

STRONG n -SHAPE THEORY

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INTRODUCTION

Let μ^{n+1} be the $(n+1)$ -dimensional universal Menger compactum. In [Chi₁], A. Chigogidze introduced the concept of n -shape and established the $(n+1)$ -dimensional analogue of Chapman's complement theorem [Cha, Theorem 2], that is, two Z -sets X and Y in μ^{n+1} have the same n -shape type if and only if their complements $\mu^{n+1} \setminus X$ and $\mu^{n+1} \setminus Y$ are homeomorphic ($\approx$), where $X \subset M$ is a Z -set in M if there are maps $f: M \rightarrow M \setminus X$ arbitrarily close to id_M . The n -shape category of compacta was discussed in [Chi₂] (cf. [Chi₃]). Later, corresponding to [Cha, Theorem 1], Y. Akaike [Aka] defined the weak proper n -homotopy category of complements of Z -sets in μ^{n+1} which is isomorphic to the n -shape category of Z -sets in μ^{n+1} . Then, as Strong Shape Theory ([EH], [DS], [KO], etc.), it is a natural attempt to define the strong n -shape category which corresponds to the proper n -homotopy category of complements of Z -sets in μ^{n+1} . Properly, one require this category to factorize the natural functor (called the n -shape functor) from the n -homotopy category to the n -shape category into two functors through it. In this paper, we introduce the $(n+1)$ -skeletal conic telescope to define the strong n -shape category of compacta.

Throughout the paper, spaces are separable metrizable and maps are continuous. It is said that two (proper) maps $f, g: X \rightarrow Y$ are (properly) n -homotopic relative to $A \subset X$ and denoted by $f \simeq^n g \text{ rel. } A$ ($f \simeq_p^n g \text{ rel. } A$) if, for any (proper) map $\varphi: Z \rightarrow X$, there is a (proper) homotopy $h: Z \times \mathbf{I} \rightarrow Y$ such that $h_0 = f\varphi$, $h_1 = g\varphi$ $h_t|_{\varphi^{-1}(A)} = f\varphi|_{\varphi^{-1}(A)}$ for each $t \in \mathbf{I}$. When $A = \emptyset$, we say that f and g are (properly) n -homotopic and denote $f \simeq g$ ($f \simeq_p g$).

A map $\varphi: M \rightarrow X$ is said to be n -invertible if any map $\psi: Z \rightarrow X$ of a space Z with $\dim Z \leq n$ lifts to M , that is, there exists a map $\tilde{\psi}: Z \rightarrow M$ such that $\varphi\tilde{\psi} = \psi$. In case φ is a proper map, if ψ is proper then $\tilde{\psi}$ is also proper. For an n -invertible map $\varphi: M \rightarrow X$ and $A \subset X$, $\varphi|_{\varphi^{-1}(A)}: \varphi^{-1}(A) \rightarrow A$ is also n -invertible. By the result of Dranishnikov [Dra, Theorem 1], for any compactum X , there exists an n -invertible map $\varphi: M \rightarrow X$ of a compactum M with $\dim M \leq n$. Then, for two (proper) maps $f, g: X \rightarrow Y$, $f \simeq^n g \text{ rel. } A$ ($f \simeq_p^n g \text{ rel. } A$) if and only if $f\varphi \simeq g\varphi \text{ rel. } \varphi^{-1}(A)$ ($f\varphi \simeq_p g\varphi \text{ rel. } \varphi^{-1}(A)$) for an invertible (proper) map $\varphi: M \rightarrow X$.

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1. THE POLYHEDRAL TELESCOPE

The n -skeleton of a simplicial complex K is denoted by $K^{(n)}$, whence $K^{(0)}$ is the set of vertices of K . The polyhedron of K is denoted by $|K|$ (i.e., $|K| = \bigcup_{\sigma \in K} \sigma$). By $\langle v_1, \dots, v_n \rangle$, we denote the simplex with vertices $v_1, \dots, v_n$. A subdivision δK of K induces the subdivision $\delta K^{(n)}$ of $K^{(n)}$. It should be remarked that $\delta K^{(n)} \subset (\delta K)^{(n)}$ but $\delta K^{(n)} \neq (\delta K)^{(n)}$ in general. The following is well known:

Fact 1. *Let L be a subcomplex of K and Z a space with $\dim Z \leq n$. Then, for any map $\varphi: Z \rightarrow |K|$, there is a map $\psi: Z \rightarrow |K^{(n)} \cup L|$ such that $\varphi \simeq \psi$ rel. $\varphi^{-1}(|K^{(n)} \cup L|)$.*

An ordered simplicial complex is a simplicial complex with an order of vertices such that the set of vertices of each simplex is totally ordered. The barycentric subdivision $\text{Sd } K$ of a simplicial complex K is an ordered simplicial complex with the following order:

$$\hat{\sigma} \leq \hat{\tau} \quad \stackrel{\text{def}}{\iff} \quad \sigma \text{ is a face of } \tau,$$

where $\hat{\sigma}$ is the barycenter of σ .

Let $I = \{0, 1, \mathbf{I}\}$ be the natural triangulation of the unit interval $\mathbf{I} = [0, 1]$. Then, I is an ordered simplicial complex with the natural order $0 < 1$. For an ordered simplicial complex K , the product simplicial complex $K \times I$ is defined as follows:

$$\begin{aligned} K \times I &= \{ \sigma \times \{0\}, \sigma \times \{1\} \mid \sigma \in K \} \\ &\cup \{ \langle (v_1, 0), \dots, (v_i, 0), (v_j, 1), \dots, (v_k, 1) \rangle \mid \langle v_1, \dots, v_k \rangle \in K \\ &\quad v_1 < \dots < v_k \in K^{(0)}, 1 \leq i \leq j \leq k \}. \end{aligned}$$

Then $K \times I$ is an ordered simplicial complex with the following order on $(K \times I)^{(0)} = K^{(0)} \times \{0, 1\}$:

$$(v, i) \leq (v', i') \quad \stackrel{\text{def}}{\iff} \quad v \leq v' \text{ and } i \leq i'.$$

Let K and L be ordered simplicial complexes and $f: K \rightarrow L$ a simplicial map. The simplicial mapping cylinder $M(f)$ is defined as follows:

$$\begin{aligned} M(f) &= K \cup L \cup \{ \langle f(v_1), \dots, f(v_i), v_j, \dots, v_k \rangle \mid \\ &\quad \langle v_1, \dots, v_k \rangle \in K, v_1 < \dots < v_k, 1 \leq i \leq j \leq k \}. \end{aligned}$$

When L is degenerate (i.e., a singleton), $M(f)$ is the simplicial cone $C(K)$ over K . We have the natural simplicial map $q_f: K \times I \rightarrow M(f)$ which is naturally defined by $q_f(v, 0) = f(v)$ and $q_f(v, 1) = v$ for $v \in K^{(0)}$. The simplicial collapsing map $c_f: M(f) \rightarrow L$ is defined by $c_f(v) = f(v)$ for $v \in K^{(0)}$ and $c_f(u) = u$ for $u \in L^{(0)}$. Then $c_f q_f = f \text{ pr}_X$ and $c_f \simeq \text{id}$ rel. $|L|$ in $|M(f)|$. Extending the orders on $K^{(0)}$ and $L^{(0)}$ to $M(f)^{(0)} = K^{(0)} \cup L^{(0)}$ so that $u < v$ for each $u \in L^{(0)}$ and $v \in K^{(0)}$, $M(f)$ is an ordered simplicial complex. Let $f^{(n)} = f|_{K^{(n)}}: K^{(n)} \rightarrow L^{(n)}$ be the restriction of f . Observe that

$$M(f)^{(n)} \subset M(f^{(n)}) \subset M(f)^{(n+1)} \subset M(f^{(n)}) \cup K \cup L$$

and $c_f|_{M(f^{(n)})} = c_{f^{(n)}} \simeq \text{id}$ rel. $|L^{(n)}|$ in $|M(f^{(n)})|$.

Fact 2. For a simplicial map $f: K \rightarrow L$, $c_f| |M(f)^{(n+1)} \cup K \cup L| \simeq^n \text{id rel. } |L|$ in $|M(f)^{(n+1)} \cup K \cup L|$, hence $f = c_f|K \simeq^n \text{id}_K$ in $|M(f)^{(n+1)} \cup K \cup L|$.

Since $K \times I$ can be regarded as $M(\text{id}_K)$, we have the following:

Fact 3. Let $p: |(K^{(n)} \times I) \cup (K \times \{0, 1\})| \rightarrow |K \times \{0\}|$ be the retraction defined by $p(x, t) = (x, 0)$. Then, $p \simeq^n \text{id rel. } |K \times \{0\}|$ in $|(K^{(n)} \times I) \cup (K \times \{0, 1\})|$, where we identify $K = K \times \{0\}$.

Let $\mathbf{K} = (|K_i|, q_{i,i+1})_{i \in \mathbb{N}}$ be an inverse sequence of ordered simplicial complexes such that each $q_{i,i+1}: K_{i+1} \rightarrow \delta K_i$ is simplicial, where δK_i is some subdivision of K_i . Let $q_i: \varprojlim \mathbf{K} \rightarrow |K_i|$ be the projection of the inverse limit of $\mathbf{K}$ to $|K_i|$ and denote

$$q_{i,j} = q_{i,i+1} \circ \cdots \circ q_{j-1,j}: |K_j| \rightarrow |K_i|, \quad i < j.$$

We define

$$\text{Tel}_{[j,\infty)}(\mathbf{K}) = \bigcup_{i=j}^{\infty} |M(q_{i,i+1})| \quad \text{and} \quad \text{Tel}_{[j,k]}(\mathbf{K}) = \bigcup_{i=j}^{k-1} |M(q_{i,i+1})|, \quad j < k,$$

where $|M(q_{i,i+1})| \cap |M(q_{i+1,i+2})| = |K_{i+1}|$ and $|M(q_{i,i+1})| \cap |M(q_{j,j+1})| = \emptyset$ for $|i - j| > 1$. The polyhedron $\text{Tel}_{[1,\infty)}(\mathbf{K})$ is called the *polyhedral telescope* for $\mathbf{K}$. One should note that $\bigcup_{i=1}^{\infty} M(q_i)$ is not a simplicial complex unless $\delta K_i = K_i$ for every $i \in \mathbb{N}$. Let

$$\text{Tel}_{[0,\infty)}(\mathbf{K}) = |C(K_1)| \cup \text{Tel}_{[1,\infty)}(\mathbf{K}) \quad \text{and} \quad \text{Tel}_{[0,k]}(\mathbf{K}) = |C(K_1)| \cup \text{Tel}_{[1,k]}(\mathbf{K}),$$

where $|C(K_1)| \cap \text{Tel}_{[1,\infty)}(\mathbf{K}) = |K_1|$. We call $\text{Tel}_{[0,\infty)}(\mathbf{K})$ the *polyhedral conic telescope*.

The simplicial collapsing map $c_{q_{i,i+1}}: M(q_{i,i+1}) \rightarrow \delta K_i$ extends to the deformation retraction

$$c_{i,i+1}^{\mathbf{K}}: \text{Tel}_{[0,i+1]}(\mathbf{K}) = \text{Tel}_{[0,i]}(\mathbf{K}) \cup |M(q_{i,i+1})| \rightarrow \text{Tel}_{[0,i]}(\mathbf{K}).$$

The following diagram is commutative:

$$\begin{array}{ccccccc} \text{Tel}_{[0,1]}(\mathbf{K}) & \xleftarrow[c]{c_{1,2}^{\mathbf{K}}} & \text{Tel}_{[0,2]}(\mathbf{K}) & \xleftarrow[c]{c_{2,3}^{\mathbf{K}}} & \text{Tel}_{[0,3]}(\mathbf{K}) & \xleftarrow[c]{c_{3,4}^{\mathbf{K}}} & \cdots \\ \cup & & \cup & & \cup & & \cdots \\ |K_1| & \xleftarrow[q_{1,2}}{} & |K_2| & \xleftarrow[q_{2,3}}{} & |K_3| & \xleftarrow[q_{3,4}}{} & \cdots \end{array}$$

The inverse limit of the upper sequence is denoted by $\text{Tel}_{[0,\infty)}(\mathbf{K})$ with the projection $c_i^{\mathbf{K}}: \text{Tel}_{[0,\infty)}(\mathbf{K}) \rightarrow \text{Tel}_{[0,i]}(\mathbf{K})$. We denote

$$c_{i,j}^{\mathbf{K}} = c_{i,i+1}^{\mathbf{K}} \circ \cdots \circ c_{j-1,j}^{\mathbf{K}}: \text{Tel}_{[0,j]}(\mathbf{K}) \rightarrow \text{Tel}_{[0,i]}(\mathbf{K}), \quad i < j.$$

Regarding $\text{Tel}_{[0,\infty)}(\mathbf{K})$ as an open subspace of $\text{Tel}_{[0,\infty)}(\mathbf{K})$, we have

$$\text{Tel}_{[0,\infty)}(\mathbf{K}) \setminus \text{Tel}_{[0,\infty)}(\mathbf{K}) = \varprojlim \mathbf{K} \quad \text{and} \quad c_i^{\mathbf{K}}| \varprojlim \mathbf{K} = q_i, \quad i \in \mathbb{N}.$$

It is easy to see that each c_i^K is a strong deformation retraction. Hence, it follows that $\text{Tel}_{[0,\infty)}(\mathbf{K})$ is homotopy dense in $\text{Tel}_{[0,\infty]}(\mathbf{K})$, that is, there is a homotopy $h: \text{Tel}_{[0,\infty]}(\mathbf{K}) \times \mathbf{I} \rightarrow \text{Tel}_{[0,\infty]}(\mathbf{K})$ such that $h_0 = \text{id}$ and $h_t(\text{Tel}_{[0,\infty]}(\mathbf{K})) \subset \text{Tel}_{[0,\infty)}(\mathbf{K})$ for $t > 0$. Since $\text{Tel}_{[0,\infty)}(\mathbf{K})$ is a polyhedron, $\text{Tel}_{[0,\infty]}(\mathbf{K})$ is an ANR by Hanner's characterization of ANR's (cf. [Hu]). Since $\text{Tel}_{[0,\infty]}(\mathbf{K})$ is contractible, it is an AR. The above construction was founded in [Ko, Theorem 1 and Corollary 1]. For each $j \in \mathbb{N}$, we can similarly define $\text{Tel}_{[j,\infty]}(\mathbf{K})$, which is an ANR and a closed subspace of $\text{Tel}_{[0,\infty]}(\mathbf{K})$. Clearly,

$$\text{Tel}_{[j,\infty]}(\mathbf{K}) \setminus \text{Tel}_{[j,\infty)}(\mathbf{K}) = \text{Tel}_{[0,\infty]}(\mathbf{K}) \setminus \text{Tel}_{[0,\infty)}(\mathbf{K}) = \varprojlim \mathbf{K}.$$

Each $d_j^K = c_j^K|_{\text{Tel}_{[j,\infty]}(\mathbf{K})}: \text{Tel}_{[j,\infty]}(\mathbf{K}) \rightarrow |K_j|$ is a strong deformation retraction and $q_{i,j}d_j^K = d_i^K|_{\text{Tel}_{[j,\infty]}(\mathbf{K})}$.

Now, we define

$$\begin{aligned} \text{Tel}_{[j,\infty)}^{n+1}(\mathbf{K}) &= \bigcup_{i=j}^{\infty} |K_i| \cup \bigcup_{i=j}^{\infty} |M(q_{i,i+1})^{(n+1)}| \quad \text{and} \\ \text{Tel}_{[j,k]}^{n+1}(\mathbf{K}) &= \bigcup_{i=j}^k |K_i| \cup \bigcup_{i=j}^{k-1} |M(q_{i,i+1})^{(n+1)}|, \quad j < k. \end{aligned}$$

These are subpolyhedra of $\text{Tel}_{[1,\infty)}(\mathbf{K})$. Recall that $\bigcup_{i=1}^{\infty} M(q_i)$ is not a simplicial complex in general. We call $\text{Tel}_{[1,\infty)}^{n+1}(\mathbf{K})$ the $(n+1)$ -skeletal telescope for $\mathbf{K}$. Let

$$\begin{aligned} \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}) &= |C(K_1)^{(n+1)}| \cup \text{Tel}_{[1,\infty)}^{n+1}(\mathbf{K}) \quad \text{and} \\ \text{Tel}_{[0,k]}^{n+1}(\mathbf{K}) &= |C(K_1)^{(n+1)}| \cup \text{Tel}_{[1,k]}^{n+1}(\mathbf{K}). \end{aligned}$$

These are n -connected. The polyhedron $\text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K})$ is called the $(n+1)$ -skeletal conic telescope for $\mathbf{K}$.

Observe that $c_i^K(\text{Tel}_{[0,i+1]}^{n+1}(\mathbf{K})) = \text{Tel}_{[0,i]}^{n+1}(\mathbf{K})$. The following diagram is commutative:

$$\begin{array}{ccccccc} \text{Tel}_{[0,1]}^{n+1}(\mathbf{K}) & \xleftarrow[c]{c_{1,2}^K|} & \text{Tel}_{[0,2]}^{n+1}(\mathbf{K}) & \xleftarrow[c]{c_{2,3}^K|} & \text{Tel}_{[0,3]}^{n+1}(\mathbf{K}) & \xleftarrow[c]{c_{3,4}^K|} & \dots \\ \cup & & \cup & & \cup & & \dots \\ |K_1| & \xleftarrow[q_{1,2}]{} & |K_2| & \xleftarrow[q_{2,3}]{} & |K_3| & \xleftarrow[q_{3,4}]{} & \dots \end{array}$$

Then the inverse limit of the upper sequence is the closed subspace

$$\text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}) = \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}) \cup \varprojlim \mathbf{K} \subset \text{Tel}_{[0,\infty]}(\mathbf{K}).$$

For each $j \in \mathbb{N}$, let $\text{Tel}_{[j,\infty)}^{n+1}(\mathbf{K}) = \text{Tel}_{[j,\infty)}^{n+1}(\mathbf{K}) \cup \varprojlim \mathbf{K}$.

Fact 4. For each $j \in \mathbb{N} \cup \{0\}$, $\text{Tel}_{[j,\infty]}^{n+1}(\mathbf{K}) \setminus \text{Tel}_{[j,\infty]}^{n+1}(\mathbf{K}) = \varprojlim \mathbf{K}$ is a Z -set in $\text{Tel}_{[j,\infty]}^{n+1}(\mathbf{K})$.

Let $\psi: Z \rightarrow \text{Tel}_{[j,\infty]}^{n+1}(\mathbf{K})$ be a map of a space Z with $\dim Z \leq n$. Then it is easy to construct a homotopy $h: Z \times \mathbf{I} \rightarrow \text{Tel}_{[j,\infty]}^{n+1}(\mathbf{K})$ such that $h_0 = \psi$ and $h_t(Z) \subset \text{Tel}_{[j,\infty]}^{n+1}(\mathbf{K})$ for $t > 0$. In general, $\text{Tel}_{[0,\infty]}^{n+1}(\mathbf{K})$ is not an ANR, but we have the following:

Fact 5. Each $\text{Tel}_{[j,\infty]}^{n+1}(\mathbf{K})$ is LC^n , hence it is an $ANE(n+1)$. Moreover, the space $\text{Tel}_{[0,\infty]}^{n+1}(\mathbf{K})$ is n -connected, so it is an $AE(n+1)$.¹

The following follows from Fact 2:

Fact 6. For $i < j \in \mathbb{N} \cup \{0\}$, $d_{i,j}^{\mathbf{K}}| \text{Tel}_{[i,j]}^{n+1}(\mathbf{K}) \simeq^n \text{id}$ in $\text{Tel}_{[i,j]}^{n+1}(\mathbf{K})$, hence $q_{i,j} \simeq^n \text{id}_{K_j}$ in $\text{Tel}_{[i,j]}^{n+1}(\mathbf{K})$. Moreover, $d_i^{\mathbf{K}}| \text{Tel}_{[i,\infty]}^{n+1}(\mathbf{K}) \simeq^n \text{id}$ in $\text{Tel}_{[i,\infty]}^{n+1}(\mathbf{K})$, so $q_i \simeq^n \text{id}_{K_j}$ in $\text{Tel}_{[i,j]}^{n+1}(\mathbf{K})$.

2. THE STRONG n -SHAPE CATEGORY Sh_S^n

Let $\mathcal{H}^n$ be the n -homotopy category of compacta and Sh^n the n -shape category of compacta. In this section, we define the strong n -shape category Sh_S^n of compacta and show that the n -shape functor from $\mathcal{H}^n$ to Sh^n is factorized into two functors through the category Sh_S^n .

Every compactum X is the limit of an inverse sequence $\mathbf{K} = (K_i, q_i)_{i \in \mathbb{N}}$ of finite simplicial complexes such that each $q_{i,i+1}: K_{i+1} \rightarrow \text{Sd } K_i$ is simplicial for the barycentric subdivision $\text{Sd } K_i$ of K_i and $\dim K_i \leq \dim X$ for all $i \in \mathbb{N}$ [Isb, Lemma 33] (cf. Proof of [Ko₂, Theorem 1]). We call $\mathbf{K}$ a *barycentric sequence associated with X* . It should be noted that $q_{i,i+1}: K_{i+1} \rightarrow K_i$ is not simplicial in general. In fact, there exists a 1-dimensional compact AR which is not the limit of any inverse sequence of simplicial complexes and *simplicial* maps [Ko₁, Theorem 1(2)] (cf. [Ko₂, p.536]). It should be also noted that a barycentric sequence associated with X is an $LC^n(n+1)$ -sequence associated with X (cf. [Chi₂]).

Theorem 1. Let X and Y be compacta and $\mathbf{K}, \mathbf{L}$ be barycentric sequences associated with X and Y , respectively.

- (1) Every map $f: X \rightarrow Y$ extends to a map $\bar{f}: \text{Tel}_{[0,\infty]}(\mathbf{K}) \rightarrow \text{Tel}_{[0,\infty]}(\mathbf{L})$ such that $\bar{f}(\text{Tel}_{[0,\infty]}^k(\mathbf{K})) \subset \text{Tel}_{[0,\infty]}^k(\mathbf{L})$ for each $k \in \mathbb{N}$.
- (2) For two maps $f, g: \text{Tel}_{[0,\infty]}^{n+1}(\mathbf{K}) \rightarrow \text{Tel}_{[0,\infty]}^{n+1}(\mathbf{L})$ with $f^{-1}(Y) = g^{-1}(Y) = X$, if $f|_X \simeq^n g|_X$ in Y then $f| \text{Tel}_{[0,\infty]}^{n+1}(\mathbf{K}) \simeq_p^n g| \text{Tel}_{[0,\infty]}^{n+1}(\mathbf{K})$ in $\text{Tel}_{[0,\infty]}^{n+1}(\mathbf{L})$.

In Theorem 1(1) above, a proper map $\bar{f}| \text{Tel}_{[0,\infty]}^{n+1}(\mathbf{K}): \text{Tel}_{[0,\infty]}^{n+1}(\mathbf{K}) \rightarrow \text{Tel}_{[0,\infty]}^{n+1}(\mathbf{L})$ is said to be *induced* by f . By Theorem 1(2), the proper homotopy class of such a map is unique. The following is a direct consequence of Theorem 1.

¹A space Y is an $AE(n+1)$ (or an $ANE(n+1)$) if every map of any closed set A in an arbitrary metrizable space X with $\dim X \leq n+1$ extends over X (or a neighborhood of A). A space Y is an $AE(n+1)$ if and only if Y is an n -connected $ANE(n)$, and Y is an $ANE(n+1)$ if and only if Y is LC^n .

Corollary 1. *Let $\mathbf{K}$ and $\mathbf{L}$ be barycentric sequences associated with the same compactum X . Then a proper map $h: \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}) \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{L})$ induced by id_X is a proper n -homotopy equivalence.*

Definition of Sh_S^n . Let X and Y be compacta. Let $\mathbf{K}, \mathbf{K}'$ be barycentric sequences associated with X and $\mathbf{L}, \mathbf{L}'$ barycentric sequences associated with Y . Two proper maps $F: \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}) \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{L})$ and $F': \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}') \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{L}')$ are n -fundamentally equivalent (written by $F \simeq_f^n F'$) if $h'F \simeq_p^n F'h$ for some proper n -homotopy equivalences $h: \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}) \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}')$ and $h': \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{L}') \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{L})$ induced by id_X and id_Y , respectively. A *strong n -shape morphism* from X to Y is the n -fundamentally equivalence class of a proper map $F: \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}) \rightarrow \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{L})$, where $\mathbf{K}$ and $\mathbf{L}$ are barycentric sequences associated with X and Y respectively. Thus, the strong n -shape category Sh_S^n of compacta can be defined.

The following follows immediately from Theorem 1 and the definition above.

Corollary 2. *There exists a functor $\Xi: \mathcal{H}^n \rightarrow \text{Sh}_S^n$ which maps objects identically.*

For simplicity, let us assign each compactum X to a barycentric sequence $\mathbf{K}^X = (K_i^X, q_{i,i+1}^X)_{i \in \mathbb{N}}$ associated with X and denote as follows:

$$\begin{aligned} \text{Tel}_{[0,\infty)}^{n+1}(X) &= \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}^X), \quad \text{Tel}_{[j,k]}^{n+1}(X) = \text{Tel}_{[j,k]}^{n+1}(\mathbf{K}^X), \\ c_{i,i+1}^X &= c_{i,i+1}^{\mathbf{K}^X} | \text{Tel}_{[0,i+1]}^{n+1}(\mathbf{K}^X), \quad c_i^X = c_i^{\mathbf{K}^X} | \text{Tel}_{[0,\infty)}^{n+1}(\mathbf{K}^X), \\ d_i^X &= d_i^{\mathbf{K}^X} | \text{Tel}_{[i,\infty)}^{n+1}(\mathbf{K}^X), \quad \text{etc.} \end{aligned}$$

Thus, X is assigned to the following commutative diagram of inverse sequences:

$$\begin{array}{ccccccc} \text{Tel}_{[0,1]}^{n+1}(X) & \xleftarrow[c]{c_{1,2}^X} & \text{Tel}_{[0,2]}^{n+1}(X) & \xleftarrow[c]{c_{2,3}^X} & \text{Tel}_{[0,3]}^{n+1}(X) & \xleftarrow[c]{c_{3,4}^X} & \dots \\ \cup & & \cup & & \cup & & \dots \\ |K_1^X| & \xleftarrow[q_{1,2}^X]{} & |K_2^X| & \xleftarrow[q_{2,3}^X]{} & |K_3^X| & \xleftarrow[q_{3,4}^X]{} & \dots \end{array}$$

Now, we prove the following:

Theorem 2. *There exists a full² functor $\Theta: \text{Sh}_S^n \rightarrow \text{Sh}^n$ such that $\Theta \circ \Xi: \mathcal{H}^n \rightarrow \text{Sh}^n$ is the n -shape functor.*

Remarks. The following proposition can be proved similarly to Theorem 1(1).

Proposition. *Let $\mathbf{K}$ and $\mathbf{L}$ be barycentric sequences associated with compacta X and Y , respectively. Every proper map $f: \text{Tel}_{[0,\infty)}(\mathbf{K}) \rightarrow \text{Tel}_{[0,\infty)}(\mathbf{L})$ is properly homotopic to a proper map $\bar{f}: \text{Tel}_{[0,\infty)}(\mathbf{K}) \rightarrow \text{Tel}_{[0,\infty)}(\mathbf{L})$ such that $\bar{f}(\text{Tel}_{[0,\infty)}^k(\mathbf{K})) \subset \text{Tel}_{[0,\infty)}^k(\mathbf{L})$ for each $k \in \mathbb{N}$.*

By the same proof, Theorem 1(2) is valid even if $\text{Tel}_{[0,\infty)}^{n+1}$ is replaced with $\text{Tel}_{[0,\infty]}$. Then, in the definition of Sh_S^n , replacing $\text{Tel}_{[0,\infty)}^{n+1}$ by $\text{Tel}_{[0,\infty]}$, we can define the

²The functor is *full* if the induced maps of the sets of morphisms are surjective.

category $\overline{\text{Sh}}_S^n$ which factorizes the n -shape functor into two functors through $\overline{\text{Sh}}_S^n$. In fact, the functor Ξ in Corollary 2 is factorized into two natural functors through $\overline{\text{Sh}}_S^n$, where the natural functor from $\overline{\text{Sh}}_S^n$ to Sh_S^n can be obtained by the proposition above. As is easily observed, the functor from $\overline{\text{Sh}}_S^n$ to Sh_S^n is injective, but it is a problem whether it is surjective or not.

$$\begin{array}{ccc} \mathcal{H}^n & \longrightarrow & \text{Sh}^n \\ \downarrow & & \uparrow \\ \overline{\text{Sh}}_S^n & \longrightarrow & \text{Sh}_S^n \end{array}$$

In the definition of Sh_S^n , replacing $\text{Tel}_{[0,\infty)}^{n+1}$ and $\simeq_p^n$ by $\text{Tel}_{[0,\infty)}$ and $\simeq_p$, we can obtain the strong shape category Sh_S (cf. [DS]). Then, we can easily obtain the natural functor from Sh_S to $\overline{\text{Sh}}_S^n$. Let $\mathcal{H}$ be the homotopy category of compacta. We have the following diagram of categories and functors:

$$\begin{array}{ccccccc} \mathcal{H} & \longrightarrow & \text{Sh}_S & \xlongequal{\quad} & \text{Sh}_S & \longrightarrow & \text{Sh} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{H}^n & \longrightarrow & \overline{\text{Sh}}_S^n & \longrightarrow & \text{Sh}_S^n & \longrightarrow & \text{Sh}^n \end{array}$$

Restricting the objects to compacta with $\dim \leq k$, we have the subcategories $\text{Sh}(k)$, $\text{Sh}^n(k)$, $\text{Sh}_S(k)$, $\text{Sh}_S^n(k)$ and $\overline{\text{Sh}}_S^n(k)$ of Sh , Sh^n , Sh_S , Sh_S^n and $\overline{\text{Sh}}_S^n$, respectively. Then, $\text{Sh}_S^n(n) = \overline{\text{Sh}}_S^n(n)$ because $\text{Tel}_{[0,\infty)}^{n+1}(X) = \text{Tel}_{[0,\infty)}(X)$ if $\dim X \leq n$. Moreover, $\text{Sh}_S^n(n-1) = \overline{\text{Sh}}_S^n(n-1) = \text{Sh}_S(n-1)$ because $\dim \text{Tel}_{[0,\infty)}(X) \leq n$ if $\dim X \leq n-1$. Although $\text{Sh}^n(n) = \text{Sh}(n)$, it is not known whether $\text{Sh}_S^n(n) = \text{Sh}_S(n)$ or not.

3. AN ISOMORPHISM BETWEEN $\text{Sh}_S^n(\mathcal{Z}(\mu^{n+1}))$ AND $\mathcal{H}_P^n(\mathcal{M}_{n+1})$

Let $\mathcal{Z}(\mu^{n+1})$ be the class of Z -sets in μ^{n+1} and $\mathcal{M}_{n+1}$ the class of μ^{n+1} -manifolds $\mu^{n+1} \setminus X$, $X \in \mathcal{Z}(\mu^{n+1})$. In this section, we prove that the strong n -shape category $\text{Sh}_S^n(\mathcal{Z}(\mu^{n+1}))$ of $\mathcal{Z}(\mu^{n+1})$ is categorically isomorphic to the proper n -homotopy category $\mathcal{H}_P^n(\mathcal{M}_{n+1})$ of $\mathcal{M}_{n+1}$.

Lemma 1. *Let $f: X \rightarrow Y$ be a map from a locally compact separable metrizable space X with $\dim X \leq n+1$ to a completely metrizable ANE($n+1$) Y . For any closed set $A \subset X$ and a Z -set $B \subset Y$, f is approximated by maps $g: X \rightarrow Y$ such that $g|_A = f|_A$ and $g(X \setminus A) \subset Y \setminus B$.*

As in §2, we assign each $X \in \mathcal{Z}(\mu^{n+1})$ to the following diagram:

$$\begin{array}{ccccccc} \text{Tel}_{[0,1]}^{n+1}(X) & \xleftarrow[c]{c_{1,2}^X} & \text{Tel}_{[0,2]}^{n+1}(X) & \xleftarrow[c]{c_{2,3}^X} & \text{Tel}_{[0,3]}^{n+1}(X) & \xleftarrow[c]{c_{3,4}^X} & \cdots \\ \cup & & \cup & & \cup & & \cdots \\ |K_1^X| & \xleftarrow[q_{1,2}^X]{} & |K_2^X| & \xleftarrow[q_{2,3}^X]{} & |K_3^X| & \xleftarrow[q_{3,4}^X]{} & \cdots, \end{array}$$

where the lower sequence is a barycentric sequence associated with X . To prove Theorem 3, we apply the construction in [Sa] to this diagram.

Let $M_1^X = C(K_1^X)^{(n+1)}$. Then $|M_1^X| = \text{Tel}_{[0,1]}^{n+1}(X)$. We inductively define a simplicial complex

$$M_{i+1}^X = (\text{Sd } M_i^X \times I)^{(n+1)} \cup M(q_{i,i+1}^X)^{(n+1)},$$

where we identify $\text{Sd } M_i^X = \text{Sd } M_i^X \times \{0\}$. So we have

$$M(q_{i,i+1}^X)^{(n+1)} \cap (\text{Sd } M_i^X \times I) = M(q_{i,i+1}^X)^{(n+1)} \cap \text{Sd } M_i^X = \text{Sd } K_i.$$

Observe that $\text{Tel}_{[0,i+1]}^{n+1}(X) = \text{Tel}_{[0,i]}^{n+1}(X) \cup |M(q_{i,i+1}^X)^{(n+1)}| \subset |M_{i+1}^X|$. The simplicial collapsing map $c_{q_{i,i+1}^X}: M(q_{i,i+1}^X)^{(n+1)} \rightarrow \text{Sd } K_i^X$ extends to the simplicial retraction

$$\tilde{c}_{i,i+1}: M_i^X = (\text{Sd } M_{i-1}^X \times I)^{(n+1)} \cup M(q_{i,i+1}^X)^{(n+1)} \rightarrow (\text{Sd } M_{i-1}^X \times I)^{(n+1)}.$$

We define $r_{i,i+1}^X = \text{pr}_i \tilde{c}_{i,i+1}: M_{i+1}^X \rightarrow M_i^X$, where $\text{pr}_i: (\text{Sd } M_i^X \times I)^{(n+1)} \rightarrow M_i^X$ is the projection. Let $\pi_1^X = \text{id}: |M_1^X| \rightarrow \text{Tel}_{[0,1]}^{n+1}(X) (= |M_1^X|)$ and inductively define the retraction $\pi_{i+1}^X: |M_{i+1}^X| \rightarrow \text{Tel}_{[0,i+1]}^{n+1}(X)$ by $\pi_{i+1}^X|_{|M(q_{i,i+1}^X)^{(n+1)}|} = \text{id}$ and $\pi_{i+1}^X|_{(\text{Sd } M_i^X \times I)^{(n+1)}} = \pi_i^X \text{pr}_i$. Thus, we obtain the following commutative diagram of the inverse sequences:

$$\begin{array}{ccccccc} |M_1^X| & \xleftarrow[\text{c}]{r_{1,2}^X} & |M_2^X| & \xleftarrow[\text{c}]{r_{2,3}^X} & |M_3^X| & \xleftarrow[\text{c}]{r_{3,4}^X} & \dots \\ \parallel & & \pi_2^X \downarrow \cup & & \pi_3^X \downarrow \cup & & \\ \text{Tel}_{[0,1]}^{n+1}(X) & \xleftarrow[\text{c}]{c_{1,2}^X} & \text{Tel}_{[0,2]}^{n+1}(X) & \xleftarrow[\text{c}]{c_{2,3}^X} & \text{Tel}_{[0,3]}^{n+1}(X) & \xleftarrow[\text{c}]{c_{3,4}^X} & \dots \\ \cup & & \cup & & \cup & & \\ |K_1^X| & \xleftarrow[\text{q}_{1,2}^X]{} & |K_2^X| & \xleftarrow[\text{q}_{2,3}^X]{} & |K_3^X| & \xleftarrow[\text{q}_{3,4}^X]{} & \dots \end{array}$$

Recall that $\text{Tel}_{[0,\infty)}^{n+1}(X) = \bigcup_{i \in \mathbb{N}} \text{Tel}_{[0,i]}^{n+1}(X)$, $\text{Tel}_{[0,\infty]}^{n+1}(X) = \text{Tel}_{[0,\infty)}^{n+1}(X) \cup X$ is the inverse limit of the middle sequence and X is the inverse limit of the bottom sequence. Let M^X be the inverse limit of the upper sequence. Then $X \subset \text{Tel}_{[0,\infty]}^{n+1}(X) \subset M^X$ but $M^X \neq X \cup \bigcup_{i \in \mathbb{N}} |M_i^X|$. Applying Bestvina's characterization of μ^{n+1} [Be], one can see that $M^X \approx \mu^{n+1}$ (cf. [Sa] and [Iwa, Proposition 2.1]). It is easily seen that X is a Z -set in M^X (it is also a Z -set in $\text{Tel}_{[0,\infty]}^{n+1}(X)$ [Sa]). Since $(M^X, X) \approx (\mu^{n+1}, X)$ by the Z -set unknotting theorem [Be], we have a homeomorphism $h_X: M^X \setminus X \rightarrow \mu^{n+1} \setminus X$. On the other hand, we have the retraction of $\pi^X: M^X \rightarrow \text{Tel}_{[0,\infty]}^{n+1}(X)$ induced by π_i^X . Observe that $\pi^X|_X = \text{id}$ and $\pi^X(M^X \setminus X) = \text{Tel}_{[0,\infty)}^{n+1}(X)$.

Lemma 2. $\pi^X|_{M^X \setminus X} \simeq_p^n \text{id}$ in $M^X \setminus X$.

Now we have the following:

Theorem 3. *There is a categorical isomorphism $\Phi: \text{Sh}_S^n(Z(\mu^{n+1})) \rightarrow \mathcal{H}_P^n(\mathcal{M}_{n+1})$ such that $\Phi(X) = \mu^{n+1} \setminus X$ for $X \in Z(\mu^{n+1})$.*

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