocated to agents that choose to join all the platforms and the remaining $q_i^X - (1 - \phi_i)q_i^L$ portion is allocated to single-homing agents. This implies that the number of

²¹In many ways, the limitations of our approach stem from the issue that we previously discuss in Subsection 3.1: How to deal with duplicated interactions between consumers and sellers that multi-home across platforms? See footnote 12 for more information on this issue.

²²We provide several examples of infeasible allocations in Figure 1 (in Section 5), where the black regions denote homing allocations that are infeasible.

unique agents who multi-home is given by $(1 - \phi_i)q_i^L$, and that the number of unique agents on side i that join at least one platform is given by

$$\tau_i^* = \underbrace{(1 - \phi_i)q_i^L}_{\text{Multi-homing Agents}} + \underbrace{\sum_{X=1}^N [q_i^X - (1 - \phi_i)q_i^L]}_{\text{Single-homing Agents}}. \quad (8)$$

Note that if outputs are equal across platforms, then a ϕ_i portion of each platform's output goes toward side i single-homers and the $(1 - \phi_i)$ portion goes toward side i multi-homers.

4 The Two-Sided Market Cournot Equilibrium

4.1 Equilibrium Pricing

Combined, equations (5) and (8) effectively capture the aggregate inverse demands for sides 1 and 2 that the platforms use to maximize profit. Given the extent of single-homing, (ϕ_1, ϕ_2) , the Cournot platform equilibrium prices are given by:

Proposition 1. *Suppose that N platforms compete in output and that a symmetric equilibrium exists. Then equilibrium prices in the symmetric equilibrium are given by:*

$$p_i^G = c_i + \frac{1}{1 + (N - 1)\phi_i} \cdot \frac{p_i^G}{-\epsilon_i^G} - \frac{1}{1 + (N - 1)\phi_j} \cdot \alpha_j(\tau_j^G)\tau_j^G, \quad (9)$$

where the superscript G (i.e., the p_i^G and τ_j^G) denotes the equilibrium across general homing allocations (ϕ_1, ϕ_2) and $\epsilon_i^G = \frac{\partial Q_i^G}{\partial p_i^G} \cdot \frac{p_i^G}{Q_i^G} < 0$ is the elasticity of demand on side i .

Much like with monopoly platform pricing, we see that the Cournot platform price on side i is given by the sum of three terms: marginal cost ($MC_i = c_i$), a markup term ($MU_i = \frac{1}{1 + (N - 1)\phi_i} \cdot \frac{p_i^G}{-\epsilon_i^G}$), and a markdown term that incorporates the network benefit to the other side ($MD_j = \frac{1}{1 + (N - 1)\phi_j} \cdot \alpha_j(\tau_j^G)\tau_j^G$). In terms of comparative statics, naturally any changes in N , ϕ_1 , or ϕ_2 will impact all equilibrium terms denoted by superscript G .

However by holding these terms fixed, we see that greater N reduces the markup term, as with traditional Cournot pricing (Section 2), and the markdown term. Similarly, greater single-homing on side i (ϕ_i) appears to reduce the markup term on side i while greater single-homing on side j (ϕ_j) appears to reduce the markdown term on side i . Given the across the platform nature of the markdown term, these results effectively coincide so that greater single-homing reduces markups and markdowns (assuming all else constant).²³ As we will show in Section 5 where we impose greater structure, this logic follows even when we allow equilibrium outcomes to change with N, ϕ_1 , and ϕ_2 .

To this point, we have assumed that a symmetric equilibrium exists. However, existence is not straightforward in two-sided markets since there are output choices for the two sides of the market. To ensure existence, we make the following assumptions:

Assumption 1. *We assume that the network benefit functions are uniformly bounded, have uniformly bounded derivatives, and are not too convex, and that platform marginal costs are zero. Specifically, $\sup_x |\alpha_i(x)| < \infty$, $\sup_x |\alpha'_i(x)| < \infty$, $\frac{-\alpha''_i(\tau_i)\tau_i}{\alpha'_i(\tau_i)} < \frac{2}{N}$ for all τ_i , $i = 1, 2$, and $c_1 = c_2 = 0$.*

It is important to note that while the assumption of $c_1 = c_2 = 0$ simplifies some of our analysis, such an assumption plays a meaningful role only for the results in this section where $N \rightarrow \infty$ (Proposition 3). Hence, we leave the c_1 and c_2 terms included whenever possible. Furthermore, these assumptions allow us to establish the symmetric outcome characterized by Proposition 1 as an equilibrium:

Proposition 2. *Under Assumption 1, there exists a symmetric equilibrium characterized by the first-order condition given by equation (9).*

²³Another interpretation is that greater multi-homing (a reduction in the ϕ) increases markup and markdown terms which may suggest greater platform market power. However, it is important to note that this is driven by our assumption that multi-homing agents join all platforms, which we make to simplify the analysis, interacts with the capacity constraints brought on by Cournot competition. In this case, more multi-homing can tighten the capacity constraints, since multi-homing agents join all platforms, resulting in more emphasis on the markups and markdowns.

4.2 The Two-Sided Market Lerner Index

By comparing traditional and two-sided approaches (equations (1) and (9)) we see that both platform competition and agent homing impact platform pricing. Notably, the normalization that is traditionally brought on by competition, the $\frac{1}{N}$, now depends on both competition and homing: $\frac{1}{1+(N-1)\phi_i}$. In addition, this normalization impacts both the markup and markdown terms. In many ways, this highlights (i) the similarities between traditional and two-sided market pricing and (ii) the consistency of the monopoly platform pricing strategy in generalizing to oligopoly platforms.

Strengthening the comparison between traditional and two-sided markets, Armstrong (2006) and Weyl (2010) derive a modified version of the Lerner index for two-sided markets with a monopoly platform. In the context of our model, they show that a monopoly platform sets prices so that $L_i^{AW} \equiv \frac{p_i - c_i + \alpha_j(\tau_j)\tau_j}{p_i} = \frac{1}{-\epsilon_i}$ for $i, j = 1, 2$ and $i \neq j$. This mirrors the traditional monopoly Lerner relationship where $L = \frac{p-c}{p} = \frac{1}{-\epsilon}$. However, adapting to traditional Cournot competition results in $L = \frac{p-c}{p} = \frac{1}{N} \cdot \frac{1}{-\epsilon}$; hence, it would be useful to have an adaptation to L_i^{AW} that accounts for platform competition. Fortunately, our pricing results from Proposition 1 generalize the Lerner indices proposed by Armstrong (2006) and Weyl (2010) to include Cournot platform competition under mixed-homing so that

$$L_i^{2SM} := \frac{p_i - c_i + \frac{1}{1+(N-1)\phi_j} \cdot \alpha_j(\tau_j)\tau_j}{p_i} = \frac{1}{1 + (N-1)\phi_i} \cdot \frac{1}{-\epsilon_i}. \quad (10)$$

As with prices, we see that the traditional $\frac{1}{N}$ normalization becomes $\frac{1}{1+(N-1)\phi_i}$ with two-sided markets so that Equation (10) bears resemblance to both the traditional Lerner index formula, $L \equiv \frac{p-c}{p} = \frac{1}{N} \cdot \frac{1}{-\epsilon}$, and the two-sided market Lerner index formula for a monopoly platform derived by Armstrong (2006) and Weyl (2010): $L_i^{AW} \equiv \frac{p_i - c_i + \alpha_j(\tau_j)\tau_j}{p_i} = \frac{1}{-\epsilon_i}$. In addition, the two-sided market Lerner definition includes the across network markdown term as in Armstrong (2006) and Weyl (2010), but with the inclusion of the normalization in the markdown term. Thus, we generalize the existing Lerner indices to consider two-sided

markets with platform competition, and this reveals that the two-sided market Lerner index formula depends on both platform competition and agent homing.

4.3 Comparative Statics

In what follows Proposition 1, we discuss the difficulties associated with directly considering comparative statics on N, ϕ_1 , and ϕ_2 . However, there are several comparative statics worth considering before we impose additional structure on the network benefit formulas. By considering equilibrium pricing, we see that greater platform competition reduces both markup and markdown terms so that in the limit, and just like with traditional firms competing à la Cournot, the platform pricing strategy approaches marginal cost pricing so long as at least some agents single-home on *both* sides of the market: $\phi_1, \phi_2 > 0$. Formally, we have the following:

Proposition 3. *If at least some agents single-home on both sides ($\phi_1, \phi_2 > 0$), then prices approach marginal costs as the number of platforms increase: $N \rightarrow \infty$ implies that $p_i^G \rightarrow 0$.*

An important caveat to the results above (and all of our results that follow) is that the equilibrium prices (p_1^G, p_2^G) and participation on the two sides (τ_1^G, τ_2^G) should satisfy the condition that all the multi-homing agents have an incentive to do so. That is, whenever $\phi_i < 1$, equalities (6) and (7) must be such that

$$U_i^M((1 - \phi_i)q_i^G) = \alpha_i((1 - \phi_i)q_i^G)\tau_j^G - Np_i^G \geq \alpha_i((1 - \phi_i)q_i^G)q_j^G - p_i^G = U_i^S((1 - \phi_i)q_i^G), \quad (11)$$

where $\tau_i^M = (1 - \phi_i)q_i^G$ represents that last agent type to multi-home. Note that if there exists a $\phi'_i \in (0, 1)$ such that equation (11) holds with equality, then the inequality fails for all $\phi_i \in [0, \phi'_i]$. This implies that it is possible that allocations with substantial multi-homing (with both ϕ_1 and ϕ_2 close to zero) will not exist.²⁴ Thus, our micro-foundations on homing

²⁴Such allocations should not be feasible: If many side 1 users multi-home (ϕ_1 close to zero), then side 2 users have little incentive to do so. Hence, as in Bakos and Halaburda (2020) and Jeitschko and Tremblay (2020), mixed-homing allocations with substantial multi-homing on both sides of the market do not occur.

imply that not all mixed-homing allocations are feasible in a symmetric Cournot platform equilibrium; in particular, those with greater multi-homing on both sides of the market.²⁵

Another important takeaway from Proposition 3 is that pricing behavior is discontinuous at the point where an entire side multi-homes. This is particularly important for models that assume that an entire side multi-homes (as is commonly done in the competitive bottleneck model). Given that an entire side always includes at least some single-homers in practice (e.g., even in the smartphone market where nearly all app developers are on both iOS and Android, some only develop for one of the two platforms), our findings suggest that caution should be taken when modeling such allocations. To see this discontinuity explicitly, consider the competitive bottleneck allocation, where all consumers single-home ($\phi_1 = 1$) and all sellers multi-home ($\phi_2 = 0$), and note that N has no impact on the consumer markdown and the seller markup terms:

Corollary 1 (The Competitive Bottleneck). *The Cournot platform prices under the competitive bottleneck allocation are given by:*

$$p_1^{CB} = c_1 + \frac{1}{N} \cdot \frac{p_1^{CB}}{-\epsilon_1^{CB}} - \alpha_2(\tau_2^{CB})\tau_2^{CB}, \quad (12)$$

$$p_2^{CB} = c_2 + \frac{p_2^{CB}}{-\epsilon_2^{CB}} - \frac{1}{N} \cdot \alpha_1(\tau_1^{CB})\tau_1^{CB}, \quad (13)$$

where superscript CB denotes the competitive bottleneck equilibrium.

In this case we see that the asymmetric homing decisions distort prices across the two sides. On the single-homing side (side 1), platform competition N normalizes the markup term but not the markdown term. In contrast, on the multi-homing side (side 2), platform competition N normalizes the markdown term but not the markup term. Thus, as competition increases, prices diverge so that the consumer price includes no markup and the seller

²⁵To see equation (11) impacting equilibrium existence in practice, note that in Figure 1 and in the Proof of Proposition 8 we show that there are linear $\alpha_i(\cdot)$ such that some allocations do not occur as Cournot platform equilibrium. These allocations are captured by the black areas in Figure 1.

price includes no markdown. Such pricing distortions across homing decisions in the competitive bottleneck equilibrium is consistent with intuition found in the previous literature: platforms compete for single-homers and have market power over the multi-homers (e.g., Rochet and Tirole (2003), Armstrong (2006) and Hagiu (2006)). Lastly, note that Corollary 1 reveals that the competitive bottleneck allocation under Cournot platform competition can result in a straddle pricing equilibrium, where $p_1 > 0$ and $p_2 < 0$, which Jeitschko and Tremblay (2020) show to be the only pricing strategy where platforms that compete in price a la Bertrand can earn profit. Thus, even in the limit, N Cournot competing platforms might earn profit under the competitive bottleneck allocation.

5 Additional Insights on Pricing and Welfare

By imposing additional structure on the network benefits, we are able to consider welfare and additional comparative statics on prices. Thus, we assume that network externalities are linear in agent type (so that $\alpha_i(\tau_i) = a_i - b_i\tau_i$, $\tau_i \in [0, a_i/b_i]$, for $i = 1, 2$) and we continue to assume that marginal costs are zero ($c_1 = c_2 = 0$) as in Assumption 1. These simplifying assumptions are not uncommon in the literature (as in for example, the base models of Katz and Shapiro (1985), Cabral (2011), Jullien (2011), Gabszewicz and Wauthy (2014), Adachi and Tremblay (2020), Bakos and Halaburda (2020), and Halaburda et al. (2020)).²⁶

It has been well established that a welfare tradeoff occurs with greater platform competition: more platforms result in (i) lowers prices which improves welfare and (ii) reduces an individual platform's size which decreases the accumulation of network effects and harms welfare. In addition, many have discussed how single- vs multi-homing impacts two-sided market pricing: the side that single-homes gets a better price than the side that multi-homes as platforms compete intensely for the single-homing side. Based on our results so far, it

²⁶Furthermore, we focus on the parameterization under which all the participating agents enjoy positive network effects in the equilibrium, that is, $\alpha_i(\tau_i^*) > 0$ for $i = 1, 2$. This restriction should be reasonable because we consider the situation where agents enjoy positive network effects, and we will make it clear whenever this restriction plays a role (Footnotes 27 and 30).

is natural to think that agent homing allocations will impact pricing and welfare. As a result, we start by considering the simple case of platform competition under the competitive bottleneck allocation and then we turn to the case of general homing allocations.

5.1 The Competitive Bottleneck Allocation

For the competitive bottleneck allocation, we have the following result:

Proposition 4. *In the competitive bottleneck equilibrium (where $\phi_1 = 1$ and $\phi_2 = 0$), linear network externalities ($\alpha_i(\tau_i) = a_i - b_i\tau_i$ for $i = 1, 2$) result in competitive bottleneck participation levels given by*

$$\tau_1^{CB} = \frac{(a_1 + a_2)N}{b_1(N + 2)} \text{ and } \tau_2^{CB} = \frac{a_1 + a_2}{b_2(N + 2)}.$$

In addition, equilibrium consumer participation increases and seller participation decreases as the number of platforms increases, $\frac{\partial \tau_1^{CB}}{\partial N} > 0$ and $\frac{\partial \tau_2^{CB}}{\partial N} < 0$, and the price to consumers decreases while the price to sellers increases with a greater number of platforms: $\frac{\partial p_1^{CB}}{\partial N} < 0$ and $\frac{\partial p_2^{CB}}{\partial N} > 0$.²⁷

Our pricing results from Corollary 1 show that platforms have an incentive to price more competitively to consumers while extracting rents from sellers under the competitive bottleneck allocation. Proposition 4 reveals that this carries through to increases in platform competition: greater platform competition decreases the consumer price and increases the seller price. Furthermore, we see that participation levels coincide with the pricing effects so that more consumers and fewer sellers participate when platform competition increases.

Given that the changes in participation are negatively correlated across the two sides, we must investigate welfare explicitly to determine how greater platform competition impacts total welfare in two-sided markets under the competitive bottleneck allocation. In the competitive bottleneck, note that every participating consumer interacts with every participating

²⁷ The comparative statics of $\frac{\partial p_2^{CB}}{\partial N} > 0$ requires $\alpha_i(\tau_i^*) \geq 0$ for $i = 1, 2$.

seller and vice versa. Thus, total welfare is given by:

$$W^{CB} = \int_0^{\tau_1^{CB}} \alpha_1(\tau_1) \tau_2^{CB} d\tau_1 + \int_0^{\tau_2^{CB}} \alpha_2(\tau_2) \tau_1^{CB} d\tau_2. \quad (14)$$

Differentiating welfare with respect to the number of platforms generates the following:

Proposition 5. *Under the competitive bottleneck allocation, welfare is decreasing in the number of platforms: $\frac{\partial W^{CB}}{\partial N} < 0$ for all $N \geq 2$.*

While platform mergers under the competitive bottleneck allocation are uncommon in practice, we have observed several industries where the number of platforms has changed due to entry and exit.²⁸ For example, Google entered the smartphone market (and Microsoft followed suit) that was originally developed by Apple; however, Microsoft began to exit the market in 2016.²⁹ Our findings from Proposition 5 suggest that these entries would actually harm welfare given that these industries adhere to the competitive bottleneck allocation. The smartphone market is the most likely to fit the competitive bottleneck bill as most consumers own a single smartphone while app providers make their apps available across platforms. This suggests that Microsoft’s exit from the smartphone market has improved welfare in the smartphone industry.

5.2 General Homing Allocations

The competitive bottleneck allocation offers an excellent starting point for our welfare analysis as it offers unambiguous comparative statics. However, it is important to consider other feasible homing allocations and determine how robust those results, and their corresponding policy recommendations, actually are. By considering general homing allocations, we also determine the extent for which single- and multi-homing distributions impact our findings; something that is often overlooked in the literature.

²⁸One potential example of a merge to monopoly is Facebook’s acquisition of Instagram in 2012 since Snapchat was in its infancy at the time. However, Instagram was not selling ads at the time and so it is difficult to argue that this was indeed a two-sided platform.

²⁹See Microsoft’s Exit From Smartphone Business Moves Into High Gear; Bloomberg 2016.

Recall that in the general homing setup, ϕ_i captures the extent of single-homing on side i so that ϕ_i closer to one (zero) corresponds to the case where the majority of side i agents single-homing (multi-home). In terms of prices and participation, we expect that the comparative statics might hinge on agent homing decisions. Indeed this is the case:

Proposition 6. *For general homing allocations characterized by $\phi_1, \phi_2 \in [0, 1]$, equilibrium consumer (seller) participation increases in the number of platforms if and only if $2\phi_1 > \phi_2$ ($2\phi_2 > \phi_1$): $\frac{\partial \tau_i^G}{\partial N} > 0$ if and only if $2\phi_i > \phi_j$. Furthermore, individual output is decreasing in the number of platforms across all allocations with at least some single-homing: $\frac{\partial q_i^G}{\partial N} < 0$ for $i = 1, 2$ so long as $\phi_1 > 0$ or $\phi_2 > 0$.*

In terms of pricing, we have that if mixed-homing occurs so that $\phi_i \in (0, 1)$ for $i = 1, 2$, then the markup and markdown terms are decreasing in N and their respective single-homing terms: If the side i markup term is given by $MU_i = \frac{1}{1+(N-1)\phi_i} \cdot \frac{p_i^G}{-\epsilon_i^G}$ and the side j markdown term is given by $MD_j = \frac{1}{1+(N-1)\phi_j} \cdot \alpha_j(\tau_j^G)\tau_j^G$ so that $p_i = c_i + MU_i - MD_j$, then $\frac{dMU_i}{d\phi_i} < 0$, $\frac{dMD_j}{d\phi_j} < 0$, $\frac{dMU_i}{dN} < 0$, and $\frac{dMD_j}{dN} < 0$.³⁰

First investigating total participation, if the extent of single-homing on side i is at least half the extent of single-homing on side j ($\phi_i > 0.5\phi_j$), then greater platform competition results in greater equilibrium participation on side i . However, if there is too little single-homing (relative to the other side), then participation decreases with greater platform competition. These results echo our results from the competitive bottleneck allocation where greater platform competition increases participation on the single-homing side but decreases participation on the multi-homing side.

One important conclusion from our findings in Proposition 6 is that there is a large mass of homing allocations where participation on *both* sides increases, a result that is not possible when focusing on the competitive bottleneck allocation. This occurs when the $\phi_1, \phi_2 \in [0, 1]$ are such that $2\phi_1 > \phi_2 > 0.5\phi_1$. However, even though the total number of participating agents can increase on both sides of the market, we see that individual platforms get smaller

³⁰The comparative statics of $\frac{\partial MD_j}{\partial N} < 0$ requires $\alpha_i(\tau_i^*) \geq 0$ for $i = 1, 2$.

as the number of platforms increases, and this may reduce the amount of network surplus generated in the market which has important implications for welfare.

Turning to the pricing results from Proposition 6, recall that we discuss how homing might impact the markup and markdown terms in the two-sided market pricing formula when mixed-homing occurs following the general homing pricing results presented in Proposition 1. There we mention that if we crudely hold all of the equilibrium terms fixed, then the markup and markdown terms are decreasing in N and their single-homing terms. However, we were unable to say more without specifying the network externality functions. Now we see that these comparative statics on the markup and markdown terms indeed hold under a linear specification which sheds light on how homing allocations impact the two-sided market pricing formula.

The results from Proposition 6 do not immediately reveal the welfare effects from greater platform competition under general homing allocations. Hence, we investigate welfare explicitly. Total welfare in the general homing case is given by

$$W^G = \sum_{i=1}^2 \left[\int_0^{(1-\phi_i)q_i^G} \alpha_i(\tau_i) \tau_j^G d\tau_i + \int_{(1-\phi_i)q_i^G}^{\tau_i^G} \alpha_i(\tau_i) q_j^G d\tau_i \right]. \quad (15)$$

The first integral is all surplus captured by multi-homers and the second term captures the surplus from single-homers.

As an extension of the competitive bottleneck welfare result in Proposition 5, we first consider the case where multi-homing inverses across the two sides so that $\phi_1 = \phi \in [0, 1]$ and $\phi_2 = 1 - \phi$. In this case, the competitive bottleneck is given by $\phi = 1$ so that $\phi_1 = 1$ (all consumers single-home) and $\phi_2 = 0$ (all sellers multi-home). There are many reasons why the inverse homing allocation is worth investigating. First, greater multi-homing on one side generates an incentive to single-home on the other side (and vice a versa). Second, many industries appear to have this inverse relationship (e.g., both the consumer and game developer sides of the video game industry experience single- and multi-homers, but the game

developer side has considerably more multi-homing than on the consumer side). Third, as we show in Figure 1, the inverse homing allocations are often feasible (note that the black areas in Figure 1 denote the infeasible allocations and that the inverse homing allocations are captured by the line between the top left corner to the bottom right corner). As we show in the following proposition, our results from Proposition 5 generalize to the case of the inverse allocation:

Proposition 7. *If homing inverses across sides so that $\phi_1 = 1 - \phi_2$, then welfare is decreasing in the number of platforms: $\frac{\partial W^G}{\partial N} < 0$ for all $N \geq 2$.*

To the extent that a platform industry has multi-homers concentrated on one side of the market (making single-homers concentrated on the other side of the market), this result implies that platform mergers would increase welfare.

Moving to the general homing case, it appears that fewer platforms is welfare improving even if ϕ_1 and ϕ_2 are not restricted to the case of inverse homing decisions. To see this explicitly, note that Figure 1 depicts the regions of $(\phi_1, \phi_2) \in [0, 1] \times [0, 1]$ for when a reduction in the number of platforms improves social welfare. Importantly, note that the black regions in Figure 1 are where at least one of these four inequalities in (11) fail (four inequalities, since there are two sides for each of the two N s in each graph). Note that the 45 degree line from (0,1) to (1,0) remains outside the black area in Figure 1 so that inverse homing allocations are feasible. Without the black regions in Figure 1 from inequality (11) there are allocations close to the origin where n platforms generate more surplus than $n - 1$ platforms for all four subgraphs. Hence, the feasibility conditions are an important feature in determining welfare results. More formally, we have the following result:³¹

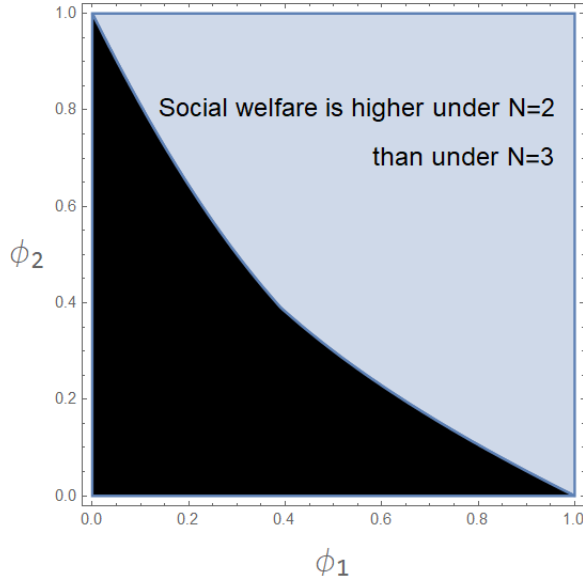
Proposition 8. *If the $\alpha_i(\cdot)$ are linear and symmetric so that $\alpha_1(\tau) = \alpha_2(\tau)$, then $\frac{\partial W^G}{\partial N} < 0$ for all $N \geq 2$ under the (ϕ_1, ϕ_2) that are feasible in the Cournot platform equilibrium.³²*

³¹Figure 1 considers the symmetric case where $\alpha_i(\tau_i) = 1 - \tau_i$. It is important to note that, as we show explicitly in Proposition 8, the qualitative results behind the welfare ranks remain for symmetric network benefit functions that are linear.

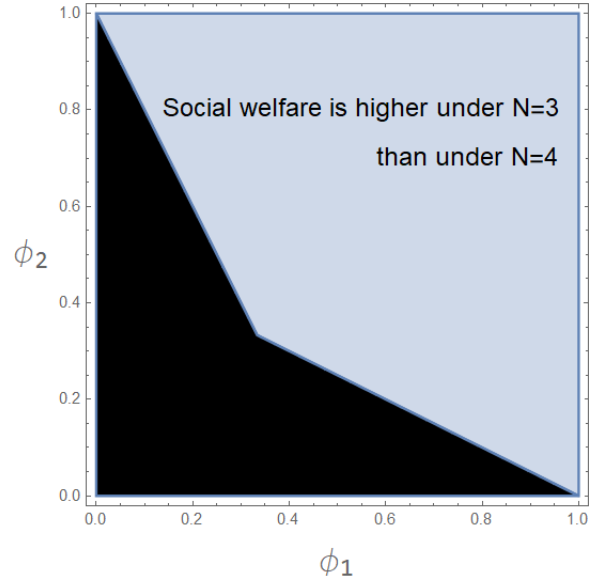
³²We find similar findings by focusing exclusively on the surplus generated by non-platform agents (consumers and firms).

Figure 1: Region for Social Welfare to be Higher

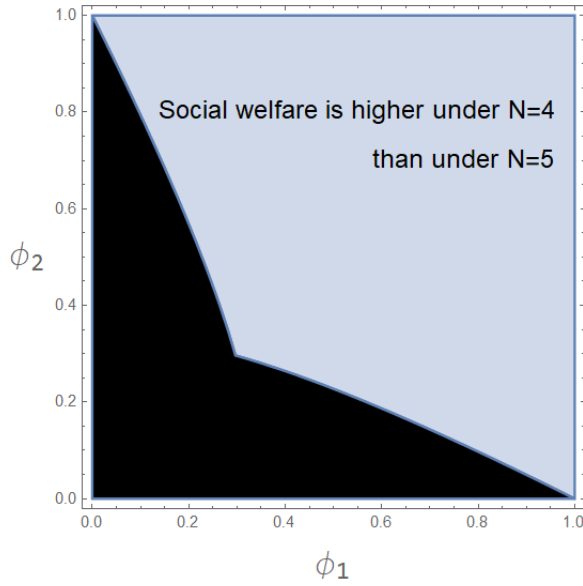
(a) $N = 2, 3$



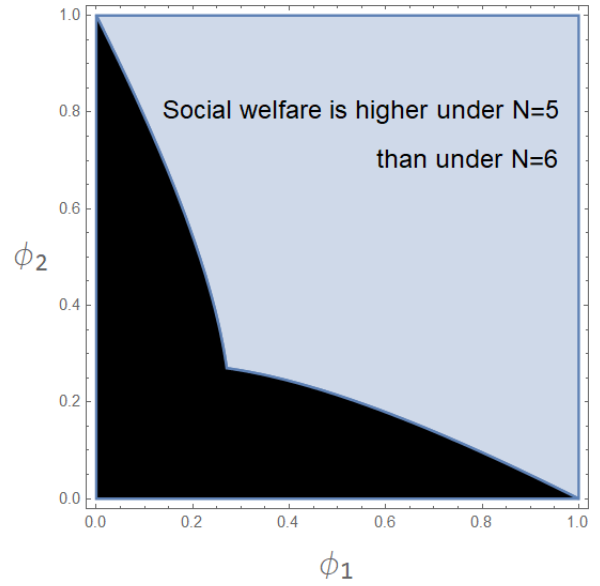
(b) $N = 3, 4$



(c) $N = 4, 5$



(d) $N = 5, 6$



While our results on welfare suggest that platform competition is harmful, there are several caveats to our analysis that should be considered when evaluating platform mergers. First, we assume symmetric network benefits across the two sides for analytical purposes; with significant asymmetries, the welfare effects become ambiguous.³³ Second, platforms are often differentiated in practice; naturally this increases surplus and makes a platform merger more detrimental than in our setting of homogeneous platforms. Finally, we abstract away from any standalone value generated by the platform (e.g., a smartphone generates considerable value outside of apps). How these standalone values change across homing decisions complicates the analysis considerably (see Jeitschko and Tremblay (2020) for details), but they may also impact the welfare effects from different levels of competition.

6 Conclusion

The study of platform competition often presents researchers with difficulties that do not arise in the study of competition amongst traditional firms. As a result, there is limited variety in the models that consider platform competition (especially for the case of more than two platforms). In particular, and, to the best of our knowledge, we are the first to expand Cournot platform competition with homing allocations to Lerner formulas. By allowing for general homing allocations on both sides of the market, we find that with mixed-homing on both sides of the market, greater platform competition results in both platform prices converging to their respective marginal cost—a result that is consistent with a traditional Cournot market. We also show that welfare is decreasing in the number of platforms across a variety of homing allocations and this highlights how the welfare losses from splitting up a network across more platforms can dominate the welfare benefits associated with greater competition.

³³However, note that symmetric network benefits was not assumed in Proposition 7.

References

- Adachi, T., Sato, S., and Tremblay, M. (2023). Platform oligopoly with endogenous homing: Implications for mergers and free entry. *Journal of Industrial Economics*, Forthcoming.
- Adachi, T. and Tremblay, M. J. (2020). Bargaining in two-sided markets. *European Economic Review*, 130:103591.
- Alexandrov, A., Deltas, G., and Spulber, D. F. (2011). Antitrust and competition in two-sided markets. *Journal of Competition Law and Economics*, 7(4):775–812.
- Anderson, S. P., Foros, Ø., and Kind, H. J. (2019). The importance of consumer multihoming (joint purchases) for market performance: Mergers and entry in media markets. *Journal of Economics & Management Strategy*, 28(1):125–137.
- Anderson, S. P., Foros, Ø., Kind, H. J., and Peitz, M. (2012). Media market concentration, advertising levels, and ad prices. *International Journal of Industrial Organization*, 30(3):321–325.
- Armstrong, M. (2006). Competition in two-sided markets. *RAND Journal of Economics*, 37(3):668–691.
- Armstrong, M. and Wright, J. (2007). Two-sided markets, competitive bottlenecks and exclusive contracts. *Economic Theory*, 32:353–380.
- Bakos, Y. and Halaburda, H. (2020). Platform competition with multi-homing on both sides: Subsidize or not? *Management Science*, 66(12):5599–5607.
- Baranes, E., Cortade, T., and Cosnita-Langlais, A. (2019). Horizontal mergers on platform markets: cost savings v. cross-group network effects? *MPRA Working Paper No. 97459*.
- Belleflamme, P. and Peitz, M. (2019). Platform competition: Who benefits from multihoming? *International Journal of Industrial Organization*, 64:1–26.

- Belleflamme, P., Peitz, M., and Toulemonde, E. (2020). The tension between market shares and profit under platform competition. *CRC TR 224 Discussion Paper Series*.
- Bertrand, J. (1883). *Review of Théorie Mathématiques de la Richesse Sociale and Recherches sur les Principes Mathématiques de la Théorie des Richesses*, volume 68.
- Bryan, K. A. and Gans, J. S. (2019). A theory of multihoming in rideshare competition. *Journal of Economics & Management Strategy*, 28(1):89–96.
- Cabral, L. (2011). Dynamic price competition with network effects. *Review of Economic Studies*, 78(1):83–111.
- Caillaud, B. and Jullien, B. (2003). Chicken & egg: Competition among intermediation service providers. *RAND Journal of Economics*, 34(2):309–328.
- Correia-da Silva, J., Jullien, B., Lefouili, Y., and Pinho, J. (2019). Horizontal mergers between multisided platforms: Insights from cournot competition. *Journal of Economics & Management Strategy*, 28(1):109–124.
- Corts, K. S. (1999). Conduct parameters and the measurement of market power. *Journal of Econometrics*, 88(2):227–250.
- Cournot, A.-A. (1838). *Recherches sur les principes mathématiques de la théorie des richesses par Augustin Cournot*. Chez L. Hachette.
- d’Aspremont, C. and Dos Santos Ferreira, R. (2009). Price–quantity competition with varying toughness. *Games and Economic Behavior*, 65(1):62–82.
- Doganoglu, T. and Wright, J. (2006). Multihoming and compatibility. *International Journal of Industrial Organization*, 24(1):45–67.
- Gabszewicz, J. J. and Wauthy, X. Y. (2014). Vertical product differentiation and two-sided markets. *Economics Letters*, 123(1):58–61.

- Gautier, A. and Lamesch, J. (2021). Mergers in the digital economy. *Information Economics and Policy*, 54:100890.
- Hagiu, A. (2006). Pricing and commitment by two-sided platforms. *RAND Journal of Economics*, 37(3):720–737.
- Halaburda, H., Jullien, B., and Yehezkel, Y. (2020). Dynamic competition with network externalities: Why history matters. *RAND Journal of Economics*, 51(1):3–31.
- Hotelling, H. (1929). Stability in competition. *Economic Journal*, 39(153):41–57.
- Ishihara, A. and Oki, R. (2021). Exclusive content in two-sided markets. *Journal of Economics & Management Strategy*, 30(3):638–654.
- Ivaldi, M. and Zhang, J. (2021). Simulating media platform mergers. *International Journal of Industrial Organization*, 79:102729.
- Jeitschko, T. D. and Tremblay, M. J. (2020). Platform competition with endogenous homing. *International Economic Review*, 61(3):1281–1305.
- Jullien, B. (2011). Competition in multi-sided markets: Divide and conquer. *American Economic Journal: Microeconomics*, 3(4):186–220.
- Katz, M. L. and Shapiro, C. (1985). Network externalities, competition, and compatibility. *American Economic Review*, 75(3):424–440.
- Lefouili, Y. and Pinho, J. (2020). Collusion between two-sided platforms. *International Journal of Industrial Organization*, 72:102656.
- Reisinger, M., Ressen, L., and Schmidtke, R. (2009). Two-sided markets with pecuniary and participation externalities. *Journal of Industrial Economics*, 57(1):32–57.
- Rochet, J.-C. and Tirole, J. (2003). Platform competition in two-sided markets. *Journal of the European Economic Association*, 1(4):990–1029.

- Salop, S. C. (1979). Monopolistic competition with outside goods. *Bell Journal of Economics*, 10(1):141–156.
- Sato, S. (2021). Horizontal mergers in the presence of network externalities. *SSRN Working Paper*.
- Schiff, A. (2003). Open and closed systems of two-sided networks. *Information Economics and Policy*, 15(4):425–442.
- Tan, G. and Zhou, J. (2021). The effects of competition and entry in multi-sided markets. *Review of Economic Studies*, 88(2):1002–1030.
- Teh, T.-H., Liu, C., Wright, J., and Zhou, J. (2023). Multihoming and oligopolistic platform competition. *American Economic Journal: Microeconomics*, Forthcoming.
- Weyl, E. G. (2010). A price theory of multi-sided platforms. *American Economic Review*, 100:1642–1672.
- Weyl, E. G. and Fabinger, M. (2013). Pass-through as an economic tool: Principles of incidence under imperfect competition. *Journal of Political Economy*, 121(3):528–583.

Appendix of Proofs

Proof of Proposition 1: Equation (5) provides consumer and seller inverse demands for each platform for the general homing case. Substituting into the platform X 's profit function yields:

$$\Pi^X = [\alpha_1(\tau_1^*) \cdot q_2^X - c_1] \cdot q_1^X + [\alpha_2(\tau_2^*) \cdot q_1^X - c_2] \cdot q_2^X.$$

Using the output functions for τ_i^* given by equation (8) we see that there are two possible first-order conditions for each output variable q_i^X (one for $q_i^X \geq q_i^{\mathcal{L}} := \min_{Y \neq X}^N \{q_i^Y\}$ and another for $q_i^X < q_i^{\mathcal{L}}$). These first-order conditions are

$$0 = \frac{\partial \Pi^X}{\partial q_i^X} = \alpha'_i(\tau_i) \cdot q_j^X \cdot q_i^X + [\alpha_i(\tau_i) \cdot q_j^X - c_i] + \alpha_j(\tau_j) \cdot q_j^X,$$

for $q_i^X \geq q_i^{\mathcal{L}}$, and

$$0 = \frac{\partial \Pi^X}{\partial q_i^X} = \phi_i \cdot \alpha'_i(\tau_i) \cdot q_j^X \cdot q_i^X + [\alpha_i(\tau_i) \cdot q_j^X - c_i] + \alpha_j(\tau_j) \cdot q_j^X,$$

for $q_i^X < q_i^{\mathcal{L}}$. Focusing on the symmetric equilibrium, we have that $q_i^X = q_i$ for all $X = 1, \dots, N$ and so the first-order condition with $q_i^X \geq q_i^{\mathcal{L}}$ provides the equilibrium.

Furthermore, symmetry and equation (8) imply that the unique number of side i agents is given by $\tau_i^G = [1 + (N - 1)\phi_i]q_i$ so that $q_i = \frac{\tau_i^G}{[1 + (N - 1)\phi_i]}$. Substituting q_i^X and q_j^X for $q_i^X = q_i = \frac{\tau_i^G}{[1 + (N - 1)\phi_i]}$ and $q_j^X = q_j = \frac{\tau_j^G}{[1 + (N - 1)\phi_j]}$ into the first-order condition above implicitly defines the solutions for (τ_1^G, τ_2^G) .

Also note that the total output on side i can be written as $Q_i = N \cdot q_i = \frac{N \cdot \tau_i^G}{[1 + (N - 1)\phi_i]}$. As a result, we can write the classic markup term given by $\frac{-p_i}{\epsilon_i}$, where the elasticity of demand on side i is given by $\epsilon_i = \frac{\partial Q_i}{\partial p_i} \cdot \frac{p_i}{Q_i} < 0$, as the following:

$$\begin{aligned} \frac{p_i}{\epsilon_i} &= \frac{\partial p_i}{\partial Q_i} \cdot Q_i = \left[\alpha'_i(\tau_i^G) \cdot q_j \cdot \frac{[1 + (N - 1)\phi_i]}{N} \right] \cdot \frac{N \cdot \tau_i^G}{[1 + (N - 1)\phi_i]} = \alpha'_i(\tau_i^G) \cdot q_j \tau_i^G \\ &= \alpha'_i(\tau_i^G) \cdot q_j \cdot [1 + (N - 1)\phi_i] q_i \end{aligned}$$

Substituting into the first-order condition reduces to

$$0 = \frac{1}{1 + (N-1)\phi_i} \cdot \frac{p_i^G}{\epsilon_i} + [p_i^G - c_i] + \alpha_j(\tau_j^G) \cdot \frac{\tau_j^G}{1 + (N-1)\phi_j}. \quad (16)$$

Rearranging terms implies that

$$p_i^G = c_i + \frac{1}{1 + (N-1)\phi_i} \cdot \frac{p_i^G}{-\epsilon_i^G} - \frac{1}{1 + (N-1)\phi_j} \cdot \alpha_j(\tau_j^G) \tau_j^G.$$

□

Proof of Proposition 2: For the equilibrium characterization (9) to be sufficient, the optimality condition must satisfy the second-order condition. In our setting, the second-order condition is satisfied when the Hessian of Π^X is negative semi-definite whenever $\partial \Pi^X / \partial q_i^X = 0$ for $i = 1, 2$. Because we have

$$\begin{aligned} \frac{\partial^2 \Pi^X}{\partial (q_i^X)^2} &= \alpha_i''(\tau_i) \cdot q_j^X \cdot q_i^X + 2\alpha_i'(\tau_i) \cdot q_j^X \\ &= \alpha_i'(\tau_i) q_j^X \left[2 + \frac{\alpha_i''(\tau_i)}{\alpha_i'(\tau_i)} q_i^X \right], \\ \frac{\partial^2 \Pi^X}{\partial q_i^X \partial q_j^X} &= \alpha_i'(\tau_i) \cdot q_i^X + \alpha_i(\tau_i) + \alpha_j'(\tau_j) \cdot q_j^X + \alpha_j(\tau_j) \\ &= \alpha_i'(\tau_i) \left[q_i^X + \frac{\alpha_i(\tau_i)}{\alpha_i'(\tau_i)} \right] + \alpha_j'(\tau_j) \left[q_j^X + \frac{\alpha_j(\tau_j)}{\alpha_j'(\tau_j)} \right], \end{aligned}$$

The Hessian of Π^X is negative semi-definite when

$$\frac{\partial^2 \Pi^X}{\partial (q_i^X)^2} < 0,$$

and

$$\frac{\partial^2 \Pi^X}{\partial (q_i^X)^2} \frac{\partial^2 \Pi^X}{\partial (q_j^X)^2} - \left(\frac{\partial^2 \Pi^X}{\partial q_i^X \partial q_j^X} \right)^2 > 0$$

Which can be written as

$$\begin{aligned} & \alpha'_i(\tau_i)\alpha'_j(\tau_j)q_i^X q_j^X \left[2 + \frac{\alpha''_i(\tau_i)}{\alpha'_i(\tau_i)}q_i^X \right] \left[2 + \frac{\alpha''_j(\tau_j)}{\alpha'_j(\tau_j)}q_j^X \right] \\ & - \left\{ \alpha'_i(\tau_i) \left[q_i^X + \frac{\alpha_i(\tau_i)}{\alpha'_i(\tau_i)} \right] + \alpha'_j(\tau_j) \left[q_j^X + \frac{\alpha_j(\tau_j)}{\alpha'_j(\tau_j)} \right] \right\}^2 > 0. \end{aligned}$$

Suppose that $c_i = c_j = 0$. In this case, $\partial\Pi^X/\partial q_i^X = 0$ if

$$\alpha'_i(\tau_i)q_i^X + \alpha_i(\tau_i) + \alpha_j(\tau_j) = 0.$$

This implies that at the point where the first-order condition is satisfied, there exists a corresponding value $\beta > 0$ such that

$$\alpha_i(\tau_i) + \alpha_j(\tau_j) = -\alpha'_i(\tau_i)q_i^X = -\alpha'_j(\tau_j)q_j^X = \beta.$$

At that point, the Hessian of Π^X is positive definite when

$$\beta^2 \left[2 + \frac{\alpha''_i(\tau_i)}{\alpha'_i(\tau_i)}q_i^X \right] \left[2 + \frac{\alpha''_j(\tau_j)}{\alpha'_j(\tau_j)}q_j^X \right] - \beta^2 > 0.$$

Let $\sigma_i(\tau_i) := -\alpha''_i(\tau_i)\tau_i/\alpha'_i(\tau_i)$, which is positive when $\alpha''_i(\tau_i) > 0$. Because $N\tau_i \geq Q_i \geq q_i^X$, we have, when $\alpha''_i(\tau_i) > 0$,

$$\frac{-\alpha''_i(\tau_i)}{\alpha'_i(\tau_i)}q_i^X \leq N\sigma_i(\tau_i).$$

Therefore, the second-order condition is satisfied when

$$\sigma_i(\tau_i) < \frac{2}{N}$$

holds for all i and τ_i , and

$$[2 - N\sigma_i(\tau_i)][2 - N\sigma_j(\tau_j)] - 1 > 0$$

for all (τ_i, τ_j) . As a special case, $\sigma_i(\tau_i) \leq 2/N$ always holds if α_i is globally concave or linear.

□

Proof of Proposition 3: We first show that if $\phi_1, \phi_2 > 0$, then $\alpha_1(\tau_1^G) + \alpha_2(\tau_2^G) \rightarrow 0$ as $N \rightarrow \infty$. To see this, note that the equilibrium condition is given by

$$\alpha'_i(\tau_i^G) \frac{\tau_i^G}{1 + (N-1)\phi_i} + \alpha_i(\tau_i^G) + \alpha_j(\tau_j^G) = 0,$$

for $i, j = 1, 2, j \neq i$.

Now suppose to the contrary that $\alpha_1(\tau_1^G) + \alpha_2(\tau_2^G) \rightarrow \bar{\alpha} \neq 0$. This implies that

$$\alpha'_i(\tau_i^G) \frac{\tau_i^G}{1 + (N-1)\phi_i} \rightarrow -\bar{\alpha}$$

as $N \rightarrow \infty$. Because $\inf_x \alpha'_i(x) > -\infty$ and $\sup_x \alpha'_i(x) < 0$, we must have $\lim_{N \rightarrow \infty} \alpha'_i(\tau_i^G) \rightarrow \underline{\alpha} \in (-\infty, 0)$. This implies that $\bar{\alpha} > 0$, which in turn implies that $\alpha_i(\tau_i^G) > 0$ and $\tau_i^G < \infty$ holds for at least one $i = 1, 2$. Then, we must have $\tau_i^G/[1 + (N-1)\phi_i] \rightarrow 0$ as $N \rightarrow \infty$ when $\phi_i > 0$, which implies that $\alpha_1(\tau_1^G) + \alpha_2(\tau_2^G) = 0$ must hold at the limit, a contradiction.

Finally, we show that $p_i^G = \alpha_i(\tau_i^G)q_i^G \rightarrow 0$. The fact that $\alpha_i(\tau_i^G) + \alpha_j(\tau_j^G) \rightarrow 0$ as $N \rightarrow \infty$ implies that

$$\frac{\tau_i^G}{1 + (N-1)\phi_i} \rightarrow 0$$

as $N \rightarrow \infty$. Therefore, by the assumption that $\sup_x |\alpha_i(x)| < \infty$, we have

$$p_i^G = \alpha_i(\tau_i^G)q_i^G = \alpha_i(\tau_i^G) \frac{\tau_i^G}{1 + (N-1)\phi_i} \rightarrow 0 = c_i.$$

□

Proof of Proposition 4: We know from the proof of Proposition 1 that $\frac{p_i^{CB}}{-\epsilon_i} = -\alpha'_i(\tau_i^{CB}) \cdot q_j^{CB} \tau_i^{CB}$, $p_i^{CB} = \alpha_i(\tau_i^{CB}) \cdot q_j^{CB}$, and $q_i = \frac{\tau_i}{1 + (N-1)\phi_i}$. Thus, equation (9) with $\phi_1 = 1, \phi_2 = 0$,

and $c_i = 0$ becomes

$$\begin{aligned}\alpha_1(\tau_1^{CB})\tau_2^{CB} &= 0 - \frac{1}{N} \cdot \alpha'_1(\tau_1^{CB})\tau_1^{CB}\tau_2^{CB} - \alpha_2(\tau_2^{CB})\tau_2^{CB}, \\ \alpha_2(\tau_2^{CB})\frac{\tau_1^{CB}}{N} &= 0 - \alpha'_2(\tau_2^{CB})\tau_2^{CB}\frac{\tau_1^{CB}}{N} - \frac{1}{N} \cdot \alpha_1(\tau_1^{CB})\tau_1^{CB}.\end{aligned}$$

With cancelations and by rearranging terms we have that

$$0 = \frac{1}{N} \cdot \alpha'_1(\tau_1^{CB})\tau_1^{CB} + \alpha_1(\tau_1^{CB}) + \alpha_2(\tau_2^{CB}),$$

$$0 = \alpha'_2(\tau_2^{CB})\tau_2^{CB} + \alpha_2(\tau_2^{CB}) + \alpha_1(\tau_1^{CB}).$$

Totally differentiating each equation when the $\alpha''_i(\cdot) = 0$ implies that

$$0 = \left(1 + \frac{1}{N}\right) \alpha'_1(Q_1^{CB})\partial Q_1^{CB} + \alpha'_2(Q_2^{CB})\partial Q_2^{CB} - \alpha'_1(Q_1^{CB}) \cdot \frac{Q_1^{CB}}{N^2} \partial N,$$

$$0 = 2\alpha'_2(Q_2^{CB})\partial Q_2^{CB} + \alpha'_1(Q_1^{CB})\partial Q_1^{CB}.$$

After some algebra, we have that $\frac{\partial \tau_1^{CB}}{\partial N} = \frac{2\tau_1^{CB}}{N(N+2)} > 0$ and $\frac{\partial \tau_2^{CB}}{\partial N} = -\frac{\alpha'_1(\tau_1^{CB})}{\alpha'_2(\tau_2^{CB})} \frac{\tau_1^{CB}}{N(N+2)} < 0$.

In terms of prices, we know that $p_1^{CB} = \alpha_1(\tau_1^{CB}) \cdot \tau_2^{CB}$ and $p_2^{CB} = \alpha_2(\tau_2^{CB}) \cdot \frac{\tau_1^{CB}}{N}$ by equation (5). Differentiating with respect to N implies that

$$\frac{\partial p_1^{CB}}{\partial N} = \alpha'_1(\tau_1^{CB})\tau_2^{CB} \frac{\partial \tau_1^{CB}}{\partial N} + \alpha_1(\tau_1^{CB}) \frac{\partial \tau_2^{CB}}{\partial N},$$

$$\frac{\partial p_2^{CB}}{\partial N} = \alpha'_2(\tau_2^{CB}) \frac{\tau_1^{CB}}{N} \frac{\partial \tau_2^{CB}}{\partial N} + \alpha_2(\tau_2^{CB}) \frac{1}{N} \frac{\partial \tau_1^{CB}}{\partial N} - \alpha_2(\tau_2^{CB}) \frac{\tau_1^{CB}}{N^2}.$$

The first equation is less than zero since $\alpha'_1(\cdot), \frac{\partial \tau_2^{CB}}{\partial N} < 0$ and $\frac{\partial \tau_1^{CB}}{\partial N} > 0$. The second equation is ambiguous at first glance. Here, however, the consideration for interior solutions might play a role. To see this, note that the linear structure of $\alpha_i(\tau_i) = a_i - b_i\tau_i$ for $i = 1, 2$ implies

that the first-order conditions for consumers and sellers that reduce to:

$$0 = a_1 - \left(1 + \frac{1}{N}\right) b_1 \cdot \tau_1^{CB} + a_2 - b_2 \cdot \tau_2^{CB},$$

$$0 = a_2 - 2b_2 \cdot \tau_2^{CB} + a_1 - b_1 \cdot \tau_1^{CB}.$$

As a result, we have that

$$\tau_1^{CB} = \frac{(a_1 + a_2)N}{b_1(N + 2)} \text{ and } \tau_2^{CB} = \frac{a_1 + a_2}{b_2(N + 2)}.$$

This implies that $p_2^{CB} = \alpha_2(\tau_2^{CB}) \frac{\tau_1^{CB}}{N}$ becomes

$$p_2^{CB} = \frac{[(N + 1)a_2 - a_1] \cdot (a_1 + a_2)}{b_1 b_2 (N + 2)^2}.$$

Differentiating implies that $\frac{\partial p_2^{CB}}{\partial N} > 0$ if and only if $2a_1 > Na_2$. In the end, an interior solution requires that $\tau_i^{CB} < \bar{\tau}_i := \frac{a_i}{b_i}$. Given the τ_1^{CB} and τ_2^{CB} above, we require that $2a_1 > Na_2$ and $(N + 1)a_1 > a_2$ for an interior solution to occur (which is satisfied to by the former). Thus, $\frac{\partial p_2^{CB}}{\partial N} > 0$. $\square$

Proof of Proposition 5: As discussed in the Proof of Proposition 4, a linear structure so that $\alpha_i(\tau_i) = a_i - b_i \tau_i$ for $i = 1, 2$ implies that

$$\tau_1^{CB} = \frac{(a_1 + a_2)N}{b_1(N + 2)} \text{ \& } \tau_2^{CB} = \frac{a_1 + a_2}{b_2(N + 2)}.$$

An interior solution occurs whenever $\tau_i^{CB} \leq \frac{a_i}{b_i}$ (so that $\alpha_i(\cdot) \geq 0$ for all τ_i). This restriction implies that we require $\frac{N}{2} \cdot a_2 \leq a_1 \leq (N + 1)a_2$ so that the network effect magnitudes cannot differ too drastically. Differentiating implies that

$$\frac{\partial \tau_1^{CB}}{\partial N} = \frac{2(a_1 + a_2)}{b_1(N + 2)^2} \text{ \& } \frac{\partial \tau_2^{CB}}{\partial N} = - \frac{a_1 + a_2}{b_2(N + 2)^2}.$$

Applying the Leibniz integral rule to equation (14) implies that

$$\frac{\partial W^{CB}}{\partial N} = \alpha_1(\tau_1^{CB})\tau_2^{CB} \cdot \frac{\partial \tau_1^{CB}}{\partial N} + \int_0^{\tau_1^{CB}} \alpha_1(\tau_1) \frac{\partial \tau_2^{CB}}{\partial N} d\tau_1 + \alpha_2(\tau_2^{CB})\tau_1^{CB} \cdot \frac{\partial \tau_2^{CB}}{\partial N} + \int_0^{\tau_2^{CB}} \alpha_2(\tau_2) \frac{\partial \tau_1^{CB}}{\partial N} d\tau_2.$$

Substituting for the τ_i^{CB} , $\frac{\partial \tau_i^{CB}}{\partial N}$, and $\alpha_i(\cdot)$, and after some algebra, we have that

$$\frac{\partial W^{CB}}{\partial N} = \frac{(a_1 + a_2)^3}{2b_1b_2(N+2)^4} \cdot (-N^2 - 2N + 6).$$

Thus, $\frac{\partial W^{CB}}{\partial N} > 0$ if and only if $-N^2 - 2N + 6 > 0$ which never holds for $N \geq 2$. $\square$

Proof of Proposition 6: We know from the proof of Proposition 1 that $\frac{p_i^G}{-\epsilon_i} = -\alpha'_i(\tau_i^G) \cdot$

$q_j^G \tau_i^G$, $p_i^G = \alpha_i(\tau_i^G) \cdot q_j^G$, and $q_i^G = \frac{\tau_i^G}{1+(N-1)\phi_i}$. Thus, equation (9) with $c_i = 0$ becomes

$$\alpha_i(\tau_i^G) \cdot \frac{\tau_j^G}{1+(N-1)\phi_j} = 0 - \frac{1}{1+(N-1)\phi_i} \cdot \alpha'_i(\tau_i^G) \cdot \frac{\tau_j^G}{1+(N-1)\phi_j} \tau_i^G - \frac{1}{1+(N-1)\phi_j} \cdot \alpha_j(\tau_j^G) \tau_j^G,$$

which reduces to

$$0 = \alpha_i(\tau_i^G) + \alpha'_i(\tau_i^G) \frac{\tau_i^G}{1+(N-1)\phi_i} + \alpha_j(\tau_j^G),$$

for $i, j = 1, 2$ and $i \neq j$.

To determine an explicit expression for the $\frac{\partial \tau_i^G}{\partial N}$, note that $\alpha_i(\tau_i) = a_i - b_i\tau_i$ implies that:

$$0 = a_i - b_i\tau_i^G - \frac{b_i\tau_i^G}{1+(N-1)\phi_i} + a_2 - b_2\tau_2^G,$$

for $i, j = 1, 2$ and $i \neq j$. Solving the system of equations for τ_1^G and τ_2^G we have that

$$\tau_i^G = \frac{(a_1 + a_2)[1 + (N-1)\phi_i]}{b_i[3 + (N-1)\phi_1 + (N-1)\phi_2]},$$

which implies that

$$q_i^G = \frac{\tau_i^G}{1+(N-1)\phi_i} = \frac{(a_1 + a_2)}{b_i[3 + (N-1)\phi_1 + (N-1)\phi_2]},$$

for $i = 1, 2$. An interior solution occurs whenever $\tau_i^G \leq \frac{a_i}{b_i}$ (so that $\alpha_i(\cdot) \geq 0$ for all τ_i). This restriction implies that we require $\frac{1+(N-1)\phi_1}{2+(N-1)\phi_2} \cdot a_2 \leq a_1 \leq \frac{2+(N-1)\phi_1}{1+(N-1)\phi_2} \cdot a_2$ so that the network effect magnitudes cannot differ too drastically. Differentiating implies that

$$\frac{\partial \tau_i^G}{\partial N} = \frac{(a_1 + a_2)}{b_i[3 + (N-1)\phi_1 + (N-1)\phi_2]^2} \cdot [\phi_i[3 + (N-1)\phi_1 + (N-1)\phi_2] - (\phi_1 + \phi_2)[1 + (N-1)\phi_i]],$$

for $i = 1, 2$. This implies that $\frac{\partial \tau_i^G}{\partial N} > 0$ if and only if $2\phi_i > \phi_j$ for $i, j = 1, 2$ and $i \neq j$. Similarly, we clearly see that $\frac{\partial q_i^G}{\partial N} < 0$ so long as $\phi_1 > 0$ or $\phi_2 > 0$.

Lastly, our work above implies that

$$MU_i = \frac{1}{1 + (N-1)\phi_i} \cdot \frac{p_i^G}{-\epsilon_i^G} = \frac{(a_1 + a_2)^2}{b_j[3 + (N-1)(\phi_1 + \phi_2)]^2}$$

and

$$MD_j = \frac{1}{1 + (N-1)\phi_j} \cdot \alpha_j(\tau_j^G) \tau_j^G = \frac{(a_1 + a_2) \cdot \{[2 + (N-1)\phi_i]a_j - [1 + (N-1)\phi_j]a_i\}}{b_j[3 + (N-1)(\phi_1 + \phi_2)]^2}.$$

Differentiating we have that $\frac{dMU_i}{d\phi_i} < 0$, $\frac{dMD_j}{d\phi_j} < 0$, $\frac{dMU_i}{dN} < 0$, and $\frac{dMD_j}{dN} < 0$. $\square$

Proof of Proposition 7: Under a linear structure, so that $\alpha_i(\tau_i) = a_i - b_i\tau_i$ for $i = 1, 2$, equation (15) simplifies to

$$\begin{aligned} W^G &= \frac{(a_1 + a_2)^3}{2b_1b_2[3 + (N-1)(\phi_1 + \phi_2)]^3} \\ &\times [4 + 5(N-1)(\phi_1 + \phi_2) + (N-1)(4N-6)\phi_1\phi_2 \\ &\quad + (N-1)^2(\phi_1^2 + \phi_2^2) - (N-1)(2N-1)\phi_1\phi_2(\phi_1 + \phi_2)]. \end{aligned}$$

Differentiating with respect to N implies that

$$\begin{aligned}\frac{\partial W^G}{\partial N} &= \frac{(a_1 + a_2)^3}{2b_1b_2[3 + (N - 1)(\phi_1 + \phi_2)]^4} \\ &\quad \times [3(\phi_1 + \phi_2) + 2(2N - 5)\phi_1\phi_2 - 4(N - 1)(\phi_1^2 + \phi_2^2) \\ &\quad - N(5N - 2)\phi_1\phi_2(\phi_1 + \phi_2) - (N - 1)^2(\phi_1^3 + \phi_2^3) \\ &\quad + 4N(N - 1)\phi_1^2\phi_2^2 + 2N(N - 1)\phi_1\phi_2(\phi_1^2 + \phi_2^2)].\end{aligned}$$

Finally, by imposing $\phi_1 = \phi$ and $\phi_2 = 1 - \phi$, this reduces to

$$\begin{aligned}\frac{\partial W^G}{\partial N} &= \underbrace{-\frac{(a_1 + a_2)^2}{2b_1b_2(2 + N)^4}}_{<0} \\ &\quad \times [\{N + [1 - 3(1 - \phi)\phi]\}^2 - \{7 - 21(1 - \phi)\phi - 9(1 - \phi)^2\phi^2\}],\end{aligned}$$

where $1 - 3(1 - \phi)\phi > 0$ for $\phi \in [0, 1]$. Thus, the positive solution for $\{N + [1 - 3(1 - \phi)\phi]\}^2 - \{7 - 21(1 - \phi)\phi - 9(1 - \phi)^2\phi^2\} = 0$ is $N^+ = -1 + 3\phi(1 - \phi) + \sqrt{7 - 21\phi + 30\phi^2 - 18\phi^3 + 9\phi^4}$. It is verified that this $N^+ \in (1, 2)$ for any $\phi \in [0, 1]$. Therefore, $\frac{\partial W^G}{\partial N} < 0$ for all $N \geq 2$. $\square$

Proof of Proposition 8: As shown in the proof of Proposition 7, we have that

$$\begin{aligned}\frac{\partial W^G}{\partial N} &= \frac{(a_1 + a_2)^3}{2b_1b_2[3 + (N - 1)(\phi_1 + \phi_2)]^4} \\ &\quad \times [3(\phi_1 + \phi_2) + 2(2N - 5)\phi_1\phi_2 - 4(N - 1)(\phi_1^2 + \phi_2^2) \\ &\quad - N(5N - 2)\phi_1\phi_2(\phi_1 + \phi_2) - (N - 1)^2(\phi_1^3 + \phi_2^3) \\ &\quad + 4N(N - 1)\phi_1^2\phi_2^2 + 2N(N - 1)\phi_1\phi_2(\phi_1^2 + \phi_2^2)].\end{aligned}$$

Imposing the constraints from (11), letting $a_1 = a_2 = a$, and letting $b_1 = b_2 = b$ we have that $\frac{\partial W^G}{\partial N} < 0$ for $N \geq 2$ across the homing allocations that exist as Cournot platform equilibria. $\square$